Option Pricing with Lévy Processes

Jump models for European-style options

Rui Monteiro

Dissertation

Master of Science in Financial Mathematics

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*Supervisor:* João Pedro Nunes

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Abstract

The goal of this dissertation is to explain how the pricing of European-style options under Lévy processes, namely jump and jump diffusion processes, can be performed and the mathematics associated with it. For this purpose, three models are exposed: Merton, Kou and Variance Gamma, each with different valuation approaches. A Monte Carlo path simulation is also explained. Finally, calibration of the models to real data takes place.
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Chapter 1

Introduction

1.1 Motivation: Why Large Oscillations

A common problem within standard financial theory is that one admits that the price of a financial asset follows a stochastic process based on a Brownian motion, in which variations of great amplitude are not likely to happen. This assumption dates back from the origins of modern financial theory, pioneered by [4]. This assumption was also the base of the first option pricing model, the well known Black-Scholes model — see [7]. However, empirical evidence shows that extreme variations in prices do occur and that the tails of the distributions of the logarithms of the returns on the assets are heavier than usually assumed — see, for instance, [22]. This has serious consequences in the evaluation of financial derivatives, namely options, since their prices derive from the underlying asset’s behavior. Path continuity is also a common assumption. It is, however, inconsistent with the existence these variations — also called jumps. These are illustrated, using real data, in Appendix A.1. One way to try and get around the problem is to consider Lévy processes, namely processes with jumps, in which non marginal variations are more likely to happen as a consequence of fat-tailed distribution based processes, being, therefore, much more realistic.
1.2 Objective and Methodology

This thesis aims to describe pricing models for European-style options, assuming that the underlying asset follows a Lévy process — jump-diffusion and pure jump — hoping that this way, one can better explain the options' market. The thesis is structured as follows. First, an introduction to the mathematical tools associated with Lévy processes. The following chapter is dedicated to two jump-diffusion models — the Merton and the Kou models — where a common introduction takes place, since both models are similar in their essence. However, the methodologies used for pricing are different. Some other fundamental mathematical methods are explained. Then, pure jump processes are introduced and one of the applications, namely the Variance Gamma model, is exposed, with the particularity that the new pricing approach is totally different from the ones used in the jump-diffusion models. A Monte Carlo path simulation is also explained and calibration to real data also takes place. Finally, I draw some conclusions and make a suggestion of further reading.

This thesis is, thus, a very modern, compact and clear exposure of the theory and applications of Lévy processes in financial modeling, and can also be viewed as an extension to the models studied during this master’s course.

I hope the reader may enjoy this work.
Chapter 2

Mathematical Tools

2.1 Stochastic Process

First of all, it is important to recall what a stochastic process is. A stochastic process is simply a collection of random variables referenced to an ordered set. This way, it can be used to construct models in which some given system might evolve in a non deterministic way, since a whole set of paths are possible. In finance, such processes are of great importance, since one is willing to model the paths of an asset’s price, say stock, which are not deterministic at all, for at each instant in time it evolves randomly. Formally a stochastic process can be defined as:

Definition 1. A stochastic process is a collection of random variables $S_t$ referenced to an ordered set \( \{ t : t \in [0, T] \} \) and defined in a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), where \( \Omega \) is the set of all possible outcomes, \( \mathcal{F} \) is a subset of \( \Omega \) and \( \mathbb{P} \) is a function which associates a number — probability — to an event.

2.2 Lévy Process

There are several types of stochastic processes. The one to be dealt with in this work is called a Lévy process. It has some particular aspects that define it. It is a sequence of random variables which are uncorrelated, e.g., the evolution to the next step is not influenced by its past, and are also stationary, which means that the increments
that take the process from one point to another within the ordered set — say time — follow the same distribution. It is quite simple, and mathematically can be defined as follows:

**Definition 2.** A Lévy process is a stochastic process $L = \{L_t : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which verifies the following properties:

1. $L$ has $\mathbb{P}$-almost surely right continuous paths with left limits;
2. $L_0 = 0$;
3. for $0 \leq s \leq t$, $L_t - L_s$ has the same distribution law of $L_{t-s}$ (stationarity);
4. for $0 \leq s \leq t$, $L_t - L_s$ is independent of $L_u : u \leq s$.

For those already familiarized with Wiener normal processes, clearly the only difference is that in the Wiener case it is assumed that the increments follow a normal distribution. Therefore, Wiener processes are a subclass of Lévy processes and generate pure diffusion, since great magnitude variations are not likely.

### 2.3 Poisson Process

Since the new component is jump like, it is important is to introduce pure jump processes. A jump process is a sequence of random variables that form a purely discontinuous path. To illustrate it, let us define a Poisson process.

**Definition 3.** A Poisson process is a sequence of random variables with jump size 1, where the occurrence of such jumps is distributed in time as

$$
\mathbb{P}(N(t + dt) - N(t) = k) = \frac{e^{-\lambda dt} (\lambda dt)^k}{k!}, \quad k \in \mathbb{N}_0^+
$$

and the increments are independent and stationary. $\lambda$ is called the intensity and $k$ is the number of events between $t$ and $t + dt$. 
2.4 Compound Poisson Process

In insurance, for example, Poisson processes are used to model the arrival of accidents for the company to cover, which arrive at a rate $\lambda$. The value to be paid by the company is another random variable, making a so called compound Poisson process. It can be formally defined as:

**Definition 4.** A compound Poisson process is a sequence of random variables where the jump size follows a certain distribution, and the occurrence of such a jump is distributed in time as a standard Poisson process.

2.5 Lévy-Khintchine Representation

A fundamental theorem concerning Lévy processes is the so called Lévy-Khintchine representation:

**Theorem 1.** Lévy-Khintchine: Let $L_t$ be a Lévy process. Then its characteristic function is of the form

$$\phi_{L_t} = \int_{\mathbb{R}} e^{i\theta L_t} \nu(dx) = e^{t\Psi(\theta)} \forall \theta \in \mathbb{R} \quad (2.1)$$

iff $\exists$ a triplet $(a, \sigma^2, \nu)$, $a \in \mathbb{R}$, $\sigma \geq 0$, $\nu$ a measure concentrated on $\mathbb{R}\setminus\{0\}$ that satisfies $\int_{\mathbb{R}} (1 \wedge x^2) \nu(dx) \leq \infty$ s.t.

$$\Psi(\theta) = ia\theta - \frac{1}{2} \sigma^2 \theta^2 + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x 1_{|x| \leq 1}\right) \nu(dx) \forall \theta \in \mathbb{R} \quad (2.2)$$

where $\Psi$ is called the characteristic exponent.

*Proof.** See [10].

This theorem is the core of the theory of Lévy processes, and deserves much attention. It states that every Lévy process has a characteristic function of the form of equation (2.1). Thus, they can be parameterized using the triplet $(a, \sigma^2, \nu)$, where $a$ is the drift,
\(\sigma^2\) is the variance of a Brownian motion and \(\nu(dx)\) is the so called Lévy measure or jump measure. As for the drift and Brownian motion components, one assumes that the reader is already familiar with them. The new component is the Lévy measure, \(\nu(dx)\), which is associated with the jumps beyond “normal” that make the process path discontinuous.

### 2.6 Finite and Infinite Activity

The rate at which such discontinuities arrive is given by the integral of the Lévy density, \(\lambda = \int_{-\infty}^{\infty} \nu(dx)\). In the particular case of compound Poisson processes, \(\nu(dx) = \lambda f(x)dx\), where \(f(x)\) is the density of the distribution of jumps. When this integral is finite the process is said to have finite activity. If the integral is infinite, then it is said to have infinite activity, which implies that the process can have infinite jumps in a finite time interval.

So, a pure jump process has a triplet of the form \((a, 0, \nu)\), since there is no diffusion component. As for a pure diffusion process, the triplet is of the form \((a, \sigma^2, 0)\) since no jump component exists, whereas in a jump-diffusion process the triplet takes the form \((a, \sigma^2, \nu)\) for it is a mixture of jump and diffusion components.

It is also important to point out that a jump-diffusion has a jump component with finite rate \(\lambda\), whereas a pure jump process has, in financial modeling, infinite activity.

