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Abstract

The aim of this thesis is to find a closed-form solution for evaluation of knock-out barrier options under the Constant Elasticity of Variance model. To achieve this goal, the pricing solution of the double knock out barrier option is written in terms of Laplace Transforms, that are inverted using the Abate and Whitt Euler method. Then the analytical solution is implemented using a C++ program.

Key words: Path-dependent options, barrier options, Constant Elasticity of Variance model, Laplace Transforms, Abate and Whitt Euler method.
1 Introduction

The goal of this thesis is to achieve an analytical solution to price barrier options under the Constant Elasticity of Variance (CEV) process.

Option pricing is one of the most important areas of modern finance. Financial historians attribute the first attempts to model stochastic processes to Bachlier. In his PhD thesis entitled *The Theory of Speculation*, published in 1900, Bachlier discussed the use of the Brownian motion to evaluate stock options. This thesis is historically the first paper to use advanced mathematics in the study of finance (Courtault et al., 2000). But the model that revolutionized the financial markets was the Black and Scholes (1973) model. The Black and Scholes (1973) model assumes that the price of the underlying asset follows a geometric Brownian motion, and as a consequence the underlying asset follows a lognormal process. Using the assumptions of Black and Scholes (1973), Merton (1973) was the first to derive a closed-form solution for the standard European call and put. Merton (1973) was also the first to get a closed-form solution for a down-and-out European call option. This was the first step to the development of further studies on the pricing of barrier options. Rubinstein and Reiner (1991) and Carr (1995) derive analytical solutions for all types of single barrier options. The study of barrier options using a binomial method was presented by Boyle and Lau (1994). Kunimoto and Ikeda (1992) covered the valuation of double barrier options expressing the probability density as a sum of normal density functions. German and Yor (1996) derive expressions for the Laplace Transform of a double barrier option price, that is inverted analytically. Later, Schroder (2000) derives the formulas of Kunimoto and Ikeda (1992) inverting the Laplace Transforms derived in German and Yor (1996). Pelsser (2000) derived analytical expressions to price double barrier options inverting the Laplace Transform using a contour integration method.

One of the assumptions of the Black and Scholes (1973) model is that the volatility of the underlying asset price is constant; however, the market shows that such volatility is not constant, and does not follow a lognormal distribution (Hull, 2002). Black (1976) and Christie (1982) consider the effect of financial leverage on the variance of the stock. A fall in
the stock price increases the debt / equity ratio of the firm, and so increases the risk and the variance of the stock. Empirical evidences shows that the constant volatility is inapplicable in real market situations. Schmalensee and Trippi (1978) find a strong negative relationship between stock prices changes and changes in implied volatility. When the implicit volatility of the asset is analyzed we see that volatility is not constant. The plot of implied volatility, known as *volatility smile* (see figure in Hull (2002), p 437) evidences that the volatility of the asset is related to the strike price. This effect of *volatility smile* is not captured by the Black and Scholes (1973) model, and therefore new models were studied with the intention of capturing this effect evidenced by the market. The Constant Elasticity of Variance (CEV) model is one model that incorporates the volatility smile. The CEV model was developed by Cox (1975) and in this model the volatility of the underlying asset is related to his price level. This fact is consistent with empirical observations in which stock volatility changes with stock prices oscillations. Cox (1975) was the first to price vanilla European call and put options under the CEV process. Cox (1975) studied the CEV model for the case $\theta \leq 2$, and Emanuel and MacBeth (1982) and Schroder (1989) extended this analysis to the case $\theta > 2$. Originally Cox (1975) has restricted the parameter $\theta$ to the interval $[0; 2]$, but Reiner (1994) and Jackweth and Rubinstein (1998) found out that the values of the CEV elasticity implicit in the post crash S&P500 stock index options prices were as low as $\theta = -4$ or $\theta = -6$.

The comparison between the Black and Scholes (1973) model and the CEV model has been made by several authors. Macbeth and Merville (1980) compared the two models and concluded that prices obtained by the CEV model were closer to market quotes than the ones obtained by Black and Scholes (1973), specially in case of $\theta < 2$. They also concluded that the volatility of the underlying asset decreases when the price increases, and this matches the specifications of the CEV model that assume the volatility of the asset as a function of the asset price. Boyle and Tian (1999) concluded that the difference of prices between the Black and Scholes (1973) and the CEV models is greater for path dependent options than for standard options.

The pricing of barrier options under CEV model is also well established. Boyle and Tian (1999) used a trinomial lattice to price single and double barrier options under the CEV
model. Later Davidov and Linetsky (2001) derived closed-form expressions for the Laplace Transforms of single and double barrier options prices in time to maturity. Davidov and Linetsky (2003) developed eigenfunctions expansions for single and double barrier options. These eigenfunctions expansions were used to invert the Laplace transform of Davidov and Linetsky (2001). All the above derivations assume the model parameters as volatility, interest rate and dividend yield are constant. However, the pricing of barrier options was also studied using time dependent parameters: Lo, Yuen and Hui (2000) priced path dependent options under the CEV model using Lie-algebraic techniques. Lo, Tang, Ku and Hui (2009) derived the analytical kernels of the pricing formulae of the CEV knockout options with time-dependent parameters for a parametric class of moving barriers.

This thesis considers the CEV model with time independent parameters, and is based on the results of Davidov and Linetsky (2001): I will derive the Laplace transform of double knock out barrier options (and also of single barrier knock out options). Then, I will implement the computation of the inversion of the Laplace Transform in a C++ program using the Abate and Whitt (1995) algorithm.

This thesis is further organized as follows: In Section 2 the barrier options and contracts will be introduced. In Section 3 the CEV diffusion and its assumptions will be described. In section 4, starting from the definition of final payoff of a double barrier option, the analytical expression of the Laplace Transform of such payoff will be computed. The same thing is done using the definition of the first hitting or exit time, and the analytical expression of the Laplace Transform of the rebate price will also be computed. In Section 5 the Abate and Whitt (1995) Euler-type method to invert Laplace Transforms will be presented, and the parameters of the method used to invert the Laplace Transform will be defined. In Section 6 the analytical closed-form solution deduced in the other chapters is implemented using a C++ program. This program will then be used to price down-and-out, up-and-out and double knock out call options with different strikes, and different betas. In Section 7 the conclusions of this work will be presented.
2 Barrier Options

Option trading has become increasingly popular over these years because of its flexibility to fulfill the needs of investors. Options with more complex payoffs than standard European or American options are called exotic options. Exotic options where the payoff depends on the path followed by the price of the underlying asset are referred as path dependent options. Barrier options are one of the most popular types of path dependent options. Barrier options are options where the final payoff depends on the values that the underlying asset reaches during the life of the option. As the name suggest, in barrier options we have a pre-defined price level, or levels, that in case of being reached will influence the final payoff of the option. The barrier options are classified in two groups according to the type of the barrier:

- Knock Out Barrier Options - these are options where the option becomes inactive if the underlying asset price reaches the value of the barrier. The barrier can be hit at any moment of the option life, and becomes inactive at the moment the barrier is hited. The payoff is a standard vanilla option if the barrier never is hited, or nothing (or a rebate value) if the barrier is hited. In knock out options we can find:
  
  1. Up-and-out Options: when the barrier is approached from below, that is, the value of the barrier is bigger than the spot price.
  2. Down-and-out Options: The value of the barrier is lower than the spot price.
  3. Double knock out Options: In this case, there exists up and down barriers that limit the price of the underlying asset. If the price goes out of the interval defined by the two barriers, the option becomes inactive. The option payoff is a standard vanilla option if the barrier is hited, or nothing (or a rebate value) if the barrier is never hited.

- Knock In Barrier Options - In this case, the option contract is only activated if the price of the underlying asset reaches the value of the barrier. If that value is never reached during the option life, then the payoff is zero. There are three cases of knock in options:
1. Up-and-in Options: The barrier value is lower than the spot price.

2. Down-and-in Options: The barrier value is greater than the spot price.

3. Double knock in Options: There exists up and down barrier levels that limit the price of the underlying asset. If the price touches one of the barriers then the option becomes active.

If knock in options never become active, or if knock out option become inactive, then there is a possibility to pre-establish a rebate value that is paid in these cases. In knock in options the rebate can only be paid at maturity, but in knock out options the price of the underlying asset reaches the barrier at any moment, and so the rebate can be paid at any moment.

There are two ways of monitoring the barrier: continuously or discretely. Continuous monitoring of the barrier happens when the barrier condition is applied at any time prior to maturity. In discrete monitoring we define a discrete time interval, and at the end of this interval we see if the barrier condition is reached, but not at any other time. This thesis is only concerned with the continuous monitoring of the barrier option.
3 CEV Diffusion

The Black and Scholes (1973) option pricing model assumes that the underlying asset price follows a Geometric Brownian motion

\[ dS_t = \mu S_t dt + \sigma S_t dw^Q_t, \tag{1} \]

where \( w^Q_t \) is a standard Brownian motion. This means that the stock price \( \frac{\partial S_t}{S_t} \) in the small interval \( dt \) is normally distributed with mean \( \mu dt \) and variance \( \sigma^2 dt \). And it is known that the lognormality assumption does not hold empirically for stock prices. Alternative stochastic processes have been studied and applied to option pricing. Cox (1973) studied a general class of stochastic processes known as the CEV process:

\[ dS_t = \mu S_t dt + \delta S_t^\theta dw^Q_t, \tag{2} \]

where \( dS_t \) is the change of the stock price \( S \) over the short increment of time \( dt \), \( \mu, \delta \) and \( \theta \) are constants, and \( dw^Q_t \) is a Wiener process increment.

Note that the difference between equations (1) and (2) consists in the diffusion term. Notice that the volatility of the percentage change in the underlying asset price in (2) is \( \sigma(S) = \delta S_t^{\theta-1} \) and is not constant. And it has an inverse relationship with the underlying asset price \( S \) as long as \( \theta < 2 \). This is, the volatility is an increasing (decreasing) function of \( S \) when \( \theta > 2 \) (\( \theta < 2 \)).

The CEV model nests, as special cases, other pricing models:

- If \( \theta = 2 \), then the volatility is the constant \( \delta \), and so we have the Black and Scholes (1973) model.

- If \( \theta = 1 \), then we have the Square Root Process that was first presented by Cox and Ross (1976) as an alternative diffusion process for options valuation.

- If \( \theta = 0 \), we have the Ornstein-Uhlenbeck process.
Cox and Ross (1975) define the elasticity factor as $\theta$, but I will use, along this thesis, $\beta + 1 = \frac{\theta}{2}$ in (2) for a simplification of the valuation formulas. And so we get:

$$dS_t = \mu S_t dt + \delta S_t^{\beta+1} dw_t^Q, \quad t \geq 0, \quad S_0 = S \geq 0$$

where:

- $Q$ is an equivalent martingale measure (risk neutral probability measure). Under $Q$ the asset price $\{S_t, t \geq 0\}$ is a time-homogeneous, non-negative diffusion process
- $\{w_t^Q, t \geq 0\}$ is a standard Brownian motion defined on a filtered probability space $(\Omega; F; \{F_t\}_{t \geq 0}; Q)$.
- $\mu$ is a constant ($\mu = r - q$, where $r \geq 0$ and $q \geq 0$ are respectively the risk-free rate and the constant dividend yield).
- $\beta$ is a constant elasticity factor.
- $\sigma(S_t) = \delta S_t^\beta$

Next some results are presented in order to evaluate knock out barriers options under the CEV model.
4 Knock Out Options Under The CEV Model

All over this thesis the variable $t$ denotes the running time, and it is also assumed that all options are written at time $t = 0$, and expire at time $t = T > 0$. The time left behind expiration is denoted by $\tau = T - t$. Suppose the initial asset price is $S$ and the lower and upper barrier levels are $L$ and $U$, where $L < S < U$. The first hitting and exit times of the barriers can be defined as:

- First hitting time of the lower barrier:
  \[ \mathcal{H}_L = \inf\{ t \geq 0 : S_t = L \}. \]  \(4\)

- First hitting time of the upper barrier:
  \[ \mathcal{H}_U = \inf\{ t \geq 0 : S_t = U \}. \]  \(5\)

- First exit time from an interval between two barriers:
  \[ \mathcal{H}_{(L,U)} = \inf\{ t \geq 0 : S_t \notin (L,U) \}. \]  \(6\)

Using Zhang (1998, p 206-207) the final payoffs of knock out options are:

- The final payoff of a down and out option (PDO) with strike $K$ and no rebate at expiration is (call if $\alpha = 1$, and put if $\alpha = -1$) is equal to
  \[ PDO = \mathbb{1}_{\{\mathcal{H}_L > T\}}(\alpha S_t - \alpha K)^+. \]  \(7\)

- The final payoff of a up and out option (PUO) with strike $K$ and no rebate at expiration is (call if $\alpha = 1$, and put if $\alpha = -1$) is equal to
  \[ PUO = \mathbb{1}_{\{\mathcal{H}_U > T\}}(\alpha S_t - \alpha K)^+. \]  \(8\)

- The final payoff of a double knock out option (PDKO) with strike $K$ and no rebate at expiration is (call if $\alpha = 1$, and put if $\alpha = -1$) is equal to
  \[ PDKO = \mathbb{1}_{\{\mathcal{H}_{(L,U)} > T\}}(\alpha S_t - \alpha K)^+. \]  \(9\)

where:
• \( \mathcal{Z}_L, \mathcal{Z}_U \) and \( \mathcal{Z}_{(L,U)} \) are defined in (4), (5) and (6).