### 2.7 Gamma Process

Another important process in financial modeling is the Gamma process. It is a purely discontinuous process with infinite activity. Since the Lévy-Khintchine representation has already been explained, this process will be stated in terms of its Lévy measure.

It can be defined as follows.

**Definition 5.** The Gamma process — see \([20]\) — \(\gamma_t(\mu, \delta)\) with mean rate \(\delta\) and variance rate \(\nu\) is the increasing process of independent Gamma increments which
has the following Lévy measure:

\[
\nu(dx) = \frac{\mu^2 e^{-\frac{\mu}{\delta} x}}{\delta x} \mathbb{1}_{x>0} dx
\]

See also \cite{[3]}. This measure has infinite integral, so the process has an infinite arrival rate of jumps, most of which are small, as one can see from the concentration of the Lévy measure at the origin. So, at every instant in time an increment is added, which follows a Gamma distribution. This process will be used in one of the option pricing models presented in this thesis.

### 2.8 Momenta of Stochastic Processes

Also recall that once one has the characteristic function, the \(n\)th moment of the process can be found through

\[
E[L_t^{(n)}] = i^{-n} \left[ \frac{d^n}{d\theta^n} \phi_{L_t}(\theta) \right]_{\theta=0}
\]

(2.3)

**Proof.** See \cite{[1]}. \qed

### 2.9 Exponential Lévy Process

Another important process to point out is the Lévy analogous to the well known geometric Brownian motion, since in finance one often works with the log returns on the asset. It is called exponential Lévy process and can be defined as follows.

**Definition 6.** Following \cite{[10]}, an exponential Lévy process is a stochastic process that can be written in the form:

\[
S_t = S_0 e^{L_t}
\]

where \(L_t\) is a Lévy process.
2.10 Lévy-Itô Lemma

Since we will be using processes with a jump component it is important to define a way of differentiating functions with stochastic jump and diffusion arguments. Traditionally this is done through the Itô lemma. However, it only admits Wiener like arguments. So, an extension of this lemma to processes with jumps needs to be stated.

**Theorem 3.** Lévy-Itô lemma: For a process of the type 
\[ L_t = L_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \sum_{i=1}^{N_t} \Delta X_i \]
the Itô formula is:

\[
\frac{df(L_t, t)}{dt} dt + b_t \frac{\partial f(L_t, t)}{\partial x} dt + \frac{\sigma_t^2}{2} \frac{\partial^2 f(L_t, t)}{\partial x^2} dt + \sigma_t \frac{\partial f(L_t, t)}{\partial x} dW_t \\
+ [f(X_{t^-} + \Delta X_t) - f(X_{t^-})]
\]  

(2.4)

*Proof.* See [10].

Notice that the only difference to the traditional Itô lemma is the last term, which accounts for the occurrence of jumps, where \( t^- \) is the left limit of the function at a given point in time, e.g., the value that the function takes just before the occurrence of a jump. This lemma is valid for finite activity processes only. For infinite activity see, for instance, [23].

I would also like to point out that in this work one will only deal with finite variance processes, since this is an essential requirement in derivative pricing. If this was not the case, then asset’s price would be likely to jump with arbitrary magnitude, which would not be realistic.

Now that one is familiarized with Lévy processes, we will proceed to the pricing models.
Chapter 3

Jump-Diffusions

3.1 The Idea

The idea of a jump-diffusion process — see [25] — is that the diffusion part takes into account the normal fluctuations in the risky asset’s price caused by, for example, a temporary imbalance between supply and demand, changes in capitalization rates, changes in the economic outlook or other information that causes marginal changes in price. As for the non-marginal variations, it is expected that the information that would cause them come in discrete points in time, and that is why a jump component is added to the traditional diffusion process.

3.1.1 Incorporation of Jumps

The goal is to start from a diffusion model, namely the Black and Scholes model — see [7] — and add up a jump component, which is modeled as a compound Poisson process.

One begins by defining the probability of a certain number of jumps $N$ occurring in
the time interval \( dt \), which we call \( dN_t \):

\[
P(N = 1 \text{ in } dt) = P(dN_t = 1) \approx \lambda dt \\
P(dN_t \geq 2) \approx 0 \\
P(dN_t = 0) \approx 1 - \lambda dt
\]

where \( \lambda \) is the jump intensity. Notice that the arrival of jumps is approximated by a Bernoulli variable, which comes from a series expansion of the Poisson distribution that models the arrival of jumps.

Suppose that in \( dt \), \( S_t \) jumps to \( V_t S_t \), where \( V_t \) is the absolute jump size.

The relative jump size is then:

\[
\frac{dS_t}{S_t} = \frac{V_t S_t - S_t}{S_t} = V_t - 1
\]

### 3.1.2 Equation of Dynamics

Thus, in order to incorporate the jump component into the model, one writes the stochastic differential equation that defines the dynamics as:

\[
\frac{dS_t}{S_t} = (\alpha - \lambda \zeta) dt + \sigma W_t + (V_t - 1) dN_t 
\]

where \( W_t \) is a standard Brownian motion, \( N_t \) is a Poisson process with intensity \( \lambda \), and it is assumed that \( S_t \), \( N_t \) and \( V_t \) are independent. \( \alpha \) is the instantaneous expected return, \( \sigma \) is the volatility associated with the Brownian motion and

\[
\zeta = \mathbb{E}[V_t - 1]
\]

Notice that

\[
\mathbb{E}\left[\frac{dS_t}{S_t}\right] = \alpha dt
\]

Thus, the term \(-\lambda \zeta\) is added in the drift of equation (3.1) in order to compensate the Poisson process, so that the log returns on the asset will have an expectation of
αdt. Thus, the process of the discounted prices becomes a martingale.

3.1.3 Solution via Lévy-Itô Lemma

**Theorem 4.** The solution for the stochastic differential equation

\[ dS_t = S_t(\alpha - \lambda \zeta)dt + S_t\sigma W_t + S_t(V_t - 1)dN_t \]

is

\[ S_t = S_0e^{(\alpha - \frac{\sigma^2}{2} - \lambda \zeta)t + \sigma W_t + \sum_{k=1}^{N_t} Y_k} \]

**Proof.** Applying the Itô formula (2.4) for jump-diffusion processes from **Theorem 3** to the log returns \(f(S_t) = \ln(S_t)\), then

\[
\begin{align*}
    df &= \frac{\partial f}{\partial t} + (\alpha - \lambda \zeta)S_t\frac{\partial f}{\partial S_t}dt + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 f}{\partial S_t^2}dt + \sigma S_t \frac{\partial f}{\partial S_t}dW_t \\
    &\quad + (f(V_t S_t) - f(S_t)) \\
    d(ln(S_t)) &= (\alpha - \lambda \zeta)S_t \frac{1}{S_t}dt - \frac{\sigma^2 S_t^2}{2}dt + \sigma S_t \frac{1}{S_t}dW_t + ln(V_t) + ln(S_t) - ln(S_t) \\
    d(ln(S_t)) &= (\alpha - \lambda \zeta)dt - \frac{\sigma^2}{2}dt + \sigma dW_t + ln(V_t) \\
    ln(S_t) &= ln(S_0) + (\alpha - \frac{\sigma^2}{2} - \lambda \zeta)t + \sigma W_t + \sum_{k=1}^{N_t} ln(V_k) \\
    S_t &= S_0e^{(\alpha - \frac{\sigma^2}{2} - \lambda \zeta)t + \sigma W_t} \prod_{k=1}^{N_t} V_k \\
    S_t &= S_0e^{(\alpha - \frac{\sigma^2}{2} - \lambda \zeta)t + \sigma W_t + \sum_{k=1}^{N_t} Y_k} \]

where \(Y_k = \ln(V_k)\).

Now, one ends up with an exponential Lévy process \(S_t = S_0e^{L_t}\) where \(L_t\) is of the form

\[
L_t = \underbrace{(\alpha - \frac{\sigma^2}{2} - \lambda \zeta)t}_{\text{drift}} + \underbrace{\sigma W_t}_{\text{Brownian Motion}} + \underbrace{\sum_{k=1}^{N_t} Y_k}_{\text{compound Poisson process}}
\]
and

\[ L_t = \ln \left( \frac{S_t}{S_0} \right) \]

### 3.1.4 Risk Neutral Dynamics

For the purpose of risk neutral valuation, it is required that the asset grows at the risk-free rate, so that the discounted prices become a martingale under the risk neutral measure. So, the logical thing to do is to refer to \( \alpha \) as the risk-free rate, and taking the other parameters as risk neutral too. Jump-diffusion models are incomplete market models, since there is more than one equivalent martingale measure \( Q \sim P \) under which the process of the discounted asset price becomes a martingale. In terms of the hedging portfolio, this is equivalent to saying that there is no way of building a completely risk-free hedging portfolio.

[25] gets an equivalent measure, \( Q \sim P \), by changing the drift of the Brownian motion and keeping the rest unchanged. Then

\[ S_t = S_0 e^{(r-q-\frac{\sigma^2}{2} - \lambda \zeta) t + \sigma W^Q_t + \sum_{k=1}^{N_t} Y_k} \]  

under \( Q \).

Notice that the only thing that happens is that \( \alpha \) becomes \( r - q \), the risk-free rate (minus the continuous dividend yield). [25] suggests that \( dN_t \) has no correlation with the market as a whole, e.g., the risk associated with jumps is non-systematic. Therefore, it can be diversified away. This is the way [25] gets around the incompleteness problem.

### 3.2 The Merton Model

This section exposes the first option pricing application of jump-diffusion processes, proposed by [25]. I shall follow [24].
3.2.1 Dynamics

Under the Merton model it is assumed that the dynamics are driven by a process generated by an equation like (3.2), with jumps log-normally distributed \( Y_k = \ln(V_k) \sim \mathcal{N}(\mu, \delta^2) \), where \( \mathcal{N}(\mu, \delta^2) \) denotes a Gaussian distribution with mean \( \mu \) and variance \( \delta^2 \). Thus

\[
\zeta = e^{\mu + \frac{\delta^2}{2}} - 1
\]

3.2.2 Characteristic Function, Momenta and Activity

Notice that the process can be characterized by the triplet \((b, \sigma^2, \nu(dx))\), where \( b = r - q - \frac{\sigma^2}{2} - \lambda \zeta \) and \( \nu(dx) = \lambda f(dx) = \lambda \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{(dx-\mu)^2}{2\delta^2}} \). Thus, by using Theorem 1, the characteristic function of the process is

\[
\phi_T = e^{ib\theta T - \frac{\sigma^2\theta^2 T}{2} + \lambda T \left( e^{i\mu\theta - \frac{\delta^2\theta^2}{2} - 1} \right)}
\]

From its Lévy density, one can check that this is indeed a finite activity process, since

\[
\int_{-\infty}^{+\infty} \nu(dx) = \lambda
\]

And, by differentiating the characteristic function using Theorem 2, the first two moments can be found to be

\[
E(L_t) = t (b + \lambda \delta)
\]

\[
Var(L_t) = t \left( \sigma^2 + \lambda \mu^2 + \lambda \delta^2 \right)
\]

As we can see, both the mean and variance are finite.

3.2.3 Option Pricing

The value of a contingent claim — say an European-style option — \( H(S_T) \) at time \( t \) is the discounted value of its expectation under the risk neutral measure:

\[
H_t^M(S_t) = e^{-r(T-t)}E^Q[H(S_T)|\mathcal{F}_t]
\]
where the superscript $M$ stands for Merton. Then,

$$H_t^M(S_t) = e^{-r(T-t)}\mathbb{E}^Q[H(S_t e^{(r-q-\frac{\sigma^2}{2} - \lambda\zeta)(T-t) + \sigma W_{T-t}^Q + \sum_{k=1}^{N_{T-t}} Y_k})] | \mathcal{F}_t]$$

$$= e^{-r(T-t)}\mathbb{E}^Q[H(S_t e^{(r-q-\frac{\sigma^2}{2} - \lambda\zeta)(T-t) + \sigma W_{T-t}^Q + \sum_{k=1}^{N_{T-t}} Y_k})]$$

For simplicity let us use $\tau = T - t$ and define the counter as $N_{T-t} = 0, 1, 2, 3... \equiv i$ so that we use index $i$ to denote the number of jumps; note also that the compound Poisson process follows the law $\sum_{k=1}^{N_{\tau}} Y_k \sim \mathcal{N}(i \mu, i \delta^2)$. And, by conditioning on the number of jumps $i$ one obtains:

$$H_t^M(S_t) = e^{-r \tau} \sum_{i \geq 0} Q(N_{\tau} = i)\mathbb{E}^Q[H(S_t e^{(r-q-\frac{\sigma^2}{2} - \lambda\zeta)\tau + \sigma W_{\tau}^Q + \sum_{k=1}^{N_{\tau}} Y_k})]$$

Notice that $Q(N_{\tau} = i)$ is the probability of occurring $i$ jumps between $t$ and $T$, which follows a Poisson distribution. Then $Q(i) = \frac{e^{-\lambda \tau} (\lambda \tau)^i}{i!}$. Thus

$$H_t^M(S_t) = e^{-r \tau} \sum_{i \geq 0} \frac{e^{-\lambda \tau} (\lambda \tau)^i}{i!} \mathbb{E}^Q[H(S_t e^{(r-q-\frac{\sigma^2}{2} - \lambda(e^{\mu+i\delta^2} - 1))\tau + \sigma W_{\tau}^Q + \sum_{k=1}^{N_{\tau}} Y_k})]$$

The term in the exponential follows

$$\{(r - q - \frac{\sigma^2}{2}) - \lambda(e^{\mu+i\delta^2} - 1)\} \tau + \sigma W_{\tau}^Q + \sum_{k=1}^{N_{\tau}} Y_k$$

$$\sim \mathcal{N}(\{r - q - \frac{\sigma^2}{2}) - \lambda(e^{\mu+i\delta^2} - 1)\} \tau + i \mu ; \sigma^2 \tau + i \delta^2)$$

and it can be rewritten so that its distribution stays the same

$$\{(r - q - \frac{\sigma^2}{2}) - \lambda(e^{\mu+i\delta^2} - 1)\} \tau + i \mu + \sqrt{\frac{\sigma^2 \tau + i \delta^2}{\tau}} W_{\tau}^Q$$

$$\sim \mathcal{N}(\{r - q - \frac{\sigma^2}{2}) - \lambda(e^{\mu+i\delta^2} - 1)\} \tau + i \mu ; \sigma^2 \tau + i \delta^2)$$

so it comes
\[ H^M_t(S_t) = e^{-r(T-t)} \sum_{i \geq 0} \frac{e^{-\lambda \tau} (\lambda \tau)^i}{i!} \mathbb{E}^Q[H(S_t e^{\{(r-q-\frac{\sigma^2}{2})-\lambda(e^{\mu+\frac{\delta^2}{2}}-1)\} \tau + i\mu + \sqrt{2 \lambda \tau + i\delta^2} W_t^Q}]] \]

and by adding and subtracting \( \frac{i\delta^2}{2\tau} \) in the exponential one gets

\[ H^M_t(S_t) = e^{-r(T-t)} \sum_{i \geq 0} \frac{e^{-\lambda \tau} (\lambda \tau)^i}{i!} \times \mathbb{E}^Q[H(S_t e^{\{(r-q-\frac{\sigma^2}{2})+\frac{i\delta^2}{2\tau}-\lambda(e^{\mu+\frac{\delta^2}{2}}-1)\} \tau + i\mu + \sqrt{\sigma^2 + \frac{i\delta^2}{\tau} W_t^Q}]}\]

Now, setting \( \sigma_i^2 = \sigma^2 + \frac{i\delta^2}{\tau} \) and rearranging it comes:

\[ H^M_t(S_t) = e^{-r(T-t)} \sum_{i \geq 0} \frac{e^{-\lambda \tau} (\lambda \tau)^i}{i!} \mathbb{E}^Q[H(S_t e^{i\mu + \frac{i\delta^2}{2\tau}-\lambda(e^{\mu+\frac{\delta^2}{2}}-1)\tau + i\mu + \sqrt{\sigma^2 + \frac{i\delta^2}{\tau} W_t^Q}]}\]

Notice that the objective is to get the evaluation formula in terms of the Black-Scholes (BS) formula:

\[ H^{BS}(\tau, S_t, \sigma, r-q) := e^{-r \tau} \mathbb{E}^Q[H(S_t e^{(r-q-\frac{\sigma^2}{2})\tau + \sigma W_t^Q}]
\]

\[ (3.3) \]