• \( \mathbbm{1}_A \) is the indicator function defined as \( \mathbbm{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \)

• \( x^+ = \max\{x, 0\} \).

As seeing before in (7), (8) and (9) the final payoff of a knock out barrier option involves hitting times that can happen at \( \theta = \mathcal{Z}_L, \mathcal{Z}_U \) and \( \mathcal{Z}_{(L,U)} \) with \( \theta \leq T \). From Ingersoll (1987, p371-372), the value of perpetual claims can help in the valuation of such hitting times. So, the first step is to analyze the valuation of three perpetual claims that pay one EUR at times \( \mathcal{Z}_L, \mathcal{Z}_U \) or \( \mathcal{Z}_{(L,U)} \) and have no expiration date. This result will be used later to find an expression for the expected value of final payoffs and also of rebates:

**Proposition 1** Suppose the risk neutral asset price process follows the diffusion (3) and the constant risk free interest rate is \( r > 0 \). Then, the prices at \( t = 0 \) of the three perpetual claims are (assuming \( S_0 = S \)):

• One EUR paid at \( \mathcal{Z}_L \):

\[
E^Q[e^{-r\mathcal{Z}_L}\mathbbm{1}_{\{\mathcal{Z}_L<\infty\}}] = \frac{\phi_r(S)}{\phi_r(L)}, \quad S \geq L
\]

(10)

One EUR paid at \( \mathcal{Z}_U \):

\[
E^Q[e^{-r\mathcal{Z}_U}\mathbbm{1}_{\{\mathcal{Z}_U<\infty\}}] = \frac{\psi_r(S)}{\psi_r(U)}, \quad S \leq U
\]

(11)

One EUR paid at \( \mathcal{Z}_{(L,U)} \):

\[
E^Q[e^{-r\mathcal{Z}_{(L,U)}}] = \frac{\Delta_r(L,S) + \Delta_r(S,U)}{\Delta_r(L,U)}, \quad L \leq S_0 \leq U,
\]

(12)

where (for any \( 0 < A < B < \infty \))

\[
\Delta_r(A,B) := \phi_r(A)\psi_r(B) - \psi_r(A)\phi_r(B),
\]

(13)

and the functions \( \psi_r(S) \) and \( \phi_r(S) \) can be characterized as the unique (up a multiplicative constant) solutions of the ordinary differential equation (ODE)

\[
\frac{1}{2}\sigma^2(S)S^2\frac{\partial^2 u}{\partial S^2} + \mu S \frac{\partial u}{\partial S} - ru = 0, \quad S \in (0, \infty)
\]

(14)
The functions $\psi_r(S)$ and $\phi_r(S)$ are characterized, and have properties described in Borodin and Salminen (2002, p.18-19), and $\psi_r(S)$ is the increasing solution and $\phi_r(S)$ is the decreasing solution.

**Proof.** For equation (10) and (11):

$\mathcal{H}_L$ and $\mathcal{H}_U$ are, respectively, the first hitting time of $L$ and $U$, so using n.10 of Borodin and Salminen (2002, p. 18) then equations (10) and (11) follow, where $\phi_r$ and $\psi_r$ can be characterized as the unique (up to a multiplicative constant) solutions of (14) demanding that $\psi_r$ is increasing and $\phi_r$ is decreasing. The boundary conditions of $\psi_r$ and $\phi_r$ are explained in Borodin and Salminen (2002, p. 19). Note that $\mathcal{H}_0 = \inf\{ t \geq 0 : S_t = 0 \}$, so this represents the default. Hence, the definition of (11) presume that the upper barrier must be hit before the default.

For equation (12):

Equation (12) can be rewritten using

$$e^{-r\mathcal{H}_L(U)} = e^{-r\mathcal{H}_L} \mathbb{1}_{\mathcal{H}_L < \mathcal{H}_U} + e^{-r\mathcal{H}_U} \mathbb{1}_{\mathcal{H}_U < \mathcal{H}_L}$$

and from the property of linearity of an expected value (Shreve, 2004, p. 55),

$$E^Q[e^{-r\mathcal{H}_L(U)}] = E^Q[e^{-r\mathcal{H}_L} \mathbb{1}_{\mathcal{H}_L < \mathcal{H}_U}] + E^Q[e^{-r\mathcal{H}_U} \mathbb{1}_{\mathcal{H}_U < \mathcal{H}_L}] .$$

$E^Q[e^{-r\mathcal{H}_L} \mathbb{1}_{\mathcal{H}_L < \mathcal{H}_U}]$ satisfies the ODE (14) because it is a Kolmogorov equation. But any solution of (14) can only be expressed as a combination of the two fundamental solutions of $\psi_r$ and $\phi_r$. Therefore, the solution of (12) will be of the form $g(s) = a_1 \psi_r(s) + a_2 \phi_r(s)$. To find the value of $a_1$ and $a_2$ the boundary conditions in the limits $L$ and $U$ are used. So,

$$\begin{cases} a_1 \psi_r(L) + a_2 \phi_r(L) = 1 \\ a_1 \psi_r(U) + a_2 \phi_r(U) = 0 \end{cases} \iff \begin{cases} a_2 = \frac{\psi_r(U)}{\phi_r(U)\psi_r(U) - \phi_r(U)\psi_r(L)} \\ a_1 = -\frac{\phi_r(L)\psi_r(U) - \phi_r(U)\psi_r(L)}{\phi_r(U)\psi_r(U) - \phi_r(U)\psi_r(L)} \end{cases}$$

Replacing $a_1$ and $a_2$ in $g(S)$ the solution is:

$$g(S) = \frac{\phi_r(U)}{\phi_r(L)\psi_r(U) - \phi_r(U)\psi_r(L)} \psi_r(S)$$

$$+ \frac{\psi_r(U)}{\phi_r(L)\psi_r(U) - \phi_r(U)\psi_r(L)} \phi_r(S)$$

$$= \frac{\phi_r(S)\psi_r(U) - \psi_r(S)\phi_r(U)}{\phi_r(L)\psi_r(U) - \phi_r(U)\psi_r(L)}$$
So doing the same for $E^Q[e^{-r\Theta} \mathbb{1}_{\{\Theta < \Theta_L\}}]$, and using the following boundary conditions in the limits $L$ and $U$

\[
\begin{align*}
    a_1\psi_r(L) + a_2\phi_r(L) &= 0, \\
    a_1\psi_r(U) + a_2\phi_r(U) &= 1,
\end{align*}
\]

we find that the solution is

\[
\frac{\phi_r(L)\psi_r(S) - \psi_r(L)\phi_r(S)}{\phi_r(L)\psi_r(U) - \phi_r(U)\psi_r(L)}
\]

Consequently,

\[
E^Q[e^{-r\Theta(L,U)}] = \frac{\phi_r(S)\psi_r(U) - \psi_r(S)\phi_r(U)}{\phi_r(L)\psi_r(U) - \phi_r(U)\psi_r(L)} + \frac{\phi_r(L)\psi_r(S) - \psi_r(L)\phi_r(S)}{\phi_r(L)\psi_r(U) - \phi_r(U)\psi_r(L)};
\]

and if $\Delta_r(A,B) = \phi_r(A)\psi_r(B) - \psi_r(A)\phi_r(B)$ then:

\[
E^Q[e^{-r\Theta(L,U)}] = \frac{\Delta_r(S,U) + \Delta_r(L,S)}{\Delta_r(L,U)}
\]

From Proposition 1 we know that $\phi_r(S)$ and $\psi_r(S)$ are the fundamental solutions of the ODE (14), but we still need to find the expression of those solutions. The next result gives a closed-form expression for the two fundamental solutions $\phi_\lambda(S)$ and $\psi_\lambda(S)$.

**Proposition 2** The fundamental solutions $\psi_\lambda(S)$ (increasing solution) and $\phi_\lambda(S)$ (decreasing solution) of the CEV ODE (14), are (up to a multiplicative constant):

\[
\psi_\lambda(S) = \begin{cases} 
S^{\beta+\frac{1}{2}}e^{\frac{1}{2}\lambda(S)}M_{k,m}(x(S)), & \beta < 0, \ \mu \neq 0 \\
S^{\beta+\frac{1}{2}}e^{\frac{1}{2}\lambda(S)}W_{k,m}(x(S)), & \beta > 0, \ \mu \neq 0 \\
S^{\beta+\frac{1}{2}}I_{\nu}(\sqrt{2\lambda(S)}), & \beta < 0, \ \mu = 0 \\
S^{\beta+\frac{1}{2}}K_{\nu}(\sqrt{2\lambda(S)}), & \beta > 0, \ \mu = 0
\end{cases}, \quad (15)
\]

and

\[
\phi_\lambda(S) = \begin{cases} 
S^{\beta+\frac{1}{2}}e^{\frac{1}{2}\lambda(S)}W_{k,m}(x(S)), & \beta < 0, \ \mu \neq 0 \\
S^{\beta+\frac{1}{2}}e^{\frac{1}{2}\lambda(S)}M_{k,m}(x(S)), & \beta > 0, \ \mu \neq 0 \\
S^{\beta+\frac{1}{2}}K_{\nu}(\sqrt{2\lambda(S)}), & \beta < 0, \ \mu = 0 \\
S^{\beta+\frac{1}{2}}I_{\nu}(\sqrt{2\lambda(S)}), & \beta > 0, \ \mu = 0
\end{cases}, \quad (16)
\]
where \( M_{k,m}(x) \) and \( W_{k,m}(x) \) are the Whittaker functions defined in Abramowitz and Stegun (1972, p 504), and \( I_\nu(\sqrt{2\lambda z}) \) and \( K_\nu(\sqrt{2\lambda z}) \) are the modified Bessel functions also defined in Abramowitz and Stegun (1972, p 374), with:

\[
x(S) = \frac{|\mu|}{\delta^2 |\beta|} S^{-2\beta}
\]

(17)

\[
z(S) = \frac{1}{\delta |\beta|} S^{-\beta}
\]

(18)

\[
\epsilon = \text{sign}(\mu\beta) = \begin{cases} 
1 & \text{if } \beta\mu > 0 \\
-1 & \text{if } \beta\mu < 0
\end{cases}
\]

(19)

\[
m = \frac{1}{4|\beta|}
\]

(20)

\[
k = \epsilon \left( \frac{1}{2} + \frac{1}{4\beta} \right) - \frac{\lambda}{2|\mu\beta|}
\]

(21)

\[
\nu = \frac{1}{2|\beta|}
\]

(22)

The Wronskian of the functions \( \psi_\lambda \) and \( \phi_\lambda \) with respect to the scale density of the CEV diffusion is

\[
\omega_\lambda = \left\{ \begin{array}{ll}
\frac{2|\mu|\Gamma(2m+1)}{8^m\Gamma(m+k+\frac{1}{2})}, & \mu \neq 0 \\
|\beta|, & \mu = 0
\end{array} \right.
\]

(23)

where \( \Gamma(x) \) is the Euler Gamma function.

Proof. See Appendix A. ■

Now that the closed-form solutions for \( \phi_\lambda(S) \) and \( \psi_\lambda(S) \) are available, it is possible to price European knock out barrier options. From now on, the focus is on the European double barrier options because the case of the single barrier option (down-and-out and up-and-out) can be obtained from the double barrier option. For instance, the down and out barrier is found if the upper barrier \( U \) tends to infinity, and the up and out barrier option is found if the down barrier \( L \) of a double barrier option is zero.

The next results are obtained only for the call case; the put case can be obtained simultaneously.

To price European double barrier calls we must find the expected value of its terminal payoff. Consequently, from (9) the price is \( e^{-rt} E^Q[\mathbb{1}_{\{S_T \geq H, S_T \geq U\}}(S_T - K)^+] \). Instead of finding
the expected value of the terminal payoff of a knock out double barrier option, its the Laplace Transform that will be obtained, and then inverted. Next result presents the expected value of the terminal payoff of a knock out double barrier option in terms of its Laplace Transform:

**Proposition 3** The Laplace Transform of $E_Q[1_{(S_T \leq L, U)}] (S_T - K)^+]$ in time to expiration $T$ is given by:

$$
\int_0^\infty e^{-\lambda T}E_Q[1_{(S_T \leq L, U)}] (S_T - K)^+]dT
$$

$$
= \frac{1}{\omega_\lambda \Delta_\lambda(L, U)} \left\{
\begin{array}{l}
\Delta_\lambda(L, S)[\psi_\lambda(U)J_\lambda(K, K, U) - \phi_\lambda(U)I_\lambda(K, K, U)], \text{ if } S \leq K \\
\Delta_\lambda(L, S)[\psi_\lambda(U)J_\lambda(K, S, U) - \phi_\lambda(U)I_\lambda(K, S, U)] \\
+ \Delta_\lambda(S, U)[\phi_\lambda(L)I_\lambda(K, K, S) - \psi_\lambda(L)J_\lambda(K, K, S)], \text{ if } S > K
\end{array}
\right
$$

where $\Delta_\lambda(A, B)$ is defined in equation (13), $\omega_\lambda$ is the Wronskian as defined in Borodin and Salminen (p. 19, 2002), and has the closed-form expression in (23). Moreover,

$$
I_\lambda(K, A, B) = \int_A^B (Y - K)\psi_\lambda(Y)m(Y)dY;
$$

$$
J_\lambda(K, A, B) = \int_A^B (Y - K)\phi_\lambda(Y)m(Y)dY;
$$

where $m(Y)$ is the speed density of the diffusion (3).