### 3.2.4 Pricing Formula

Thus, Merton’s pricing formula can be viewed as a weighted average of BS price conditioned to the number of jumps:

\[ H^M_t(S_t) = \sum_{i \geq 0} \frac{e^{-\lambda \tau} (\lambda \tau)^i}{i!} H^{BS}(\tau, S_t \equiv S_t e^{i\mu + \frac{i\delta^2}{2\tau}-\lambda(e^{\mu+\frac{\delta^2}{2}}-1)\tau} ; \sigma_i \equiv \sqrt{\sigma^2 + \frac{i\delta^2}{\tau}} ; r-q) \]

Alternatively,
\begin{align*}
H_t^M(S_t) &= e^{-r\tau} \sum_{i \geq 0} \frac{e^{-\lambda \tau (\lambda \tau)^i}}{i!} \mathbb{E}^Q[H(S_t e^{(r-q-\lambda(e^\mu + \frac{\delta^2}{2} - 1)\tau + \frac{i\mu + \frac{i\delta^2}{2}}{\tau} - \frac{\sigma^2}{2} + \sigma_i W^Q_\tau})] \\
&= \sum_{i \geq 0} \frac{e^{-\tilde{\lambda} \tau (\tilde{\lambda} \tau)^i}}{i!} H^{BS}(\tau; S_t; \sigma_i; (r-q)_i)
\end{align*}

where

\begin{align*}
\tilde{\lambda} &= \lambda(1 + \zeta) = \lambda(e^\mu + \frac{\delta^2}{2}) \\
\sigma_i &= \sqrt{\sigma^2 + \frac{i\delta^2}{2}} \\
(r-q)_i &= r - q - \lambda(e^\mu + \frac{\delta^2}{2} - 1) + \frac{i\mu + \frac{i\delta^2}{2}}{\tau}
\end{align*}

Notice that a martingale approach has been performed. One could as well get this very same formula by creating a hedging portfolio, as originally done by [25].
3.3 The Kou Model

In the Kou model, the idea is pretty much the same as in Merton’s. It incorporates a geometric drifted Brownian motion and a jump part. However, the latter follows a double exponential distribution, instead of Merton’s normal law.

Some advantages of this model are that it captures some important features of the dynamics of financial assets in the real market, namely the asymmetric leptokurtosis and the volatility smile, and yet provides closed form solutions for the most important types of options.

3.3.1 Dynamics

Following [17], under the risk neutral measure the asset’s price is modeled by the following stochastic differential equation:

$$\frac{dS_t}{S_t} = (r - q - \lambda \zeta) dt + \sigma dW_t + d \left( \sum_{i=1}^{N_t} (V_i - 1) \right) \quad (3.4)$$

being $W_t$ a Brownian motion, $N_t$ a Poisson process with intensity $\lambda$ and $V_i$ a sequence of independent identically distributed non negative random variables s.t. $Y_i = ln(V_i)$ follows an asymmetric double exponential distribution whose density is given by

$$f_Y(x) = p\eta_1 e^{-\eta_1 x} 1_{x \geq 0} + d\eta_2 e^{\eta_2 x} 1_{x < 0}$$

with $\eta_1 \geq 1$ (so $S_t$ has finite expected value), $\eta_2 \geq 0$, $p,d \geq 0$ and $p + d = 1$.

Therefore, the compensator $\zeta$ is

$$\zeta = E^Q[e^Y - 1] = \frac{p\eta_1}{\eta_1 - 1} + \frac{p\eta_2}{\eta_2 + 1} - 1$$

Notice that equation (3.4) is the same as (3.1), with a slightly different notation for the jump component. Thus, as we saw previously, by using Theorem 4 the solution is:

$$S_t = S_0 e^{(r-q-\frac{\sigma^2}{2}-\lambda \zeta)t + \sigma W_t} \prod_{i=1}^{N_t} V_i$$
Then, the logarithm of the return on the asset $L_t \equiv ln(S_t/S_0)$ follows

$$L_t = \left( r - q - \frac{\sigma^2}{2} - \lambda \zeta \right) t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \quad L_0 = 0 \quad (3.5)$$

### 3.3.2 Characteristic Function, Momenta and Activity

The process can be characterized by the triplet $\left(b, \sigma^2, \nu(dx)\right)$, where $b = r - q - \frac{\sigma^2}{2} - \lambda \zeta$ and $\nu(dx) = \lambda f(dx) = \lambda \left\{ p \eta_1 e^{-\eta_1 dx} \mathbb{1}_{\{x \geq 0\}} + d \eta_2 e^{\eta_2 dx} \mathbb{1}_{\{x < 0\}} \right\}$. Thus, by using [Theorem 1](#), the characteristic function of this process is

$$\phi_t(\theta) = e^{ib\theta t - \frac{\sigma^2 \theta^2}{2} + i\theta \lambda \left( \frac{p}{\eta_1} + \frac{d}{\eta_2} \right)}$$

From its Lévy density, one can check that this is, again, a finite activity process, since

$$\int_{-\infty}^{+\infty} \nu(dx) = \lambda$$

And, by differentiating the characteristic function using [Theorem 2](#), the first two moments can be found to be

$$\mathbb{E}(L_t) = t \left( b + \lambda \left( \frac{p}{\eta_1} + \frac{d}{\eta_2} \right) \right)$$

$$Var(L_t) = t \left( \sigma^2 + \lambda \left( \frac{p^2}{\eta_1^2} + \frac{d^2}{\eta_2^2} \right) \right)$$

As we can see, both the mean and variance are finite.

### 3.3.3 Option Pricing

As we know, in order to get the price of an option, one has to find the present value of its final payoff. For a call option’s in particular, the final payoff is $H(S_T) = (S_T - X)^+$. Then, its present value is given by

$$C_t(S_t, X, T) = e^{-r(T-t)} \mathbb{E}^Q[(S_T - X) \mathbb{1}_{\{S_T \geq X\}} | \mathcal{F}_t] \quad (3.6)$$
Now, introduce the notation for a probability as follows

$$\Psi(r - q, \sigma, \lambda, p, \eta_1, \eta_2, a, T) = \mathbb{Q}(Z_T \geq a | \mathcal{F}_t)$$  \hspace{1cm} (3.7)$$

with $$Z_t := (r - q)t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$$, and where $$Y$$ has the double exponential distribution defined previously.

Now, by expanding equation (3.6) it follows

$$C_t(S_t, X, T) = e^{-r(T-t)}\mathbb{E}_\mathbb{Q}[S_T \mathbb{1}_{\{S_T \geq X\}} | \mathcal{F}_t] - e^{-r(T-t)}X\mathbb{E}_\mathbb{Q}[\mathbb{1}_{\{S_T \geq X\}} | \mathcal{F}_t]$$

$$= e^{-r(T-t)}\mathbb{E}_\mathbb{Q}[S_T \mathbb{1}_{\{S_T \geq X\}} | \mathcal{F}_t] - e^{-r(T-t)}X\mathbb{Q}[S_T \geq X] | \mathcal{F}_t]$$  \hspace{1cm} (3.8)$$

### 3.3.4 Change of Numeraire

In order to compute the first expectation in the right hand side of equation (3.8), one needs to get rid of the $$S_T$$ term that multiplies the indicator function. This is accomplished through a change of numeraire — see [14]. Notice that this technique is not the one used by [17]. The idea is to change to a new numeraire associated with a new martingale measure. Then, one finds the Radon-Nikodym derivative and, by comparison to the Girsanov theorem, some parameters can be extracted and, based on them, a new process can be built. Thus, let us proceed. The numeraire of the “money market account” is defined as

$$\mathbb{Q} \leftrightarrow \beta_t = \beta_0 e^{rt}$$

and the new numeraire can be defined as

$$\widetilde{\mathbb{Q}} \leftrightarrow M_t = S_t e^{qt}$$

where $$q$$ refers to the asset’s dividend yield. Then, the change of numeraire is done through the formula:

$$\beta_t \mathbb{E}_\mathbb{Q}[\frac{X_T}{\beta_T} | \mathcal{F}_t] = M_t \mathbb{E}_{\widetilde{\mathbb{Q}}}[\frac{X_T}{M_T} | \mathcal{F}_t]$$
where $X_T$ represents the time-$T$ price of any financial asset. Therefore,

$$C_t(S_t, X, T) = e^{rt}\mathbb{E}^Q[{S_T\mathbb{1}_{\{S_T\geq x\}}}|\mathcal{F}_t] - e^{-r(T-t)}X\mathbb{Q}[S_T \geq X|\mathcal{F}_t]$$

$$= S_te^{qt}\mathbb{E}^{\tilde{Q}}\left[\frac{S_T}{S_te^{qT}}\mathbb{1}_{\{S_T\geq x\}}|\mathcal{F}_t\right] - e^{-r(T-t)}X\mathbb{Q}[S_T \geq X|\mathcal{F}_t]$$

$$= S_te^{-q(T-t)}\tilde{Q}[S_T \geq X|\mathcal{F}_t] - e^{-r(T-t)}X\mathbb{Q}[S_T \geq X|\mathcal{F}_t]$$