**Proof.** Transforming the expected value in to an integral having in mind that $1_{(S_T \leq L, U)}$, and $S_T \geq K$, and using the Fubbini Theorem:

$$
\int_0^\infty e^{-\lambda T}E_Q[1_{(S_T \leq L, U)}] (S_T - K)^+]dT = \int_0^\infty e^{-\lambda T} \left[ \int_K^U (Y - K)p(T; S_T; Y)dY \right] dT
$$

$$
= \int_K^U (Y - K) \left[ \int_0^\infty e^{-\lambda T}p(T; S_T; Y)dT \right] dY,
$$

where $p(T; S_T; Y)$ is the transition density of $S_T$ with respect to the speed measure (see Borodin and Salminen (2002, p.13)).

The Green function defined in Borodin and Salminen (2002, p.19) is present in the second integral. So,

$$
\int_K^U (Y - K) \left[ \int_0^\infty e^{-\lambda T}p(T; S_T; Y)dT \right] dY = \int_K^U (Y - K)G_\lambda(S, Y)dY
$$
The solution of the Green function in Borodin and Salminen (2002, p.19), with modification introduced by Davidov and Linetsky (2001, p 962) is

\[ G_\lambda(S,Y) = \begin{cases} \frac{m(Y)}{W_\lambda} \Psi_\lambda(S)\Phi_\lambda(Y) & \text{if } S \leq Y \\ \Psi_\lambda(Y)\Phi_\lambda(S) & \text{if } S > Y \end{cases} \] (29)

where \( W_\lambda \) is the Wronskian of the function \( \Psi_\lambda \) and \( \Phi_\lambda \), which is defined by (see Borodin and Salminen (p 19, 2002)):

\[ W_\lambda = \Phi_\lambda(S)\Psi_\lambda(S) - \Psi_\lambda(S)\Phi_\lambda(S), \] (30)

and \( \Psi_\lambda(S) \) and \( \Phi_\lambda(S) \) can be expressed only in terms of the fundamental solutions of equation (14) \( \phi_\lambda(S) \) and \( \psi_\lambda(S) \).

From Davidov and Linetsky (2001, p 962) \( \Phi_\lambda(S) = \Delta_\lambda(S,U) \) and \( \Psi_\lambda(S) = \Delta_\lambda(L,S) \), where \( \Delta_\lambda(A,B) \) is defined in (13). And, consequently,

\[ \Phi_\lambda(S) = \phi_\lambda'(S)\psi_\lambda(U) - \psi_\lambda'(S)\phi_\lambda(U) \] (31)

\[ \Psi_\lambda(S) = \phi_\lambda(L)\psi_\lambda'(S) - \psi_\lambda(L)\phi_\lambda'(S) \] (32)

Using (30), (31) and (32) the Wronskian - \( W_\lambda \) - is:

\[ W_\lambda = \phi_\lambda(S)\psi_\lambda(U) - \psi_\lambda(S)\phi_\lambda(U)(\phi_\lambda(L)\psi_\lambda'(S) - \psi_\lambda(L)\phi_\lambda'(S)) \]

\[ -\phi_\lambda(L)\psi_\lambda(S) - \psi_\lambda(L)\phi_\lambda(S)(\phi_\lambda'(S)\psi_\lambda(U) - \psi_\lambda'(S)\phi_\lambda(U)) \]

\[ = \phi_\lambda(S)\psi_\lambda(U)\phi_\lambda'(L)\psi_\lambda(S) - \phi_\lambda(S)\psi_\lambda'(U)\psi_\lambda'(S)\phi_\lambda(L)\psi_\lambda(U) + \phi_\lambda(L)\psi_\lambda(S)\psi_\lambda'(S)\phi_\lambda(U) \]

\[ +\psi_\lambda(L)\phi_\lambda(S)\phi_\lambda'(S)\psi_\lambda(U) - \psi_\lambda(L)\phi_\lambda(S)\psi_\lambda'(S)\phi_\lambda(U) \]

Hence,

\[ W_\lambda = \phi_\lambda(L)\psi_\lambda(U)(\phi_\lambda(S)\psi_\lambda'(S) - \psi_\lambda(S)\phi_\lambda'(S)) \]

\[ +\phi_\lambda(U)\psi_\lambda(L)(\psi_\lambda(S)\phi_\lambda'(S) - \phi_\lambda(S)\psi_\lambda'(S)) \]

so,

\[ W_\lambda = (\phi_\lambda(S)\psi_\lambda'(S) - \psi_\lambda(S)\phi_\lambda'(S))(\phi_\lambda(L)\psi_\lambda(U) - \psi_\lambda(L)\phi_\lambda(U)) \] (33)

\[ = \omega_\lambda \Delta_\lambda(L,U), \]

14
where
\[ \omega_\lambda = \phi_\lambda(S)\psi'_\lambda(S) - \psi_\lambda(S)\phi'_\lambda(S) \]

Then (29) can be rewritten as:
\[ G_\lambda(S, Y) = \begin{cases} m(Y) & \text{if } S \leq Y \\ \omega_\lambda \Delta_\lambda(L, U) \{ \Delta_\lambda(L, Y)\Delta_\lambda(S, U) \} & \text{if } S > Y \end{cases} \tag{34} \]

Going back to (28) and (34):
\[ \int_K^U (Y - K)G_\lambda(S, Y)dY = \begin{cases} \int_K^U (Y - K)m(Y) \Delta_\lambda(L, S)\Delta_\lambda(Y, U) dY & \text{if } S \leq Y \\ \int_K^U (Y - K)m(Y) \Delta_\lambda(L, Y)\Delta_\lambda(S, U) dY & \text{if } S > Y \end{cases} \tag{35} \]

**For the case** \( S \leq K \):

Using that if \( S \leq K \) and \( Y \in [K, U] \), then \( S \leq Y \) and (35)
\[ \int_K^U (Y - K)G_\lambda(S, Y)dY = \int_K^U (Y - K)m(Y) \frac{\omega_\lambda \Delta_\lambda(L, U)}{\omega_\lambda \Delta_\lambda(L, U)} \Delta_\lambda(L, S)\Delta_\lambda(Y, U)dY \]

passing the factors that do not depend on \( Y \) out of the integral, using (13),
\[ \int_K^U (Y - K)G_\lambda(S, Y)dY = \frac{\Delta_\lambda(L, S)}{\omega_\lambda \Delta_\lambda(L, U)} \int_K^U (Y - K)m(Y)(\phi_\lambda(Y)\psi_\lambda(U) - \psi_\lambda(Y)\phi_\lambda(U))dY \]

\[ = \frac{\Delta_\lambda(L, S)}{\omega_\lambda \Delta_\lambda(L, U)} \left[ \psi_\lambda(U) \int_K^U (Y - K)m(Y)\phi_\lambda(Y)dY - \phi_\lambda(U) \int_K^U (Y - K)m(Y)\psi_\lambda(Y)dY \right] \]

and using the definitions (26) and (25)
\[ = \frac{1}{\omega_\lambda \Delta_\lambda(L, U)} \Delta_\lambda(L, S) \left[ \psi_\lambda(U)J_\lambda(K, K, U) - \phi_\lambda(Y)I_\lambda(K, K, U) \right] \tag{36} \]

**For the case** \( S > K \):

As \( S \in [L, U] \) (by definition of double barrier option), and \( S > K \), using the integral properties in Theorem 1.17 of Apostol (1994, p 98)
\[ \int_K^U (Y - K)G_\lambda(S, Y)dY = \int_K^S (Y - K)G_\lambda(S, Y)dY + \int_S^U (Y - K)G_\lambda(S, Y)dY \]

15
As \( Y \in [K, S] \implies S > Y \), and \( Y \in [S, U] \implies S \leq Y \). Applying this to the first and second integral above, using (34)

\[
\int_{K}^{S} (Y - K) G_{\lambda}(S, Y) \, dY + \int_{S}^{U} (Y - K) G_{\lambda}(S, Y) \, dY
\]

\[
= \int_{K}^{S} (Y - K) \frac{m(Y)}{\omega_{\lambda} \Delta_{\lambda}(L, U)} \Delta_{\lambda}(L, Y) \Delta_{\lambda}(S, U) \, dY
\]

\[
+ \int_{S}^{U} (Y - K) \frac{m(Y)}{\omega_{\lambda} \Delta_{\lambda}(L, U)} \Delta_{\lambda}(L, S) \Delta_{\lambda}(Y, U) \, dY
\]

\[
= \frac{\Delta_{\lambda}(S, U)}{\omega_{\lambda} \Delta_{\lambda}(L, U)} \int_{K}^{S} (Y - K) m(Y) \Delta_{\lambda}(L, Y) \, dY
\]

\[
+ \frac{\Delta_{\lambda}(L, S)}{\omega_{\lambda} \Delta_{\lambda}(L, U)} \int_{S}^{U} (Y - K) m(Y) \Delta_{\lambda}(Y, U) \, dY
\]

\[
= \frac{1}{\omega_{\lambda} \Delta_{\lambda}(L, U)} \left[ \Delta_{\lambda}(S, U) \int_{K}^{S} (Y - K) m(Y) (\phi_{\lambda}(L) \psi_{\lambda}(Y) - \psi_{\lambda}(L) \phi_{\lambda}(Y)) \, dY 
\right.
\]

\[
+ \Delta_{\lambda}(L, S) \int_{S}^{U} (Y - K) m(Y) (\phi_{\lambda}(L) \psi_{\lambda}(U) - \psi_{\lambda}(L) \phi_{\lambda}(U)) \, dY \right]
\]

\[
= \frac{1}{\omega_{\lambda} \Delta_{\lambda}(L, U)} \left\{ \Delta_{\lambda}(S, U) \left[ \phi_{\lambda}(L) \int_{K}^{S} (Y - K) m(Y) \psi_{\lambda}(Y) \, dY 
\right.
\right.
\]

\[
- \psi_{\lambda}(L) \int_{K}^{S} (Y - K) m(Y) \phi_{\lambda}(Y) \, dY 
\left.
\right]
\]

\[
+ \Delta_{\lambda}(L, S) \left[ \psi_{\lambda}(U) \int_{S}^{U} (Y - K) m(Y) \phi_{\lambda}(Y) \, dY - \phi_{\lambda}(U) \int_{S}^{U} (Y - K) m(Y) \psi_{\lambda}(Y) \, dY \right] \right\}
\]

and using the definitions (25) and (26) we have

\[
= \frac{1}{\omega_{\lambda} \Delta_{\lambda}(L, U)} \left[ \Delta_{\lambda}(S, U) \left[ \phi_{\lambda}(L) I_{\lambda}(K, K, S) - \psi_{\lambda}(L) J_{\lambda}(K, K, S) \right] 
\right.
\]

\[
+ \Delta_{\lambda}(L, S) \left[ \psi_{\lambda}(U) I_{\lambda}(K, S, U) - \phi_{\lambda}(U) J_{\lambda}(K, S, U) \right] \right]
\]

Therefore, equation (24) arises immediately from this last equation and from equation (36).

The derivation of a closed-form expression for integrals \( I_{\lambda}(K, A, B) \) and \( J_{\lambda}(K, A, B) \) is given in Appendix B. Note that the close-form expression for integrals \( I_{\lambda}(K, A, B) \) and \( J_{\lambda}(K, A, B) \) deduced in this thesis are different from the solutions in Davidov and Linetsky (2001, p 964) for the case \( \mu = 0 \) and \( \beta > 0 \).

Now, to find the value in time \( t=0 \) of an European double barrier option. To do that we only have to invert the Laplace Transform (24).

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4.1 Pricing Rebates

As seen before, rebates are fixed cash amounts that can be paid if the barrier is hit (for knock out options). Many barrier options have a rebate implicit that is paid in case of knock out. To value the cash rebates included in knock-out options it is necessary to value claims that pay one EUR at times $\mathcal{X}_L$, $\mathcal{X}_U$ and $\mathcal{X}_{(L,U)}$. (as seen in Proposition 1) This means that expectations of the form $E^Q[\mathbb{1}_{\{\theta \leq T\}} e^{-r\theta}]$ must be evaluated. To calculate these expectations the Laplace Transforms will be used, like for the terminal payoff in Proposition 3.

**Proposition 4** For any $\lambda > 0$, the Laplace transform of the rebate price in time to expiration $T$ is:

$$\int_0^{\infty} e^{-\lambda T} E^Q[\mathbb{1}_{\{\theta \leq T\}} e^{-r\theta}]dT = \frac{1}{\lambda} E^Q[e^{-(r+\lambda)\theta}]$$ (37)

**Proof.** The Laplace transform of the rebate price is

$$\int_0^{\infty} e^{-\lambda T} E^Q[\mathbb{1}_{\{\theta \leq T\}} e^{-r\theta}]dT.$$

Using the Fubbini theorem:

$$\int_0^{\infty} e^{-\lambda T} E^Q[\mathbb{1}_{\{\theta \leq T\}} e^{-r\theta}]dT = E^Q[e^{-r\theta} \int_0^{\infty} e^{-\lambda T} \mathbb{1}_{\{\theta \leq T\}}dT] = E^Q[e^{-r\theta} \int_0^{\infty} e^{-\lambda T}dT],$$

and, using Barrow’s formula,

$$= E^Q[e^{-r\theta} \left(-\frac{1}{\lambda} (0 - e^{-\theta\lambda})\right)]dT = \frac{1}{\lambda} E^Q[e^{-(r+\lambda)\theta}]$$

The next result gives a closed-form solution for the rebates of knock-out barrier options.