By combining equations (3.7) and (3.5) one gets

$$\mathbb{Q}(Z_T \geq \ln\left(\frac{X}{S_t}\right)|\mathcal{F}_t) = \Psi(r - q - \frac{\sigma^2}{2} - \lambda\zeta, \lambda, p, \eta_1, \eta_2, \ln\left(\frac{X}{S_t}\right), T)$$

Now, all that is left to find is the probability under $\tilde{Q}$. Again, by the previous change of numeraire, the Radon-Nikodym is:

$$\frac{d\tilde{Q}}{d\mathbb{Q}} = \frac{S_Te^{qr}}{S_te^{rt}}$$

$$= \frac{S_T}{S_t}e^{-(r-q)(T-t)}$$

$$= e^{-(r-q)(T-t)}e^{LT}$$

$$= e^{-\frac{\sigma^2}{2}(T-t)+\sigma(W_{T-t}+\sum_{i=1}^{N_{T-t}}Y_i)}e^{-\lambda \zeta(T-t)+\sum_{i=1}^{N_{T-t}}Y_i}$$

(3.9)

### 3.3.5 Girsanov in Jump-Diffusion

Since one is dealing with a jump-diffusion model, an extended version of the Girsanov theorem is required.

**Theorem 5.** Girsanov for jump-diffusion: Let $\mathbb{Q}$ be a probability measure and $\tilde{\mathbb{Q}}$ an
equivalent martingale measure. If the Radon-Nikodym derivative is given in the form
\[
\frac{d\tilde{Q}}{dQ} = e^{-\frac{\sigma^2}{2} t - \sigma W_t - \lambda \zeta + \sum_{i=1}^{N_t} (\alpha Y_i + \theta)}
\]
where
\[
\zeta = e^{\theta} E^Q[e^{\alpha Y}] - 1
\]
then, under the new measure \( \tilde{Q} \)
\[
\tilde{W}_t = W_t - \sigma t \quad \tilde{\lambda} = \lambda (1 + \zeta)
\]
\[
\tilde{f}_Y(x) = \frac{e^{\alpha x} f_Y(x)}{\int_{-\infty}^{+\infty} e^{\alpha x} f_Y(x) dx}
\]
being \( f_Y(x) \) the distribution function of jumps and \( \tilde{f}_Y(x) \) its so called Esscher transform — see [15].

**Proof.** See [9].

Applying Theorem 5 to equation (3.9) — with \( t \to T - t, \alpha = 1 \) and \( \theta = 0 \) — and following [18] and [9], the rate under the new measure is
\[
\tilde{\lambda} = \lambda (1 + \zeta) = \lambda \left\{ \frac{p \eta_1}{\eta_1 - 1} + \frac{p \eta_2}{\eta_2 + 1} \right\} = \lambda E^Q(e^Y)
\]
Furthermore, following [18], the jump sizes are identical and independently distributed with density
\[
\tilde{f}_Y(x) = \frac{1}{E^Q(e^Y)} e^x f_Y(x) = \frac{1}{E^Q(e^Y)} e^x p \eta_1 e^{-\eta_1 x} \mathbb{1}_{\{x \geq 0\}} + \frac{1}{E^Q(e^Y)} e^x d \eta_2 e^{\eta_2 x} \mathbb{1}_{\{x < 0\}}
\]
\[
= p \left( \frac{p \eta_1}{\eta_1 - 1} + \frac{p \eta_2}{\eta_2 + 1} \right)^{-1} e^x \frac{\eta_1}{\eta_1 - 1} (\eta_1 - 1) e^{-(\eta_1 - 1) x} \mathbb{1}_{\{x \geq 0\}}
\]
\[
+ d \left( \frac{p \eta_1}{\eta_1 - 1} + \frac{p \eta_2}{\eta_2 + 1} \right)^{-1} e^x \frac{\eta_2}{\eta_2 + 1} (\eta_2 + 1) e^{-(\eta_2 + 1) x} \mathbb{1}_{\{x < 0\}}
\]
\[
= \tilde{p} \eta_1 e^{-\tilde{\eta}_1 x} \mathbb{1}_{\{x \geq 0\}} + \tilde{d} \eta_2 e^{-\tilde{\eta}_2 x} \mathbb{1}_{\{x < 0\}}
\]
where
\[ \tilde{p} = \left( \frac{p\eta_1}{\eta_1 - 1} + \frac{d\eta_2}{\eta_2 + 1} \right)^{-1} \frac{\eta_1}{\eta_1 - 1} \]
\[ \tilde{d} = \left( \frac{p\eta_1}{\eta_1 - 1} + \frac{d\eta_2}{\eta_2 + 1} \right)^{-1} \frac{\eta_2}{\eta_2 + 1} \]
\[ \tilde{\eta}_1 = \eta_1 - 1, \tilde{\eta}_2 = \eta_2 + 1 \]

Then
\[ \tilde{\mathcal{Q}}(Z_T \geq \ln(X_{S_t})|\mathcal{F}_t) = \Psi(r - q + \frac{\sigma^2}{2} - \lambda \zeta, \sigma, \tilde{\lambda}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2, \ln(X_{S_0})/T) \]
and finally
\[ C_t(S_t, X, T) = e^{-r(T-t)}S_t \Psi(r - q + \frac{\sigma^2}{2} - \lambda \zeta, \sigma, \tilde{\lambda}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2, \ln(X_{S_t})/T) \]
\[ - e^{-r(T-t)}\Psi(r - q - \frac{\sigma^2}{2} - \lambda \zeta, \sigma, \lambda, p, \eta_1, \eta_2, \ln(X_{S_t})/T) \]

3.3.6 Pricing Formula

In order to achieve the pricing formula, the probabilities of exercise \( \Psi \) need to be specified. In order to do so, it is necessary to find the density for the sum of a normal random variable and a double exponential. The derivation is extensive and can be found in [17]. To specify the probabilities, first it is necessary to define some special functions:
\[ Hh_{-1}(x) = e^{-\frac{x^2}{2}} \]
\[ Hh_0(x) = \sqrt{2\pi}\Phi(-x) \]
\[ Hh_n(x) = \int_{-\infty}^{+\infty} Hh_{n-1}(y)dy = \frac{1}{n!} \int_{-\infty}^{+\infty} (t - x)^n e^{-\frac{t^2}{2}} dt \]
\[ I_n(c, \alpha, \beta, \gamma) = \int_{c}^{+\infty} e^{\alpha x} Hh_n(\beta x - \gamma)dx \quad \forall n \geq 1 \]
where $\Phi$ is the standard normal cumulative distribution. Then

$$nHh_n(x) = Hh_{n-2}(x) - xHh_{n-1}(x) \quad \forall n \geq 1$$

and $\forall n \geq -1$

$$I_n(c, \alpha, \beta, \gamma) = -e^{\alpha c} \left( \frac{\beta}{\alpha} \right)^{n-1} Hh_i(\beta c - \gamma) + \left( \frac{\beta}{\alpha} \right)^{n+1} \frac{\sqrt{2\pi}}{\beta} e^{\frac{\alpha^2}{2\beta \gamma}} \Phi \left( \theta - \beta c + \gamma + \frac{\alpha}{\delta} \right)$$

with

$$\theta = \begin{cases} 1 & \iff \beta > 0, \ \alpha \neq 0 \\ -1 & \iff \beta < 0, \ \alpha \leq 0 \end{cases}$$