**Proposition 5** The close-form solution $(R)$ of the Laplace Transform in time to expiration $T$ of the unit rebate price for knock out barrier option is:
5.1. Down-and-out Barrier Option:

\[
R = \begin{cases} 
1 & \frac{U}{\lambda(1+\frac{x}{2}\frac{1}{\beta})} \mu < 0, \beta < 0 \\
\frac{1}{\lambda} & \frac{U}{\lambda(1+\frac{x}{2}\frac{1}{\beta})} \mu < 0, \beta > 0 \\
\frac{1}{\lambda} & \frac{U}{\lambda(1+\frac{x}{2}\frac{1}{\beta})} \mu > 0, \beta < 0 \\
1 & \frac{U}{\lambda(1+\frac{x}{2}\frac{1}{\beta})} \mu > 0, \beta > 0
\end{cases}
\]

where \( L \) is the lower barrier.

5.2. Up-and-out barrier option:

\[
R = \begin{cases} 
1 & \frac{U}{\lambda(1+\frac{x}{2}\frac{1}{\beta})} \mu < 0, \beta < 0 \\
\frac{1}{\lambda} & \frac{U}{\lambda(1+\frac{x}{2}\frac{1}{\beta})} \mu < 0, \beta > 0 \\
\frac{1}{\lambda} & \frac{U}{\lambda(1+\frac{x}{2}\frac{1}{\beta})} \mu > 0, \beta < 0 \\
1 & \frac{U}{\lambda(1+\frac{x}{2}\frac{1}{\beta})} \mu > 0, \beta > 0
\end{cases}
\]

where \( U \) is the upper barrier.

5.3. Double barrier option:

\[
R = \frac{1}{\lambda} \phi_{r}(L)\psi_{r}(S) - \psi_{r}(L)\phi_{r}(S) + \phi_{r}(S)\psi_{r}(U) - \psi_{r}(S)\phi_{r}(U) - \frac{1}{\lambda} \phi_{r}(U)\psi_{r}(L) - \psi_{r}(L)\phi_{r}(U)
\]

\( M(a,b,x) \) and \( U(a,b,x) \) are the Kummer’s function defined in 13.1.2 and 13.1.3 of Abramowitz and Stegun (1972, p 504), \( K_{\nu}(z) \) and \( I_{\nu}(z) \) are the modified Bessel function and \( \phi_{r}(S) \) and \( \psi_{r}(S) \) are defined in (16) and (15). And

\[
\alpha = r + \lambda
\]

Proof. See Appendix C. \( \blacksquare \)
Then, to find the present value of the rebate it is only necessary to invert the Laplace transforms (38), (39) or (40).
5 Laplace Transform

The Laplace Transform method is very useful to solve linear differential equations. Thus, Laplace transformation is one of the most popular methods for solving diffusion equations. On the other hand, the inversion of the Laplace Transform is the most difficult step in applying the Laplace Transform technique. However, there are algorithms to provide the inverse of the Laplace Transforms numerically, and they are powerful tools that can be used to extend the applicability of the Laplace Transform technique. As a result, in recent years, numerical transform inversion has been recognized as an important technique for calculating probability distributions in stochastic models. Over the years many different algorithms have been proposed, and studied, for numerically inverting Laplace Transforms. See, for instance, Stehfest (1970), Abate and Whitt (1991), Abate and Whitt (1995) and Fu et al (1999).

In this work the Euler Abate and Whitt method mentioned in Abate and Whitt (1995, p. 37-39) will be used. This method is called "Euler Abate and Whitt" because it employs the Euler summation, and is based on the Bromwich contour inversion integral, which can be expressed as the integral of a real valued function of a real variable by choosing a specific contour. The Abate and Whitt (1995) Euler method (see the proof in Abate and Whitt (1995, p 37-39)) calculates values of a real-valued function $f(t)$ of a positive real variable $t$ for different $t$ from the Laplace transform

$$F(s) = \int_0^\infty e^{-st} f(t) dt,$$

where $s$ is a complex variable with nonnegative real part, and

$$f(t) = E(m, n, t)$$

where $E(m, n, t)$ is the Euler sum.

The Euler sum is defined as,

$$E(m, n, t) = \sum_{k=0}^{m} \binom{m}{k} 2^{-m} S_{n+k}(t), \quad (42)$$
with
\[ S_n(t) = \frac{e^{\frac{A}{2t}}}{2t} \text{Re}(F(\frac{A}{2t})) + \frac{e^{\frac{A}{2t}}}{t} \sum_{k=0}^{n} (-1)^k a_k(t), \]  
(43)

where
\[ a_k(t) = \text{Re}(F(\frac{A + 2k\pi i}{2t})). \]  
(44)

In a Euler sum the Euler summation is applied to \( m \) terms after an initial \( n \). It is applied as an acceleration technique that can be described as the weighted average of the last \( m \) partial sums by a binomial probability distribution with parameters \( m \) and \( p = \frac{1}{2} \).

The parameter \( A \) describes the discretization error. To produce a \( 10^{-k} \) accuracy \( A = k \times \log(10) \). In Abate and Whitt (1995), the authors suggest the values of \( K = 8 \), \( m = 11 \) and \( n = 15 \).

In this work the Abate and Whitt Euler method is used to invert the Laplace Transforms (24), (38), (39) and (40). However the parameters to use are \( K = 10 \), \( m = 12 \) and \( n = 35 \), as suggested in Skachkov (2002), to achieve a better precision in the computational application.
6 Numerical Implementation

In Proposition 3 we found the expression of the Laplace Transform for a double knock out barrier option. To find the value of a double knock out barrier call option it is necessary invert that Laplace Transform. In this section the C++ program "Pricing_Knock_Out_Barrier_Call_Under_CEV_Diffusion.cpp" (see Appendix D) is implemented and will invert the Lapalce Transform (24). As a result, it will allow us to find the value of the double knock out barrier call option. In this program the inversion of the Laplace Transform is made using Euler Abate and Whitt method mentioned in Abate and Whitt (1995, p. 37-39).

The program "Pricing_Knock_Out_Barrier_Call_Under_CEV_Diffusion.cpp" is composed by functions that will implement equations (19), (15), (16), (25), (26), (23) and (24). The program also includes a function named "dfAbateWhittEu()" that inverts the Laplace Transform using the Euler Abate and Whitt method. This function was based in Skachkov (2002). Some changes have been made to the original function, specially one that allows the inversion of functions with more than one variable. The program also includes the functions Gamma, modified Bessel $I_v(x)$ and $K_v(x)$ and the Whittaker functions:

- $\text{Gamma}()$; $\text{cgamma}()$, $\text{cbessIv}()$ and $\text{cbessKv}()$ were adapted from the similar functions programed in Fortran in Zhang and Gin (1996, p. 49, 51, 230-233).

- The Whittaker functions were programed using the definitions 1.9.7 and 1.9.11 of Slater (1960, p. 14) and 1.3.5 of Slater (1960, p. 5). The Kummer $M(a,b,x)$ function present in those definitions was programed in the function $\text{KummerM\_sum\_max}()$. This function was adapted from the $\text{greens\_specfunc.cpp}$ function in Koval (2004). The function originally computes the Kummer $M(a,b,x)$ with first parameter and argument complex, and second argument real, and it has been adapted to compute the Kummer $M(a,b,x)$ function with first parameter complex and the second parameter and argument as real numbers.

The "Pricing Knock Out Barrier Call Under CEV Diffusion.cpp" program was compiled using the freeware Dev-C++ compiler.
6.1 Numerical Results of the C++ Program

The program "Pricing_Knock_Out_BARRIER_Call_Under_CEVDiffusion.cpp" to price
knock out barrier options without rebate was implemented using similar parameters to the
ones used by Davidov and Linetsky (2001, p 957), with some changes in order to include
some positive values for $\beta$ and the dividend yield as 10% per annum. So, the initial asset
price is $S_0 = 100$, the instantaneous volatility at this price is $\sigma_0 = 25\%$ per annum (so,
$\delta = \sigma_0 S_0^{-\beta}$). The risk free rate and the dividend yield are 10% per annum, and all options
have six months to expiration ($T=0.5$). And $\beta = -3; -2; -1; -0.5; 0.5; 1; 2; 3$. The results of
the program are shown in the next table:

<table>
<thead>
<tr>
<th>$U$</th>
<th>$L$</th>
<th>$K$</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>-0.5</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>% Dif.</th>
</tr>
</thead>
<tbody>
<tr>
<td>N/A</td>
<td>90</td>
<td>95</td>
<td>7.408</td>
<td>7.453</td>
<td>7.499</td>
<td>7.522</td>
<td>7.570</td>
<td>7.594</td>
<td>7.496</td>
<td>7.386</td>
<td>3%</td>
</tr>
<tr>
<td>N/A</td>
<td>90</td>
<td>100</td>
<td>5.406</td>
<td>5.512</td>
<td>5.624</td>
<td>5.683</td>
<td>5.806</td>
<td>5.870</td>
<td>5.861</td>
<td>4.853</td>
<td>21%</td>
</tr>
<tr>
<td>120</td>
<td>N/A</td>
<td>95</td>
<td>4.861</td>
<td>3.946</td>
<td>3.191</td>
<td>2.869</td>
<td>2.325</td>
<td>2.097</td>
<td>1.718</td>
<td>1.423</td>
<td>242%</td>
</tr>
<tr>
<td>120</td>
<td>N/A</td>
<td>100</td>
<td>2.840</td>
<td>2.230</td>
<td>1.733</td>
<td>1.524</td>
<td>1.176</td>
<td>1.034</td>
<td>0.800</td>
<td>0.623</td>
<td>356%</td>
</tr>
<tr>
<td>120</td>
<td>N/A</td>
<td>105</td>
<td>1.359</td>
<td>1.029</td>
<td>0.766</td>
<td>0.658</td>
<td>0.483</td>
<td>0.414</td>
<td>0.303</td>
<td>0.222</td>
<td>512%</td>
</tr>
<tr>
<td>120</td>
<td>90</td>
<td>95</td>
<td>2.974</td>
<td>2.388</td>
<td>1.904</td>
<td>1.699</td>
<td>1.360</td>
<td>1.222</td>
<td>0.998</td>
<td>0.832</td>
<td>257%</td>
</tr>
<tr>
<td>120</td>
<td>90</td>
<td>100</td>
<td>1.848</td>
<td>1.437</td>
<td>1.102</td>
<td>0.963</td>
<td>0.735</td>
<td>0.643</td>
<td>0.497</td>
<td>0.389</td>
<td>375%</td>
</tr>
<tr>
<td>120</td>
<td>90</td>
<td>105</td>
<td>0.933</td>
<td>0.697</td>
<td>0.511</td>
<td>0.436</td>
<td>0.316</td>
<td>0.269</td>
<td>0.195</td>
<td>0.144</td>
<td>548%</td>
</tr>
</tbody>
</table>

In the previous table the prices for knock out call options are presented. The first three
lines, where the upper barrier is N/A, contain the prices of down-and-out call options. These
values are found using large values for $U$, this is, making $U \to \infty$. In the same way, making
$L \to 0$, the values for up-and-out call options can be found. These values are in the next
three columns where $L$ is N/A. The last three lines have the values of double barrier call
options. For each type of knock out option three different strikes ($K$) were considered: in
the money (ITM), out of the money (OTM) and at the money (ATM). The last column
has the maximum difference, in percentage, between the minimum price and the maximum
price for each strike, and each type of knock out barrier option, for different betas. As it can be seen from the last column the beta value has a great influence in the price of the option. The impact of the parameter beta is bigger for the up-and-out and double knock out barrier options, and the maximum difference happens for OTM options. This result of greater impact of parameter beta for up-and-out and double knock out barrier options, in case of the equal risk free rate and the dividend yield, confirms the results of Boyle and Tian (1999) and Davidov and Linetsky (2001) for the cases where no dividends are paid.
7 Conclusion

On this thesis a closed-form solution for the Laplace Transform of the knock out barrier option is obtained. After that, this Laplace Transform is inverted and this allows the evaluation of the knock out barrier options under the CEV diffusion. The solution obtained differs from the expression achieved by Davidov and Linetsky (2001) for integrals $I_\lambda(K, A, B)$ and integral $J_\lambda(K, A, B)$ for the case $\mu = 0$ and $\beta > 0$. The closed-form solution found was implemented computationally using the C++ language, and a freeware compiler. The analytical formulae obtained allows a fast and accurate calculation of prices under the CEV diffusion on a normal PC.