Finally, the exercise probability is defined as

$$\Psi(\alpha, \sigma, \lambda, p, \eta_1, \eta_2, a, T)$$

$$= \frac{e^{\sigma \eta_1^2 \tau}}{\sqrt{2\pi \sigma \tau}} \sum_{n=1}^{+\infty} \sum_{k=1}^{n} \pi_n \sum_{k=1}^{n} P_{n,k}(\sigma \sqrt{T} \eta_1)^k \times I_{k-1}(a - \alpha T; -\eta_1; -\frac{1}{\sigma \sqrt{T}}; -\sigma \eta_1 \sqrt{T})$$

$$+ \frac{e^{\sigma \eta_2^2 \tau}}{\sqrt{2\pi \sigma \tau}} \sum_{n=1}^{+\infty} \sum_{k=1}^{n} D_{n,k}(\sigma \sqrt{T} \eta_2)^k \times I_{k-1}(a - \alpha T; -\eta_2; -\frac{1}{\sigma \sqrt{T}}; -\sigma \eta_2 \sqrt{T})$$

$$+ \pi_0 \Phi \left( -\frac{a - \alpha}{\sigma \sqrt{T}} \right)$$

where

$$P_{n,i} := \sum_{j=1}^{n-1} p^j d^{n-j} \left( \frac{n-i-1}{j-i} \right) \left( \frac{\eta_1}{\eta_1 + \eta_2} \right)^{j-i} \left( \frac{\eta_2}{\eta_1 + \eta_2} \right)^{n-j}, \quad 1 \leq i \leq n - 1$$

$$D_{n,i} := \sum_{j=1}^{n-1} d^j p^{n-j} \left( \frac{n-i-1}{j-i} \right) \left( \frac{\eta_2}{\eta_1 + \eta_2} \right)^{j-i} \left( \frac{\eta_1}{\eta_1 + \eta_2} \right)^{n-j}, \quad 1 \leq i \leq n - 1$$

$$P_{n,n} := p^n \quad D_{n,n} := d^n \quad \pi_n = \frac{e^{-\lambda T} \lambda^n}{n!}$$
Chapter 4

Pure Jumps

4.1 The Idea

Up until now we have been dealing with finite activity models, namely jump-diffusions. Now the idea is to use a jump process with infinite activity. Unlike the previous models, these processes generate purely discontinuous paths, e.g., there is no diffusive component. Instead of having a drifted Brownian motion generating the paths and, once in a while, a jump occurring, the paths are generated by infinite jumps, most of them small.

Moreover, a totally different valuation approach will be used. In the previous models, closed form solutions were obtained for pricing European-style options. In the Merton model the solution is expressed in terms of Black-Scholes values and in Kou is expressed by using special mathematical functions. But a closed form solution is not always possible to obtain and, in some cases, for instance in the Kou model, it requires a laborious mathematical treatment. For a more general exponential Lévy process a stronger formalism is required, and the use of characteristic functions and numerical integration turns out to be the most cost-effective procedure for pricing European-style options.
4.2 The Variance Gamma Model

4.2.1 Dynamics

The Variance Gamma process — see [20] — is a three parameter generalization of the Brownian motion and is obtained by evaluating a Brownian motion with constant drift and volatility at a random time change given by a Gamma process, with constant drift and volatility. Each unit of calendar time can be considered as having an economically relevant time length given by an independent random variable which has a density with unit mean and positive variance. Under the Variance Gamma process the unit period continuously compounded return is normally distributed, conditional on the realization of a random time change. In addition to the volatility of the Brownian motion there are two additional parameters that control both kurtosis and skewness. The Variance Gamma process has no continuous martingale component, contrary to the previous models. It is a pure jump process with infinite activity. See also [21]. The dynamics for the log returns proposed by [20] is:

\[ L_t = \ln \left( \frac{S_t}{S_0} \right) = (r - q)t + Y_t(\sigma, \theta, \nu) + \omega t \]  

(4.1)

and \( \omega \) is set to be \( \omega = \frac{1}{\nu} \ln(1 - \theta \nu - \frac{\sigma^2 \nu}{2}) \) so that the expectation of the log returns equals \( r \), the risk free rate. Notice that there is a drift \( (r - q)t \), a pure jump component \( Y_t \) and \( \omega t \) plays the role of a compensator, in analogy with the previous models. However, one question remains. What is \( Y_t \)?

Consider a drifted Brownian motion \( b_t(\mu, \sigma) = \theta t + \sigma W_t \). Now, replace time by a Gamma process with mean rate \( \theta = 1 \) and variance rate \( \nu \), so \( t \to \gamma_t(1, \nu) \). This procedure, replacing time with a stochastic process, is called subordination. Then define \( Y_t(\sigma, \theta, \nu) = b_{\gamma_t(1, \nu)}(\theta, \sigma) \) as a Variance Gamma process. Replacing time by a Gamma process is the way of giving every unit of time an economical relevant time length, as described earlier.

So, equation (4.1) indicates that the log returns on the asset follow a drifted Variance Gamma motion.
4.2.2 Option Pricing

Assuming that the characteristic function of a random variable — say $ln(S_T)$ — is known analytically, [5] provide a formula to numerically determine the price of an European-style option.

**Theorem 6.** Under the Variance Gamma model, the price of an European-style call option is given by

$$C_t(S_t, X, T) = S_t e^{-q(T-t)} \Pi_1 - e^{-r(T-t)} X \Pi_2$$

where

$$\Pi_1 = \tilde{Q}(S_T > X) = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} Re \left( \frac{e^{-iuln(X)} \phi_{ln(S_T)}(u)}{iu} \right) du$$

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} Re \left( \frac{e^{-iuln(X)} \phi_{ln(S_T)}(u-i)}{i u \phi_{ln(S_T)}(-i)} \right) du$$

The price of an European-style put option can be calculated through put-call parity.

The integrals in **Theorem 6** can be calculated numerically, using, for example, a Gaussian quadrature — see, for instance, [26].

4.2.3 Characteristic Function

According to [20], the characteristic function of a Variance Gamma process is given by

$$\phi_{Y_t}(u) = E[e^{iuY_t}] = \left( \frac{1}{1 - iu\theta \nu + \frac{\sigma^2 u^2}{2}} \right)^{\frac{1}{\nu}}$$

However, one needs the characteristic function for the log returns. We have defined previously that

$$L_t = ln \left( \frac{S_t}{S_0} \right)$$

and

$$ln(S_t) = ln(S_0) + (r - q + \omega)t + Y_t$$
Thus,

\[
\phi_{\ln(S_t)} = \mathbb{E}[e^{iu\ln(S_t)}] = e^{iu(\ln(S_0)+(r-q+w)t+Y_t)}
\]

See also [19]. Now, all that is left to do is to apply Theorem 6. Thus, it is very easy to price European-style options using this formalism. One first gets the characteristic function of the log returns and then applies Theorem 6. As simple as that. The disadvantage of this method is that it requires a numerical integration. Nevertheless, it is the most cost-effective formalism. Notice that in the Merton and Kou models, the value of an European-style option could as well be found using this method, and it would have been quite simpler. For the purpose, one would have to use the characteristic functions of the respective models.

One could ask: what about using Fast Fourier Transform (FFT) in order to calculate the integrals? In fact, FFT is a very fast method. Its application to option pricing can be found in [8]. However, there are certain stability conditions to be taken into account, and these are not as simple as [8] states. A more detailed study on the stability conditions for FFT and comparison with other methods can be found in [2].

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Chapter 5

Monte Carlo

Another way to price a contingent claim is through Monte Carlo simulations. There is no need for using Monte Carlo simulations in the previous models, since there are closed form solutions, which are much quicker to compute. Nevertheless, a demonstration of this method will be performed, since it can be very useful on more complex contexts where there is no closed form solution and because it is important to show how to simulate paths from the stochastic differential equations that define the asset’s dynamics. The idea is to simulate the paths of the asset’s price, averaging the final payoff for a large number of simulations and then discount it using the risk-free rate.