From the results obtained by the implementation of the computational program, the impact of the beta value in the pricing of options is evident, and this impact is much more clear for up-and-out and double knock out barrier options. These results confirm the weaknesses appointed in literature to the Black and Scholes model, and they attest the importance that models for pricing options should incorporate the volatility smile, or incorporate volatility as a time dependent parameter.
Appendix - Proof of Proposition 2

Has seen before in CEV model $\sigma^2(S) = \delta S^\beta$. So replacing in (14), then it becomes

$$\frac{1}{2} \delta^2 S^{2\beta+2} \frac{\partial^2 u}{\partial S^2} + \mu S \frac{\partial u}{\partial S} - \lambda u = 0.$$  \hfill (A-1)

- Proof of equations (15) and (16)

Next all possible cases are considered:

1. **Case $\mu \neq 0$**

First, we make the change of variable,

$$u(S) = S^{\frac{1}{2}+\beta} e^{\frac{\epsilon}{2} x(S)} w(x(S)), \hfill (A-2)$$

where $\epsilon$ and $x(S)$ are defined by (19) and (17) respectively.

Before replacing $u(S)$ in the ODE (A-1), $\frac{\partial u}{\partial S}$ and $\frac{\partial^2 u}{\partial S^2}$ will be calculated as

$$\frac{\partial u}{\partial S} = \left( \frac{1}{2} + \beta \right) S^{\beta - \frac{1}{2}} e^{\frac{\epsilon}{2} x(S)} w(x(S)) + S^{\frac{1}{2}+\beta} e^{\frac{\epsilon}{2} x(S)} \frac{\mu}{2 |\beta| \delta^2} (-2\beta) S^{-2\beta-1} w(x(S)) \hfill (A-3)$$

$$
= \left\{ \left[ \left( \frac{1}{2} + \beta \right) S^{\beta - \frac{1}{2}} - \frac{\epsilon |\mu| |\beta| \delta^2 S^{-\beta - \frac{1}{2}}}{2 |\beta| \delta^2} \right] w(x(S)) - 2 \frac{|\mu| |\beta| \delta^2 S^{-\beta - \frac{1}{2}}}{|\beta| \delta^2} \frac{\partial w}{\partial x} \right\} e^{\frac{\epsilon}{2} x(S)}
\right.$$
\[
\frac{\partial^2 u}{\partial S^2} = \left[ \left( \frac{1}{2} + \beta \right) \left( -\frac{1}{2} + \beta \right) S^{\beta - \frac{1}{2}} - \varepsilon \frac{\mu}{|\beta| \delta^2} \left( -\frac{1}{2} - \beta \right) S^{-\beta - \frac{3}{2}} \right] e^{\hat{z}(S)} w(x(S)) \tag{A-4}
\]
\[
+ \left[ \left( \frac{1}{2} + \beta \right) S^{\beta - \frac{1}{2}} - \frac{\varepsilon |\mu| \beta}{|\beta| \delta^2} S^{-\beta - \frac{1}{2}} \right] e^{\hat{z}(S)} \frac{\varepsilon |\mu|}{2 |\beta| \delta^2} (2\beta) S^{-2\beta - 1} w(x(S))
\]
\[
+ \left[ \left( \frac{1}{2} + \beta \right) S^{\beta - \frac{1}{2}} - \frac{\varepsilon |\mu| \beta}{|\beta| \delta^2} S^{-\beta - \frac{1}{2}} \right] e^{\hat{z}(S)} \frac{\mu}{\delta x \frac{|\beta| \delta^2} (2\beta) S^{-2\beta - 1} \frac{\partial w}{\partial x}
\]
\[
- 2 \frac{\mu}{|\beta| \delta^2} \frac{|\beta| \delta^2}{S^{\beta - \frac{1}{2}} e^{\hat{z}(S)} \frac{\partial^2 w}{\partial x^2} \frac{|\beta| \delta^2} (2\beta) S^{-2\beta - 1}
\]
\[
= \left\{ \left[ \left( \frac{1}{2} + \beta \right) \left( -\frac{1}{2} + \beta \right) S^{\beta - \frac{1}{2}} + \varepsilon^2 \frac{|\mu|^2 \beta^2}{|\beta|^2 \delta^4} S^{-3\beta - \frac{3}{2}} \right] w(x(S))
\]
\[
+ 4 \varepsilon \frac{|\mu|^2 \beta^2}{|\beta|^2 \delta^4} S^{-3\beta - \frac{3}{2}} \frac{\partial w}{\partial x} + 4 \frac{|\mu|^2 \beta^2}{|\beta|^2 \delta^4} S^{-3\beta - \frac{3}{2}} \frac{\partial^2 w}{\partial x^2} \right\} e^{\hat{z}(S)}
\]

Now replacing (A-2), (A-3) and (A-4) in (A-1):

\[
\left( As \ x(S) = \frac{|\mu|}{|\beta| \delta^2} S^{-2\beta} \Rightarrow S = \left( \frac{|\beta|^2}{|\mu|} x(S) \right)^{-\frac{1}{2\beta}} \right)
\]
\[
\frac{1}{2} \delta^2 S^{2\beta + 2} \left\{ \left[ \left( \frac{1}{2} + \beta \right) \left( -\frac{1}{2} + \beta \right) S^{\beta - \frac{1}{2}} + \varepsilon^2 \frac{|\mu|^2 \beta^2}{|\beta|^2 \delta^4} S^{-3\beta - \frac{3}{2}} \right] w(x(S)) +
\]
\[
4 \varepsilon \frac{|\mu|^2 \beta^2}{|\beta|^2 \delta^4} S^{-3\beta - \frac{3}{2}} \frac{\partial w}{\partial x} + 4 \frac{|\mu|^2 \beta^2}{|\beta|^2 \delta^4} S^{-3\beta - \frac{3}{2}} \frac{\partial^2 w}{\partial x^2} \right\} e^{\hat{z}(S)} +
\]
\[
+ \mu S \left\{ \left[ \left( \frac{1}{2} + \beta \right) S^{\beta - \frac{1}{2}} - \frac{\varepsilon |\mu| \beta}{|\beta| \delta^2} S^{-\beta - \frac{1}{2}} \right] w(x(S))
\]
\[
- 2 \frac{|\mu| \beta}{|\beta| \delta^2} S^{-\beta - \frac{1}{2}} \frac{\partial w}{\partial x} \right\} e^{\hat{z}(S)} - \lambda S^{\frac{1}{2} + \beta} e^{\hat{z}(S)} w(x(S)) = 0.
\]

Since \( e^{\hat{z}(S)} \neq 0 \) and \( S^{\frac{1}{2}} \neq 0 \),

\[
2 \frac{|\mu|^2 \beta^2}{|\beta|^2 \delta^2} S^{-\beta} \frac{\partial^2 w}{\partial x^2} + \left[ 2 \varepsilon \frac{|\mu|^2 \beta^2}{|\beta|^2 \delta^2} S^{-\beta} - 2 \frac{\mu}{|\beta| \delta^2} S^\beta \right] \frac{\partial w}{\partial x}
\]
\[
+ \left[ \frac{1}{2} \delta^2 \left( \frac{1}{2} + \beta \right) \left( -\frac{1}{2} + \beta \right) S^{3\beta}\right.
\]
\[
+ \frac{1}{2} \varepsilon^2 \frac{|\mu|^2 \beta^2}{|\beta|^2 \delta^2} S^{-\beta} + \mu \left( \frac{1}{2} + \beta \right) S^\beta - \varepsilon \frac{|\mu| \beta}{|\beta| \delta^2} S^{-\beta - \lambda S^\beta} \right] w(x(S)) = 0,
\]
Hence,
\[
\frac{\partial^2 w}{\partial x^2} + \left\{-\frac{1}{4} + \left[\mu(\beta + \frac{1}{2}) - \lambda\right] \frac{\delta^2 |\beta|^2}{2 |\mu|^2 \beta^2 \delta^2 |\beta| x} + \frac{\delta^4 |\beta|^2}{2 |\mu|^2 \beta^2} \left(\beta^2 - \frac{1}{4}\right) \frac{|\mu|^2}{\delta^4 |\beta|^2 x(S)^2}\right\} w(x(S)) = 0,
\]
and finally,
\[
\frac{\partial^2 w}{\partial x^2} + \left\{-\frac{1}{4} + \left[\frac{1}{2} \frac{|\beta|}{|\mu|} \frac{\mu |\beta|}{\beta^2} (1 + \frac{1}{2}) - \frac{\lambda}{\frac{1}{2} |\mu|}\right] \frac{1}{x} + \left(\frac{1}{4} - \frac{1}{4 \mu^2 |\beta| x(S)^2}\right)\right\} w(x(S)) = 0. \tag{A-5}
\]
In equation (A-5) \( \frac{1}{|\mu|} \frac{|\beta|}{|\mu|} = \{\begin{array}{cl} 1 & \beta_{\mu>0} \\ -1 & \beta_{\mu<0} \end{array} \), which is \( \epsilon \) as defined in (19). So, \( \frac{1}{|\mu|} \frac{|\beta|}{|\mu|} \) can be replaced by \( \epsilon \).

Considering \( k = \epsilon \left(\frac{1}{2} + \frac{1}{4\beta}\right) - \frac{\lambda}{2 |\mu|} \) as in (21), and \( m = \frac{1}{4 |\beta|} \) as in (20), equation (A-5) becomes:
\[
\frac{\partial^2 w}{\partial x^2} + \left[-\frac{1}{4} + \frac{k}{x} + \left(\frac{1}{4} - m^2\right) \frac{1}{x^2}\right] w(x(S)) = 0 \tag{A-6}
\]
Equation (A-6) is the Whittaker form of the confluent hypergeometric equation - see equation 13.1.31 of Abramowitz and Stegun (1972, p 505). The complete solution of the ODE (A-6) is:
\[
w(x(S)) = A \times M_{k,m}(x(S)) + B \times W_{k,m}(x(S)), \tag{A-7}
\]
where \( M_{k,m}(x) \) and \( W_{k,m}(x) \) are defined respectively in equations 13.1.32 and 13.1.33 of Abramowitz and Stegun (1972, p 505). So, the complete solutions of the ODE (A-1) are
\[
u(S) = S^{\frac{1}{2}+\beta} e^{\frac{x}{2}x(S)} [A \times M_{k,m}(x(S)) + B \times W_{k,m}(x(S))]. \tag{A-8}
\]
Using equations 1.9.7 and 4.1.7 in Slater (1960, p 14 and 60) for \( M_{k,m}(x) \), and equations 1.9.11 and 4.1.12 in Slater (1960, p 14 and 60) for \( W_{k,m}(x) \) we arrive at the following boundaries conditions:
\[
\lim_{x \to \infty} M_{k,m}(x) = \lim_{x \to \infty} \left(x^{\frac{1}{2}+m} e^{-\frac{1}{2}x} F_1(\frac{1}{2}+m-k,1+2m,x) \right) = \infty,
\]
\[
\lim_{x \to 0} M_{k,m}(x) = \lim_{x \to 0} \left(x^{\frac{1}{2}+m} e^{-\frac{1}{2}x} F_1(\frac{1}{2}+m-k,1+2m,x) \right) = 0,
\]
\[
\lim_{x \to \infty} W_{k,m}(x) = \lim_{x \to \infty} \left( x^{1/2+m}e^{-x/2}U[\frac{1}{2} + m - k, 1 + 2m, x] \right) = 0,
\]
\[
\lim_{x \to 0} W_{k,m}(x) = \lim_{x \to 0} \left( x^{1/2+m}e^{-x/2}U[\frac{1}{2} + m - k, 1 + 2m, x] \right) = \infty,
\]
where \( _1F_1[\frac{1}{2} + m - k, 1 + 2m, x] \) and \( U[\frac{1}{2} + m - k, 1 + 2m, x] \) are solutions of the Kummer’s equation as defined in 1.1.7 and 1.3.5 of Slater (1960, p 2 and p 5) respectively. We need to consider two cases:

1.1. **Case \( \beta > 0 \)**

If \( S \to \infty \), then \( x(S) \to 0 \); and when \( S \to 0 \), then \( x(S) \to \infty \). So, using the limit properties of the Whittakers function described above, \( M_{k,m}(x(S)) \) is the decreasing solution, and \( W_{k,m}(x(S)) \) is the increasing solution (\( x(S) \) defined in (17)).

1.2. **Case \( \beta < 0 \)**

If \( S \to \infty \), then \( x(S) \to \infty \); and when \( S \to 0 \) then \( x(S) \to 0 \). So, using the limit properties of the Whittakers function described above, \( W_{k,m}(x(S)) \) is the decreasing solution, and \( M_{k,m}(x(S)) \) is the increasing solution (\( x(S) \) defined in (17)).

Consequently, using the boundaries conditions of \( \psi_\lambda \) and \( \phi_\lambda \) explained in Borodin and Salminen (2002, p. 19), \( \psi_\lambda(S) \) is the increasing and \( \phi_\lambda(S) \) is the decreasing solution of (A-1). Then, (up a multiplicative constant):

\[
\psi_\lambda(S) = \begin{cases} 
S^{\beta+\frac{1}{2}}e^{\frac{x}{2}x(S)}M_{k,m}(x(S)), & \beta < 0, \ \mu \neq 0 \\
S^{\beta+\frac{1}{2}}e^{\frac{x}{2}x(S)}W_{k,m}(x(S)), & \beta > 0, \ \mu \neq 0
\end{cases}
\]
\[
\phi_\lambda(S) = \begin{cases} 
S^{\beta+\frac{1}{2}}e^{\frac{x}{2}x(S)}W_{k,m}(x(S)), & \beta < 0, \ \mu \neq 0 \\
S^{\beta+\frac{1}{2}}e^{\frac{x}{2}x(S)}M_{k,m}(x(S)), & \beta > 0, \ \mu \neq 0
\end{cases}
\]

2. **Case \( \mu = 0 \)**

Let us make the change of variable,

\[
u(S) = S^{\frac{1}{2}}g(\sqrt{2\lambda z}(S)), \quad (A-9)\]
where \( z(S) = \frac{1}{\delta S^{\beta}} S^{-\beta} \). Consider \( y = \sqrt{2\lambda} z(S) \Rightarrow z = \frac{y}{\sqrt{2\lambda}} \).

Before substituting \( u(S) \) in the ODE (A-1), we first need to compute \( \frac{\partial u}{\partial S} \) and \( \frac{\partial^2 u}{\partial S^2} : 
\)
\[
\frac{\partial u}{\partial S} = \frac{1}{2} S^{-\frac{3}{2}} g(\sqrt{2\lambda} z(S)) + \frac{3}{8} \frac{\partial g}{\partial z} \frac{\sqrt{2\lambda}}{\delta |\beta|} (-\beta) S^{-\beta - 1} 
\]
\[
= \frac{1}{2} S^{-\frac{3}{2}} g(\sqrt{2\lambda} z(S)) + \frac{3}{8} \frac{\partial g}{\partial z} \frac{1}{\sqrt{2\lambda} \delta |\beta|} (-\beta) S^{-\beta - 1},
\]

i.e.
\[
\frac{\partial u}{\partial S} = S^{-\frac{3}{2}} \left[ \frac{1}{2} g(\sqrt{2\lambda} z(S)) - \frac{\beta}{\delta |\beta|} S^{-\beta} \frac{\partial g}{\partial z} \right] \quad (A-10)
\]

finally,
\[
\frac{\partial^2 u}{\partial S^2} = S^{-\frac{3}{2}} \left[ -\frac{1}{4} g(\sqrt{2\lambda} z(S)) + \frac{\beta^2}{\delta |\beta|} S^{-\beta} \frac{\partial g}{\partial z} + \frac{\beta^2}{\delta^2 |\beta|^2} S^{-2\beta} \frac{\partial^2 g}{\partial z^2} \right] - \lambda S^{-\frac{1}{2}} g(\sqrt{2\lambda} z(S)) = 0 \quad (A-11)
\]

Now replacing (A-9), (A-10) and (A-11) in (A-1) but with \( \mu = 0 \):
\[
\frac{1}{2} \delta^2 S^{2\beta + 2} S^{-\frac{3}{2}} \left[ -\frac{1}{4} g(\sqrt{2\lambda} z(S)) + \frac{\beta^2}{\delta |\beta|} S^{-\beta} \frac{\partial g}{\partial z} + \frac{\beta^2}{\delta^2 |\beta|^2} S^{-2\beta} \frac{\partial^2 g}{\partial z^2} \right] - \lambda S^{-\frac{1}{2}} g(\sqrt{2\lambda} z(S)) = 0
\]

that is,
\[
S^\frac{1}{2} \left[ \frac{1}{2} \frac{\beta^2}{|\beta|^2} \frac{\partial^2 g}{\partial z^2} + \frac{1}{2} \frac{\delta^2}{|\beta|^2} S^\beta \frac{\partial g}{\partial z} - \frac{1}{8} \delta^2 S^{2\beta} g(\sqrt{2\lambda} z(S)) - \lambda g(\sqrt{2\lambda} z(S)) \right] = 0
\]

Multiplying both members for \( 2^\frac{1}{2\beta} S^{2-\beta} \) with attention to \( S^\frac{1}{2} \neq 0 \):
\[
\frac{1}{\delta^2 |\beta|^2} S^{-2\beta} \frac{\partial^2 g}{\partial z^2} + \frac{1}{\delta |\beta|} S^{-\beta} \frac{\partial g}{\partial z} - \left[ \frac{2\lambda}{\delta^2 |\beta|^2} S^{-2\beta} + \frac{1}{4\beta^2} \right] g(\sqrt{2\lambda} z(S)) = 0 \quad (A-12)
\]

But \( z(S) = \frac{1}{\delta |\beta|} S^{-\beta} \), and we can make \( \nu = \frac{1}{2|\beta|} \) (like in (22)). So, replacing in equation (A-12):
\[
z^2 \frac{\partial^2 g}{\partial z^2} + z \frac{\partial g}{\partial z} - \left[ 2\lambda z^2 + \nu^2 \right] g(\sqrt{2\lambda} z(S)) = 0 \quad (A-13)
\]

The differential equation (A-13) is the modified Bessel ODE - see equation 9.6.1 of Abramowitz and Stegun (1972, p 374). This ODE has two independent solutions \( I_\nu(\sqrt{2\lambda} z) \) and \( K_\nu(\sqrt{2\lambda} z) \) that are defined, respectively, in equations 9.6.10 and 9.6.2 of Abramowitz and Stegun (1972, p. 375).

Consequently, the complete solutions of the ODE (A-1) with \( \mu = 0 \) is
\[
u(S) = A \times S^\frac{1}{2} I_\nu(\sqrt{2\lambda} z(S)) + B \times K_\nu(\sqrt{2\lambda} z(S)). \quad (A-14)
\]
Using equations 9.6.7 and 9.7.1 in Abramowitz and Stegun (1972, p 375 and p 377) for $I_\nu(z)$ and equations 9.6.9 and 9.7.2 in Abramowitz and Stegun (1972, p 375 and p 378) for $K_\nu(z)$ we get the boundaries conditions:

\[
\lim_{z \to -\infty} I_\nu(z) = \infty \\
\lim_{z \to 0} I_\nu(z) = 0 \\
\lim_{z \to \infty} K_\nu(z) = 0 \\
\lim_{z \to 0} K_\nu(z) = \infty
\]

We need to consider two cases:

2.1. **Case $\beta > 0$**

If $S \to -\infty$, then $z(S) \to 0$; and when $S \to 0$, then $z(S) \to \infty$. So, using the limit properties of the Modified Bessel function described above, $I_\nu(\sqrt{2\lambda z}(S))$ is the decreasing solution, and $K_\nu(\sqrt{2\lambda z}(S))$ is the increasing solution ($z(S)$ defined in (18)).

2.2. **Case $\beta < 0$**

If $S \to -\infty$, then $z(S) \to \infty$; and when $S \to 0$, then $z(S) \to 0$. Consequently, using the limit properties of the Modified Bessel function described above $K_\nu(\sqrt{2\lambda z})$ is the decreasing solution, and $I_\nu(\sqrt{2\lambda z}(S))$ is the increasing solution ($z(S)$ defined in (18)).

Thus, using the boundaries conditions of $\psi_\lambda$ and $\phi_\lambda$ explained in Borodin and Salminen (2002, p. 19), $\psi_r(S)$ is the increasing and $\phi_r(S)$ is the decreasing solution of (A-1). Then, (up to a multiplicative constant):

\[
\psi_\lambda(S) = \begin{cases} 
S^{\beta + \frac{1}{2}} I_\nu(\sqrt{2\lambda z}(S)), & \beta < 0, \mu = 0 \\
S^{\beta + \frac{1}{2}} K_\nu(\sqrt{2\lambda z}(S)), & \beta > 0, \mu = 0 
\end{cases}
\]

and

\[
\phi_\lambda(S) = \begin{cases} 
S^{\beta + \frac{1}{2}} K_\nu(\sqrt{2\lambda z}(S)), & \beta < 0, \mu = 0 \\
S^{\beta + \frac{1}{2}} I_\nu(\sqrt{2\lambda z}(S)), & \beta > 0, \mu = 0 
\end{cases}
\]
Proof of the wronskian expression (23):

From Borodin and Salminen (2002, p.17) the speed density \( m(S) \) is

\[
m(S) = 2\delta^{-2} S^{-2\beta-2} \int_0^S Y^{-2\beta-2} \mu Y dY
\]

(A-15)

Computing \( \int_0^S 2\delta^{-2} Y^{-2\beta-2} \mu Y dY \):

\[
\int_0^S 2\delta^{-2} Y^{-2\beta-2} \mu Y dY = 2\delta^{-2} \mu \int_0^S Y^{-2\beta-1} dY = 2\delta^{-2} \mu \left[ \frac{Y^{-2\beta}}{-2\beta} \right]_0^S = -\frac{\mu}{\beta \delta^2} S^{-2\beta} = -\epsilon \frac{|\mu|}{|\beta| \delta^2} S^{-2\beta}
\]

(A-16)

So replacing the last result in (A-15):

\[
m(S) = \frac{2}{\delta^2 S^{2\beta+2}} e^{-\epsilon x(S)},
\]

(A-17)

where \( x(S) \) and \( \epsilon \) are respectively defined in (17) and (19).

To compute the scale density it can be used (A-16):

\( \varrho(S) \) is the scale density defined in Borodin and Salminen (2002, p.17) as

\[
\varrho(S) = \exp\left(-\int_0^S 2\delta^{-2} Y^{-2\beta-2} \mu Y dY\right).
\]

(A-18)

Using (A-16)

\[
\varrho(S) = e^{\epsilon x(S)}
\]

(A-19)

where \( \epsilon \) and \( x(S) \) are respectively defined in (19), and (17).

From Borodin and Salminen (2002, p.19), the Wronskian (wro) with respect to scale density - \( \varrho(S) \) - is

\[
wro \varrho \times \varrho(S) = \psi'_{\lambda}(S) \varrho_{\lambda}(S) - \psi_{\lambda}(S) \varrho'_{\lambda}(S)
\]

(A-20)

where \( \varrho(S) \) is defined in (A-19)

1. \( \varrho \) if \( \beta < 0 \) and \( \mu \neq 0 \)
Using (15) and (16),

\[ \phi_\lambda(S) = S^{\beta+\frac{1}{2}} e^{\frac{\epsilon}{2} x(S)} W_{k,m}(x(S)), \]

\[ \psi_\lambda(S) = S^{\beta+\frac{1}{2}} e^{\frac{\epsilon}{2} x(S)} M_{k,m}(x(S)), \]

where \( x(S) \) and \( \epsilon \) are defined in (17) and (19). Consequently,

\[ x(S) = -\frac{\mu}{\sigma^2} S^{-2\beta} \Rightarrow x'(S) = \frac{2\mu}{\sigma^2} S^{-2\beta - 1} \]

\[ \psi'_\lambda(S) = (\beta + \frac{1}{2}) S^{\beta+\frac{1}{2}} e^{\frac{\epsilon}{2} x(S)} M_{k,m}(x(S)) + S^{-\beta-\frac{1}{2}} e^{\frac{\epsilon}{2} x(S)} \left( \frac{\epsilon}{\sigma^2} M_{k,m}(x(S)) + \frac{2}{\sigma^2} M'_{k,m}(x(S)) \right) \]

\[ \phi'_\lambda(S) = (\beta + \frac{1}{2}) S^{\beta+\frac{1}{2}} e^{\frac{\epsilon}{2} x(S)} W_{k,m}(x(S)) + S^{-\beta-\frac{1}{2}} e^{\frac{\epsilon}{2} x(S)} \left( \frac{\epsilon}{\sigma^2} W_{k,m}(x(S)) + \frac{2}{\sigma^2} W'_{k,m}(x(S)) \right) \]

So,

\[ \psi'_\lambda(S) \phi_\lambda(S) = (\beta + \frac{1}{2}) S^{2\beta} e^{\epsilon x(S)} M_{k,m}(x(S)) W_{k,m}(x(S)) + e^{\epsilon x(S)} \frac{\epsilon}{\sigma^2} M_{k,m}(x(S)) W_{k,m}(x(S)) \]

\[ + e^{\epsilon x(S)} \frac{2}{\sigma^2} M'_{k,m}(x(S)) W_{k,m}(x(S)) \]

\[ \psi_\lambda(S) \phi'_\lambda(S) = (\beta + \frac{1}{2}) S^{2\beta} e^{\epsilon x(S)} M_{k,m}(x(S)) W_{k,m}(x(S)) + e^{\epsilon x(S)} \frac{\epsilon}{\sigma^2} M_{k,m}(x(S)) W_{k,m}(x(S)) \]

\[ + e^{\epsilon x(S)} \frac{2}{\sigma^2} M_{k,m}(x(S)) W_{k,m}(x(S)) \]

Replacing (A-21) and (A-22) in equation (A-20), and eliminating the symmetric expressions,

\[ \text{wro} \times g(S) = e^{\epsilon x(S)} \frac{2}{\sigma^2} M'_{k,m}(x(S)) W_{k,m}(x(S)) - e^{\epsilon x(S)} \frac{2}{\sigma^2} M_{k,m}(x(S)) W'_{k,m}(x(S)) \]

\[ = -e^{\epsilon x(S)} \frac{2}{\sigma^2} \left( M_{k,m}(x(S)) W'_{k,m}(x(S)) - W_{k,m}(x(S)) M'_{k,m}(x(S)) \right) \]