5.1 Merton

For the Merton model, the paths can be simulated through the following equation:

\[ \ln(S_{t+\Delta t}) = \ln(S_t) + (r - q - \frac{\sigma^2}{2} - \lambda \zeta) \Delta t + \sigma \epsilon \sqrt{\Delta t} + J \]

where

\[ J = \begin{cases} 
  u \sim \mathcal{N}(\mu, \delta^2), & \text{with probability } \lambda \Delta t \\
  0, & \text{with probability } 1 - \lambda \Delta t
\end{cases} \]
which is a Bernoulli variable, and \( \epsilon \sim N(0,1) \). Notice that the Bernoulli variable works as an asymptotical approximation for the arrival of jumps in the Poisson process.

See some simulations of this model in Appendix A.3.

5.2 Kou

As for Kou model, the method is very similar. However, the jump component is different. It can be modeled as

\[
J = \begin{cases} 
    u \sim \eta_1 e^{-\eta_1 x} 1_{\{x \geq 0\}}, & \text{with probability } p \\
    v \sim \eta_2 e^{\eta_2 x} 1_{\{x < 0\}}, & \text{with probability } d \\
    0 & \text{with probability } 1 - \lambda \Delta t
\end{cases}
\]

5.3 Variance Gamma

The paths can be simulated through:

\[
\ln(S_{t+\Delta t}) = \ln(S_t) + (r - q + \omega)\Delta t + \theta G_{t+\Delta t} + \sigma \epsilon \sqrt{G_{t+\Delta t}}
\]

where

\[
G \sim \Gamma \left( \frac{t + \Delta t}{\nu}, \nu \right)
\]

\( \Gamma \) being the gamma distribution. See, for instance, [13].
Chapter 6

Calibration

6.1 Method

Now it is time to test the models and to check if they fit the real data and what advantages they bring over the traditional diffusion models. In order to so, one shall minimize the sum of the squared percentage errors for a given set of N options:

$$\min_v \sum_{i=1}^{N} \left( \frac{\text{ModelPrice}_i(v) - \text{MarketPrice}_i}{\text{MarketPrice}_i} \right)^2$$

The data contains options on stock and stock market indices, namely, Apple, Nikkei, Ibovespa, FTSE and Nasdaq. Additionally, local and stochastic volatility models with no jump component are calibrated, namely the CEV — see [11] — and Heston — see [16] — models, so one can compare them all and see what advantages may come from each of them.

6.2 Results

The results are from Appendix [B.1] They are presented as MAPE — Mean Average Percentage Error — so that it is easier to check the quality of the fitting. The parameters can be found in Appendix [B.2] and the corresponding options in Appendix [C]
6.3 Analysis

In all cases some model turns out to fit the data better than the Black-Scholes model. However it is not always the same model that brings better results and sometimes the improvement is not that significant. This happens because the models are constructed assuming that the market “thinks” that a given asset price follows a determined stochastic differential equation, which may not be the most correct one. For different underlying assets the market might have different assumptions. One must keep in mind that one is trying to create a model that may explain how the market prices the options, independently of what one might think about a given asset’s historical behavior, although the market should act mainly according to that, at least in theory. There is no model which is absolutely the most efficient, however, in some cases the stochastic volatility models show better results, though they have no jump component. So, the fact that the volatility is constant in time as well as the rate arrival of jumps is a limitation to be taken into account in the models studied, since it is unrealistic to think that the market implicitly assumes the same volatility and the same rate arrival of jumps for every moment in time.

The optimization algorithm is something to be considered as well, since, as one can see from Appendix B.1 it is possible to find different minima using different algorithms. It is also important to notice that the errors in the fitting arise not only from the limitations of the models alone, but also from the possibility of non verification of certain basic conditions that are widely assumed in financial modeling, for instance, the non existence of arbitrage, the fact that one can buy or (short) sell anything at anytime, the assumption that there are no transaction costs or a bid-ask spread. There can also be the problem that when the prices are extracted, they may not be up to date with the spot, due to some lag between the pricing of the derivative by the market and the spot price.

Nevertheless, one needs to say that the results, although not perfect, they bring some improvements over the traditional Black-Scholes model.
Chapter 7

Conclusions

7.1 Remarks

In this thesis several models and methods for pricing European-style options were exposed. One started by introducing some important methods on stochastic calculus with jumps. Then, three models were presented: the Merton, the Kou and the Variance Gamma. Two of them, the Merton and the Kou, are finite activity jump-diffusions. The Variance Gamma, though, is an infinite activity pure jump model. For each model, a different pricing approach was performed. In the Merton case, the options’ prices were given by a weighted average of Black-Scholes prices. In the Kou model, the solution was given in terms of special mathematical functions and, in the Variance Gamma model, a semi-closed form solution was obtained, though quite simple and understandable.

As for the methods used, in the Merton model one expressed the price of an option in terms of Black-Scholes prices by conditioning on the number of jumps. This method is applicable for the Merton model alone and gives closed form solutions for European-style options only. In the Kou model, one recurred to a change of numeraire technique and to a Girsanov theorem for jump-diffusions. Moreover, complicated mathematical functions were required in order to define the probabilities of exercise. However, although only European-style options were studied, it is important to remark that the Kou model presents closed form solutions for the majority of option styles. In the
Variance-Gamma model, one used characteristic functions and numerical integration, in order to perform a Fourier inversion. From my point of view, the formalism applied to the Variance Gamma model is the most cost-effective, since it only requires the characteristic function of a process, which, in the case of Lévy processes, can be easily built via the Lévy-Khintchine representation. This is a great advantage of the Lévy process with respect to, for instance, the stochastic volatility models, since for the latter the computation of the characteristic function can be much more complicated, for the increments of the process are correlated, which is not verified in Lévy processes. Then, it was shown how these models’ paths can be simulated and how, therefore, the options can be priced via Monte-Carlo simulations. It was not necessary to apply it to our cases, since closed form solutions were obtained. Nevertheless it is an important methodology that can be applied under more complicated contexts. Finally, the models were calibrated with real data from the options’ market, and we saw how well these models fit the market for the given cases. No model showed to be absolutely the best. However, all of them brought improvements with respect to the traditional pure diffusion Black-Scholes model.

In this work, I also provide Matlab® code for the valuation formulae, which may be very useful to the reader. It can be found in Appendix D.

Thus, one can conclude that the Lévy processes bring together some improvements in explaining the market as well as a variety of solid frameworks to achieve pricing formulae.

7.2 Further Reading

For those interested in investigating even further, I want to suggest some reading. It would be interesting to extend these models, and study models that combine both jumps and stochastic volatility. There is a model proposed by [6] that combines the Merton jump-diffusion model and the Heston stochastic volatility model, by defining two stochastic differential equations: one for the asset’s price and another for the volatility associated with the Brownian motion. Another interesting model is one
that also includes a stochastic differential equation for the rate arrival of jumps — see [12]. Since stochastic volatility implies correlation between the increments of the process, these models lie beyond the scope of Lévy processes.

With respect to methodology, an interesting field is the option valuation approach via partial integro-differential equations and their solution through finite difference methods — see for instance, [27].

7.3 Final Note

This is a field with lots of research yet to be done, and I hope that with the reading of this thesis the reader may feel motivated to move to further investigation.
Appendix A

Graphics

A.1 Empirical Evidence

Figure A-1 shows the daily logarithm of Apple’s stock price between 01/01/2003 and 01/01/2013. Figure A-2 shows the daily logarithmic returns on Apple’s stock price between 01/01/2003 and 01/01/2013.

Large variations can be better seen in Figure A-2. Notice that although most of the values lie in a small region, sometimes some spikes show up. These represent large variations, which we call jumps.
A.2 Merton Prices

Figure A-3: Merton Model: Value vs Strike - varying lambda

Figure A-4: Merton Model: Value vs Strike - varying delta

Figure A-3 shows the value of a call option as a function of the strike, varying the parameter $\lambda$ of the Merton model, keeping the other parameters fixed. The parameters used are: $\lambda \in \{1, 3, 10\}$, $\mu = -0.1$ and $\delta = 0.1$. Figure A-4 shows the value of a call option as a function of the strike, varying the parameter $\delta$ of the Merton model, keeping the other parameters fixed. The parameters used are: $\lambda = 1$, $\mu = -0.1$ and $\delta \in \{0.2, 0.5, 0.8\}$.

As we can check, the value of the option under the Merton model relative to the value under the Black-Scholes model increases as the rate arrival of jumps $\lambda$ increases and as the variance of the jump distribution $\delta$ increases. This was expected, since more jumps with greater variance mean more uncertainty in the expected final payoff.

A.3 Path Simulation

Figure A-5 shows a simulation of the logarithm of the stock price under the Merton model for 360 days. Figure A-6 shows the first differences of the logarithm of stock the price under the Merton model for 360 days. Figure A-7 shows a simulation of the logarithm of the stock price under the Black-Scholes model for 360 days. Figure A-8 shows the first differences of the logarithm of the stock price under the Black-Scholes model for 360 days.
It is an empirical evidence that large variations do happen. By comparing Figures A-5 and A-6 with Figures A-7 and A-8 we can check that the inclusion of jumps brings indeed the likeliness of large variations in an asset price. It can be seen more clearly from Figures A-5 and A-7. Notice that in Figure A-5 all the values lie in a tiny interval. In Figure A-7 most of the values are places in a tiny interval as well. However, we see some spikes of large variation. Thus, if we compare both simulations to Figures A-1 and A-2 we can clearly see that the model that admits jumps — the Merton model — is more realistic than the Black-Scholes diffusion model.
Appendix B

Calibration Results

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## B.2 Parameters

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Appendix C

Data

Figure C-1: Options on Apple's
### Figure C-2: Options on Nikkei

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### APPENDIX C. DATA

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Figure C-4: Options on FTSE 1/3
Figure C-5: Options on FTSE 2/3
### Figure C-6: Options on FTSE 3/3

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**Exch** NYSE, LSE, Euronext

**Option Monitor:** Custom Option Monitor

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**Strikes** 5
**Exch** NYSE, LSE, Euronext

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Figure C-7: Options on Nasdaq 1/3
### Figure C-8: Options on Nasdaq 2/3

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*Note: Default color legend*

- **Zeal** -
- **None** -

*Source:* The data is from the *The Wall Street Journal* and is not intended for financial advice.
**Figure C-9: Options on Nasdaq 3/3**

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NDX Index: 2942.982 +33.383

At 18:26 0.2933, 484 H 2950.995 L 2932.884 Prev 2932.595

**APPENDIX C. DATA**

Nasdaq 100 Stock Index: 2942.982 +33.383 1.1473% / Hi 2960.995 Lo 2932.884 Volm: HV 14.04

News (CN)

Earnings Calendar (ACDR)

Exch: U.S. Composite
Figure C-10: Options on Apple's on 23/10/2012
Appendix D

Matlab® Code

The Merton Model

Valuation Function

```matlab
function c = MJD(type, So, X, T, delta, nu, lambda, vola, r, N) % r->r-q
    K = exp(nu + 0.5*delta^2) - 1;
    c = 0;
    for n = 0 : N
        sigma_n = sqrt(vola^2 + n*delta^2/T);
        r_n = r - lambda*K + n*log(1+K)/T;
        f_n = BS(type, So, X, r_n, 0, sigma_n, T);
        c = c + exp(-lambda*(1+K)*T) * (lambda*(1+K)*T)^n * f_n / factorial(n);
    end
end
```

The Kou Model

Hh Functions
APPENDIX D. MATLAB® CODE

1 function out = Hh(n,x)
2 if n==-1
3     out=sqrt(2*pi)*normpdf(x,0,1); end
4 if n>=0
5     out=sqrt(2*pi)*normcdf(-x,0,1); end
6 if n>0
7     v(1)=sqrt(2*pi)*normpdf(x,0,1);
8     v(2)=sqrt(2*pi)*normcdf(-x,0,1);
9     for i=3:n+2
10        v(i)=(1/(i-2))*(v(i-2)-x*v(i-1)); end
11     out=v(n+2); end
12 end
13

P Functions

1 function out = P(n,i,p,q,eta1,eta2)
2 if i==n
3     out=p^n; else
4     sum=0;
5     for j=1:n-1
6        sum = sum + p^j*q^(n-j)*nchoosek(n-i-1,j-1)*(eta1/(eta1+eta2))^(j-1)*(eta2/(eta1+eta2))^(n-j); end
7     out=sum; end
8 end
9
10 end
11
12
D Functions

```matlab
function out = Q(n, i, p, q, eta1, eta2)
    if i==n
        out=q^n;
    else
        sum=0;
        for j=1:n-1
            sum = sum + q^j*p^(n-j)*nchoosek(n-i-1,j-1)*(eta2/(eta1+eta2))^(j-1)*(eta1/(eta1+eta2))^(n-j);
        end
        out=sum;
    end
end
```

I Functions

```matlab
function out = I(n, c, alpha, beta, delta)
    sum=0;
    for i=1:n
        sum=sum+(beta/alpha)^(n-i)*Hh(i, beta*c-delta);
    end
    out=sum;
    if beta>0 && alpha>=0
        out=-(exp(alpha*c)/alpha)*sum+(beta/alpha)^(n+1)*sqrt(2*pi)/beta)*exp(alpha*delta/beta+alpha^2/(2*beta^2))*normcdf(-beta*c+delta+alpha/beta, 0, 1);
    elseif beta<0 && alpha<0
        out=-(exp(alpha*c)/alpha)*sum+(beta/alpha)^(n+1)*sqrt(2*pi)/beta)*exp(alpha*delta/beta+alpha^2/(2*beta^2))*normcdf(beta*c-delta-alpha/beta, 0, 1);
```
APPENDIX D. MATLAB® CODE

11       else
12           out=0;'% non valid arguments'
13       end
14   end

π Functions

1 function out=Pi(n,lambda,T)
2     out=(exp(-lambda*T)^lambda^n)/(factorial(n));
3 end

Ψ Functions

1 function out = Psi(mu,sigma,lambda,p,q,eta1,eta2,a,T)
2     sum_old= Pi(0,lambda,T)*normcdf(-(a-mu*T)/(sigma*sqrt(T))
3             ,0,1);
4     n=1;
5     dif=1;
6     out=sum_old;
7 end

Valuation Function

1 function out = Kou(type,So,X,r,sigma,lambda,p,eta1,eta2,T)  
2     r-> r-q
3     q=1-p;
4     if   eta1<=1 || eta2<=0
5           out=0;
6     else
7         zeta=(p*eta1)/(eta1-1)+(q*eta2)/(eta2+1)-1;
8         p_tilde=(p/(1+zeta))*((eta1/(eta1-1)));
9         q_tilde=1-p_tilde;
10        eta1_tilde=eta1-1;
11 end
etatilde = eta2 + 1;

lamdatilde = lambda * (zeta + 1);

call = So * Psi(r + sigma^2 / 2 - lambda * zeta, sigma, lamdatilde,
p_tile, q_tile, etatilde1, etatilde2, log(X/So), T) -
X * exp(-r * T) * Psi(r - sigma^2 / 2 - lambda * zeta, sigma, 
lambda, p, q, etatilde1, etatilde2, log(X/So), T);

if type == 1
    out = call;
else
    put = X * exp(-r * T) - So + call;
    out = put;
end
end

The Variance Gamma Model

Valuation Function

function out = VG(type, So, X, r, T, theta, nu, sigma)

CF_VG = @(u) (1./(1 - 1i * u * theta * nu + sigma^2 * nu / 2 * u.^2)) .^ (T/nu);

CF_VG/lgST = @(u) exp(1i * u * log(So) + r * T + T/nu * log(1 - theta * nu - sigma^2 * nu / 2)) .^ CF_VG(u);

integrand1 = @(u) real(exp(-1i * u * log(X)) .* CF_VG/lgST(u - 1i)) ./ (1i * u .* CF_VG/lgST(-1i));

integrand2 = @(u) real(exp(-1i * u * log(X)) .* CF_VG/lgST(u) ./ (1i * u));

integral1 = quadgk(integrand1, 0.01, inf, 'RelTol', 1e-8, 'AbsTol', 1e-12); %'MaxIntervalCount', 200000

integral2 = quadgk(integrand2, 0.01, inf, 'RelTol', 1e-8, 'AbsTol', 1e-12);
Prob1 = 0.5 + (1/pi)*integral1;
Prob2 = 0.5 + (1/pi)*integral2;
call = So*Prob1 - X*exp(-r*T)*Prob2;
if type==1
    out = call;
else
    out = call - So + exp(-r*T)*X;
end
end
Bibliography