Using the result 2.4.27 of Slater (1960, p 2 and p 26)

\[ \text{wro}_\lambda \times g(S) = -e^{\epsilon x(S)} \frac{2}{\sigma^2} \left( \frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)} \right) \]  

\[ (A-24) \]

where \( \Gamma(x) \) is the Euler Gamma function. From equation (A-19), \( g(S) = e^{\epsilon x(S)} \), where \( x(S) \) is defined in (17). Replacing it in equation (A-24) we obtain

\[ \text{wro}_\lambda = \frac{2}{\sigma^2} \left( \frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)} \right) \]  

\[ (A-25) \]
2. If $\beta > 0$ and $\mu \neq 0$

Using (15) and (16),

\[ \phi_\lambda(S) = S^{\beta + \frac{1}{2}} e^{\frac{1}{2} x(S)} M_{k,m}(x(S)), \]

\[ \psi_\lambda(S) = S^{\beta + \frac{1}{2}} e^{\frac{1}{2} x(S)} W_{k,m}(x(S)), \]

where $x(S)$ and $\epsilon$ are defined in (17) and (19). Consequently,

\[ x(S) = \frac{|\mu|}{\beta^2} S^{-2\beta} \Rightarrow x'(S) = -\frac{2|\mu|}{\beta^2} S^{-2\beta - 1} \]

\[ \psi_\lambda'(S) = (\beta + \frac{1}{2}) S^{\beta - \frac{3}{2}} e^{\frac{1}{2} x(S)} W_{k,m}(x(S)) + S^{-\beta - \frac{1}{2}} e^{\frac{1}{2} x(S)} \left( -\frac{\epsilon}{\beta^2} M_{k,m}(x(S)) - \frac{2|\mu|}{\beta^2} W_{k,m}'(x(S)) \right) \]

\[ \phi_\lambda'(S) = (\beta + \frac{1}{2}) S^{\beta - \frac{3}{2}} e^{\frac{1}{2} x(S)} M_{k,m}(x(S)) + S^{-\beta - \frac{1}{2}} e^{\frac{1}{2} x(S)} \left( -\frac{\epsilon}{\beta^2} W_{k,m}(x(S)) - \frac{2|\mu|}{\beta^2} M_{k,m}'(x(S)) \right) \]

So,

\[ \psi_\lambda'(S) \phi_\lambda(S) = (\beta + \frac{1}{2}) S^{2\beta} e^{x(S)} W_{k,m}(x(S)) M_{k,m}(x(S)) - e^{x(S)} \frac{\epsilon}{\beta^2} W_{k,m}(x(S)) M_{k,m}(A-29) \]

\[ -e^{x(S)} \frac{2|\mu|}{\beta^2} W_{k,m}'(x(S)) M_{k,m}(x(S)) \]

\[ \psi_\lambda(S) \phi_\lambda'(S) = (\beta + \frac{1}{2}) S^{2\beta} e^{x(S)} W_{k,m}(x(S)) M_{k,m}(x(S)) - e^{x(S)} \frac{\epsilon}{\beta^2} W_{k,m}(x(S)) M_{k,m}(A-27) \]

\[ -e^{x(S)} \frac{2|\mu|}{\beta^2} W_{k,m}(x(S)) M_{k,m}'(x(S)) \]

Replacing (A-26) and (A-27) in equation (A-20), and eliminating the symmetric expressions,

\[ wro_\lambda \times g(S) = -e^{x(S)} \frac{2|\mu|}{\beta^2} W_{k,m}'(x(S)) M_{k,m}(x(S)) + e^{x(S)} \frac{2|\mu|}{\beta^2} W_{k,m}(x(S)) M_{k,m}'(A-28) \]

\[ = -e^{x(S)} \frac{2|\mu|}{\beta^2} (M_{k,m}(x(S)) W_{k,m}'(x(S)) - W_{k,m}(x(S)) M_{k,m}'(x(S))) \]

Using the result 2.4.27 of Slater (1960, p 2 and p 26)

\[ wro_\lambda \times g(S) = -e^{x(S)} \frac{2|\mu|}{\beta^2} \left( -\frac{\Gamma(1 + 2m)}{\Gamma(\frac{1}{2} + m - k)} \right) \quad (A-29) \]

From equation (A-19), $g(S) = e^{x(S)}$, where $x(S)$ is defined in (17). Replacing (A-19) in (A-29) it follows that

\[ wro_\lambda = \frac{2|\mu|}{\beta^2} \left( \frac{\Gamma(1 + 2m)}{\Gamma(\frac{1}{2} + m - k)} \right) \quad (A-30) \]
3. If $\beta < 0$ and $\mu = 0$

Using (15) and (16),

$$\phi_{\lambda}(S) = S^{\frac{1}{2}}K_{\nu}(\sqrt{2\lambda z}(S)),$$

$$\psi_{\lambda}(S) = S^{\frac{1}{2}}I_{\nu}(\sqrt{2\lambda z}(S)),$$

where $z$ is defined in (18) as $z(S) = -\frac{1}{\delta z}S^{-\beta}$. Making $\sqrt{2\lambda z}(S) = w \Rightarrow w = -\frac{\sqrt{2\lambda}}{\delta z}S^{-\beta}$, then

$$\frac{\partial w}{\partial S} = \frac{\sqrt{2\lambda}}{\delta z}S^{-\beta - 1}.$$ Hence,

$$\phi_{\lambda}(S) = S^{\frac{1}{2}}K_{\nu}(\sqrt{2\lambda z}(S)),$$

and

$$\psi_{\lambda}(S) = S^{\frac{1}{2}}I_{\nu}(\sqrt{2\lambda z}(S)).$$

So,

$$\psi'_{\lambda}(S) = \frac{1}{2}S^{-\frac{1}{2}}I_{\nu}(w) + S^{-\beta - \frac{1}{2}} \frac{\sqrt{2\lambda}}{\delta} I'_{\nu}(w)$$

$$\phi'_{\lambda}(S) = \frac{1}{2}S^{-\frac{1}{2}}K_{\nu}(w) + S^{-\beta - \frac{1}{2}} \frac{\sqrt{2\lambda}}{\delta} K'_{\nu}(w)$$

and,

$$\psi'_{\lambda}(S)\phi_{\lambda}(S) = \frac{1}{2}I_{\nu}(w)K_{\nu}(w) + S^{-\beta} \frac{\sqrt{2\lambda}}{\delta} I'_{\nu}(w)K_{\nu}(w) \quad (A-31)$$

$$\psi_{\lambda}(S)\phi'_{\lambda}(S) = \frac{1}{2}I_{\nu}(w)K_{\nu}(w) + S^{-\beta} \frac{\sqrt{2\lambda}}{\delta} I_{\nu}(w)K'_{\nu}(w) \quad (A-32)$$

Replacing (A-31) and (A-32) in equation (A-20), and eliminating the symmetric expressions,

$$w\rho_{\lambda} = g(S) = S^{-\beta} \frac{\sqrt{2\lambda}}{\delta} \left[ K_{\nu}(w)I_{\nu}'(w) - K'_{\nu}(w)I_{\nu}(w) \right]$$

Using the result 9.6.15 of Abramowitz and Stegun (1972, p. 375),

$$w\rho_{\lambda} = g(S) = S^{-\beta} \frac{\sqrt{2\lambda}}{\delta} \frac{1}{w} \quad (A-33)$$

Since $\mu = 0$, then $g(S) = 1$. Replacing $g(S)$ and $w$ in (A-33) it follows that

$$w\rho_{\lambda} = S^{-\beta} \frac{\sqrt{2\lambda}}{\delta} \left( \frac{-\frac{\sqrt{2\lambda}}{\delta S^{-\beta}}}{S^{-\beta}} \right)^{-1} = -\beta \quad (A-34)$$

4. If $\beta > 0$ and $\mu = 0$
Using (15) and (16),

\[ \phi_\lambda(S) = S^{\frac{1}{2}} I_\nu(\sqrt{2\lambda} z(S)), \]

\[ \psi_\lambda(S) = S^{\frac{1}{2}} K_\nu(\sqrt{2\lambda} z(S)), \]

where \( z(S) \) is defined in (18) as \( z(S) = \frac{1}{\delta^2} S^{-\beta} \). Making \( \sqrt{2\lambda} z(S) = w \Rightarrow w = \frac{\sqrt{2\lambda}}{\delta^2} S^{-\beta} \), then

\[ \frac{\partial w}{\partial \beta} = -\frac{\sqrt{2\lambda}}{\delta^2} S^{-\beta - 1}. \]

Hence,

\[ \psi'_\lambda(S) = S^{-\frac{1}{2}} K_\nu(w) - S^{-\beta - \frac{1}{2}} \frac{\sqrt{2\lambda}}{\delta} K'_\nu(w) \]

\[ \phi'_\lambda(S) = S^{-\frac{1}{2}} I_\nu(w) - S^{-\beta - \frac{1}{2}} \frac{\sqrt{2\lambda}}{\delta} I'_\nu(w) \]

So,

\[ \psi'_\lambda(S) \phi_\lambda(S) = \frac{1}{2} S^{-\frac{1}{2}} K_\nu(w) I_\nu(w) - S^{-\beta - \frac{1}{2}} \frac{\sqrt{2\lambda}}{\delta} K'_\nu(w) I_\nu(w) \]

(A-35)

\[ \psi_\lambda(S) \phi'_\lambda(S) = \frac{1}{2} S^{-\frac{1}{2}} K_\nu(w) I_\nu(w) - S^{-\beta - \frac{1}{2}} \frac{\sqrt{2\lambda}}{\delta} K_\nu(w) I'_\nu(w) \]

(A-36)

Replacing (A-35) and (A-36) in equation (A-20), and eliminating the symmetric expressions,

\[ w \rho_\lambda \times \phi(S) = S^{-\beta} \frac{\sqrt{2\lambda}}{\delta} \left[ K_\nu(w) I'_\nu(w) - I_\nu(w) K'_\nu(w) \right]. \]

Using the result 9.6.15 of Abramowitz and Stegun (1972, p 375), then

\[ w \rho_\lambda \times \phi(S) = S^{-\beta} \frac{\sqrt{2\lambda}}{\delta} \frac{1}{w} \]

(A-37)

Using the result 9.6.15 of Abramowitz and Stegun (1972, p 375), then

Since \( \mu = 0 \), then \( \phi(S) = 1 \). Replacing \( \phi(S) \) and \( w \) in (A-37) it follows that

\[ w \rho_\lambda = S^{-\beta} \frac{\sqrt{2\lambda}}{\delta} \left( \frac{\sqrt{2\lambda}}{\delta^2} S^{-\beta} \right)^{-1} = \beta \]

(A-38)

As a result, from (A-25), (A-30), (A-34) and (A-38), we conclude that

\[ w \rho_\lambda = \begin{cases} \frac{2|m|}{\delta^2} \left( \frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)} \right) & \text{if } \mu \neq 0 \\ |\beta| & \text{if } \mu = 0 \end{cases} \]
B Appendix - Closed-Form for Integrals \( I_\lambda(K, A, B) \) and \( J_\lambda(K, A, B) \)

A. \( I_\lambda(K, A, B) = \int_A^B (Y - K)\psi_\lambda m(Y)dY \)

1. **Case \( \mu \neq 0 \)**

\( m(Y) \) was calculated in equation (A-17) of Appendix A.

A - 1. \( I_\lambda(K, A, B) = \int_A^B (Y - K)\psi_\lambda(Y)m(Y)dY \), with \( \mu \neq 0 \)

A - 1.1 **If \( \mu > 0 \) and \( \beta < 0 \)**

In this case

\[ \epsilon = -1, \]

\( m(Y) = 2\delta^{-2}Y^{-2\beta-2}e^{x(Y)}, \]

\( \psi_\lambda(Y) = Y^{\beta+\frac{1}{2}}e^{-\frac{1}{2}x(Y)}M_{k,m}(x(Y)), \]

\( x(Y) = \frac{\mu}{(\beta)\delta^2}Y^{-2\beta}, \]

\( m = -\frac{1}{4\beta}, \) where \( \epsilon, m(Y), x(Y), k \) and \( m \) are defined in (19), (A-17), (17), (21) and (20), respectively. Hence,

\[
I_\lambda(K, A, B) = \int_A^B (Y - K)Y^{\beta+\frac{1}{2}}e^{-\frac{1}{2}x(Y)}M_{k,m}(x(Y))2\delta^{-2}Y^{-2\beta-2}e^{x(Y)}dY \quad (B-1)
\]

\[
= 2\delta^{-2} \left[ \int_A^B Y^{-\beta-\frac{1}{2}}e^{x(Y)}M_{k,m}(x(Y))dY \right] -2K\delta^{-2} \left[ \int_A^B Y^{-\beta-\frac{3}{2}}e^{x(Y)}M_{k,m}(x(Y))dY \right]
\]

Let us consider the first integral \( 2\delta^{-2} \left[ \int_A^B Y^{-\beta-\frac{1}{2}}e^{x(Y)}M_{k,m}(x(Y))dY \right] \). Making the change of variable,

\[ x(Y) = \frac{\mu}{(-\beta)\delta^2}Y^{-2\beta} \Rightarrow Y = \left( \frac{\beta\delta^2}{\mu}x(Y) \right)^{-\frac{1}{\beta}}, \]
\[
\frac{\partial Y}{\partial x} = \frac{\delta^2}{2\mu} \left( -\frac{\beta \delta^2}{\mu} x \right)^{-\frac{1}{2\delta} - 1}
\]

and then,
\[
2\delta^{-2} \left[ \int_A^B Y^{-\beta - \frac{1}{2}} e^{\frac{x(Y)}{2}} M_{k,m}(x(Y)) dY \right]
\]
\[
= 2\delta^{-2} \left\{ \int_{x(A)}^{x(B)} \left[ \left( -\frac{\beta \delta^2}{\mu} x \right)^{-\frac{1}{2\delta} - \frac{1}{2}} \right] e^{\frac{x}{2}} M_{k,m}(x) \frac{\delta^2}{2\mu} \left( -\frac{\beta \delta^2}{\mu} x \right)^{-\frac{1}{2\delta} - 1} dx \right\}
\]
\[
= \frac{1}{\mu} \left( -\frac{\beta \delta^2}{\mu} \right)^{-\frac{1}{2\delta} - \frac{1}{2}} \int_{x(A)}^{x(B)} x^{m - \frac{1}{2}} e^{\frac{x}{2}} M_{k,m}(x) dx
\]

Applying the result of equation 2.4.7 of Slater (1960, p 24), the last equation become:
\[
2\delta^{-2} \left[ \int_A^B Y^{-\beta - \frac{1}{2}} e^{\frac{x(Y)}{2}} M_{k,m}(x(Y)) dY \right]
\]
\[
= \frac{1}{\mu} \left( -\frac{\beta \delta^2}{\mu} \right)^{-\frac{1}{2\delta} - \frac{1}{2}} \left[ \frac{1}{2m + 1} e^{\frac{x}{2}} M_{k+\frac{1}{2},m+\frac{1}{2}}(x) \right]_{x(A)}^{x(B)}
\]
\[
= \frac{1}{\mu} \left( -\frac{\beta \delta^2}{\mu} \right)^{-\frac{1}{2\delta} - \frac{1}{2}} \left[ \frac{1}{2m + 1} e^{\frac{x}{2}} \left( \frac{\mu}{(-\beta) \delta^2} Y^{-2\beta} \right)^{-\frac{1}{2\delta}} M_{k+\frac{1}{2},m+\frac{1}{2}}(x) \right]_{x(A)}^{x(B)}
\]
\[
= \frac{1}{\delta \sqrt{\beta \mu}} \left[ \frac{Y^{1/2}}{2m + 1} e^{\frac{y}{2}} M_{k+\frac{1}{2},m+\frac{1}{2}}(x) \right]_{x(A)}^{x(B)}
\]

but as \( x(Y) \) is a function of \( Y \), the last equation can be rewritten as
\[
2\delta^{-2} \left[ \int_A^B Y^{-\beta - \frac{1}{2}} e^{\frac{x(Y)}{2}} M_{k,m}(x(Y)) dY \right] = \frac{1}{\delta \sqrt{\beta \mu}} \left[ \frac{Y^{1/2}}{2m + 1} e^{\frac{x(Y)}{2}} M_{k+\frac{1}{2},m+\frac{1}{2}}(x(Y)) \right]_{Y=A}^{Y=B}
\]

Now we consider the second integral \( 2K\delta^{-2} \left[ \int_A^B Y^{-\beta - \frac{3}{2}} e^{\frac{x(Y)}{2}} M_{k,m}(x(Y)) dY \right] \):

Making the same change of variable that it was done for the first integral:
\[
2K\delta^{-2} \left[ \int_A^B Y^{-\beta - \frac{3}{2}} e^{\frac{x(Y)}{2}} M_{k,m}(x(Y)) dY \right]
\]
\[
= 2K\delta^{-2} \int_{x(A)}^{x(B)} \left[ \left( -\frac{\beta \delta^2}{\mu} x \right)^{-\frac{1}{2\delta} - \frac{3}{2}} \right]^{-\beta - \frac{3}{2}} e^{\frac{x}{2}} M_{k,m}(x) \frac{\delta^2}{2\mu} \left( -\frac{\beta \delta^2}{\mu} x \right)^{-\frac{1}{2\delta} - 1} dx
\]
\[
= \frac{K}{\mu} \left( -\frac{\beta \delta^2}{\mu} \right)^{-\frac{1}{2\delta} - \frac{1}{2}} \int_{x(A)}^{x(B)} x^{m - \frac{1}{2}} e^{\frac{x}{2}} M_{k,m}(x) dx
\]

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Applying equation 2.4.8 of Slater (1960, p 24) to the last equation:

\[
2K\delta^{-2} \left[ \int_A^B Y^{-\beta-\frac{3}{2}} e^{-\frac{x(Y)}{2}} M_{k,m}(x(Y))dY \right]
\]

\[
= \frac{K}{\mu} \left( -\frac{\beta\delta^2}{\mu} \right)^{\frac{1}{2} - \frac{1}{2}} \left[ \frac{2m}{m-k} e^{\frac{2}{\mu}} x^{-m} M_{k+\frac{1}{2},m-\frac{1}{2}}(x) \right]_{x(A)}^{x(B)}
\]

\[
= \frac{1}{\mu} \left( -\frac{\beta\delta^2}{\mu} \right)^{\frac{1}{2} - \frac{1}{2}} \left[ \frac{2mK}{m-k} e^{\frac{2}{\mu}} \left( \frac{\mu}{(\beta)\delta^{2}} \right)^{\frac{1}{2}} Y^{-2\beta-2} M_{k+\frac{1}{2},m-\frac{1}{2}}(x) \right]_{x(A)}^{x(B)}
\]

\[
= \frac{1}{\delta\sqrt{\mu\beta}} \left[ \frac{2mKY^{-\frac{1}{2}}}{m-k} e^{\frac{2}{\mu}} M_{k+\frac{1}{2},m-\frac{1}{2}}(x) \right]_{Y=A}^{Y=B}
\]

Replacing (B-2) and (B-3) in (B-1):

\[
I_\lambda(K, A, B) = \frac{1}{\delta\sqrt{\mu\beta}} \left[ \frac{Y^{\frac{1}{2}}}{2m+1} e^{\frac{x(Y)}{2}} M_{k+\frac{1}{2},m+\frac{1}{2}}(x(Y)) \right]_{Y=A}^{Y=B}
\]

A - 1.2  \( \mu < 0 \) and \( \beta < 0 \)

In this case

\[
\epsilon = 1,
\]

\[
m(Y) = 2\delta^{-2} Y^{-2\beta-2} e^{-x(Y)},
\]

\[
\psi_\lambda(Y) = Y^{\frac{\beta+1}{2}} e^{\frac{1}{2} x(Y)} M_{k,m}(x(Y)),
\]

\[
x(Y) = \frac{\mu}{\beta\delta^2} Y^{-2\beta},
\]

\[
m = -\frac{1}{4\beta}, \text{ where } \epsilon, m(Y), x, k \text{ and } m \text{ are defined in (19), (A-17), (17), (21) and (20), respectively. Hence,}
\]

\[
I_\lambda(K, A, B) = \int_A^B (Y - K) Y^{\beta+\frac{1}{2}} e^{\frac{1}{2} x(Y)} M_{k,m}(x(Y)) 2\delta^{-2} Y^{-2\beta-2} e^{-x(Y)} dY
\]

\[
= 2\delta^{-2} \left[ \int_A^B Y^{-\beta-\frac{3}{2}} e^{-\frac{x(Y)}{2}} M_{k,m}(x(Y)) dY \right]
\]

\[
-2K\delta^{-2} \left[ \int_A^B Y^{-\beta-\frac{3}{2}} e^{-\frac{x(Y)}{2}} M_{k,m}(x(Y)) dY \right]
\]
Consider the first integral $2\delta^{-2} \left[ \int_{A}^{B} Y^{-\beta - \frac{1}{2}} e^{-\frac{z(Y)}{2}} M_{k,m}(x(Y)) dY \right]$:

Doing the same change of variable that was done in case A - 1.1 for the first integral, and then applying equation 2.4.9 of Slater (1960, p 24):

\[
2\delta^{-2} \left[ \int_{A}^{B} Y^{-\beta - \frac{1}{2}} e^{-\frac{z(Y)}{2}} M_{k,m}(x(Y)) dY \right] = \frac{1}{\delta \sqrt{|\beta \mu|}} \left[ \frac{Y^{\frac{1}{2}}}{2m + 1} e^{-\frac{2}{4} M_{k,-\frac{1}{2},m + \frac{1}{2}}(x)} \right]_{Y=A}^{Y=B} \quad (B-7)
\]

For the second integral $2K\delta^{-2} \left[ \int_{A}^{B} Y^{-\beta - \frac{3}{2}} e^{-\frac{z(Y)}{2}} M_{k,m}(x(Y)) dY \right]$ we can apply the same change of variable that was done in case A - 1.1 for the second integral, and next applying equation 2.4.10 of Slater (1960, p 24):

\[
2K\delta^{-2} \left[ \int_{A}^{B} Y^{-\beta - \frac{3}{2}} e^{-\frac{z(Y)}{2}} M_{k,m}(x(Y)) dY \right] = \frac{1}{\delta \sqrt{|\beta \mu|}} \left[ \frac{2mKY^{\frac{1}{2}}}{m + k - \frac{7}{2}} e^{-\frac{2}{4} M_{k,-\frac{1}{2},m - \frac{1}{2}}(x(Y))} \right]_{Y=A}^{Y=B} \quad (B-8)
\]

Replacing (B-7) and (B-8) in (B-5):

\[
I_{\lambda}(K, A, B) = \frac{1}{\delta \sqrt{|\beta \mu|}} \left[ \frac{Y^{\frac{1}{2}}}{2m + 1} e^{-\frac{2}{4} M_{k,-\frac{1}{2},m + \frac{1}{2}}(x(Y))} \right]_{Y=A}^{Y=B} - \frac{2mKY^{\frac{1}{2}}}{m + k - \frac{7}{2}} e^{-\frac{2}{4} M_{k,-\frac{1}{2},m - \frac{1}{2}}(x(Y))} \quad (B-9)
\]

A - 1.3 \text{ If } \mu > 0 \text{ and } \beta > 0

In this case

\[
\varepsilon = 1,
\]

\[
m(Y) = 2\delta^{-2} Y^{-2\beta - 2} e^{-x(Y)},
\]

\[
\psi_{\lambda}(Y) = Y^{\beta + \frac{1}{2}} e^{\frac{1}{2} x(Y)} W_{k,m}(x(Y)),
\]

\[
x(Y) = \frac{\mu}{\beta \varepsilon} Y^{-2\beta},
\]
\[ m = \frac{1}{4^3}, \text{ where } \epsilon, m(Y), x, k \text{ and } m \text{ are defined in (19), (A-17), (17), (21) and (20), respectively. Hence,} \]
\[ I_\lambda(K, A, B) = \int_A^B (Y - K)Y^{\beta + \frac{1}{2}}e^{\frac{1}{2}x(Y)}W_{k, m}(x(Y))2\delta^{-2}Y^{-2\beta - 2}e^{-x(Y)}dY \] (B-10)
\[ = 2\delta^{-2} \left[ \int_A^B Y^{-\beta - \frac{1}{2}}e^{-\frac{x(Y)}{2}}W_{k, m}(x(Y))dY \right] -2K\delta^{-2} \left[ \int_A^B Y^{-\beta - \frac{3}{2}}e^{-\frac{3x(Y)}{2}}W_{k, m}(x(Y))dY \right] \]

Making the same changes of variables before and applying equations 2.4.22 and 2.4.23 of Slater (1960, p 25) to the first and second integrals respectively, in equation (B-10), we find:
\[ I_\lambda(K, A, B) = \frac{1}{\delta \sqrt{|\beta \mu|}} \left[ Y^{\frac{1}{2}}e^{-\frac{x(Y)}{2}}W_{k - \frac{1}{2}, m - \frac{1}{2}}(x(Y)) + KY^{-\frac{1}{2}}e^{-\frac{3x(Y)}{2}}W_{k - \frac{1}{2}, m + \frac{1}{2}}(x(Y)) \right]_{Y=A}^{Y=B} \] (B-11)

\[ \text{A - 1.4 } \mu < 0 \text{ and } \beta > 0 \]

In this case
\[ \epsilon = -1, \]
\[ m(Y) = 2\delta^{-2}Y^{-2\beta - 2}e^{x(Y)}, \]
\[ \psi_\lambda(Y) = Y^{\beta + \frac{1}{2}}e^{-\frac{1}{2}x(Y)}W_{k, m}(x(Y)), \]
\[ x(Y) = -\frac{\mu}{\beta \delta}Y^{-2\beta} \]
\[ m = \frac{1}{4^3} \text{ where } \epsilon, m(Y), x, k \text{ and } m \text{ are defined in (19), (A-17), (17), (21) and (20), respectively. Hence,} \]
\[ I_\lambda(K, A, B) = \int_A^B (Y - K)Y^{\beta + \frac{1}{2}}e^{-\frac{1}{2}x(Y)}W_{k, m}(x(Y))2\delta^{-2}Y^{-2\beta - 2}e^{x(Y)}dY \]

Making the same change of variables as before, and applying equations 2.4.19 and 2.4.20 of Slater (1960, p 25) to the first and second integrals, respectively, we find:
\[ I_\lambda(K, A, B) = \frac{1}{\delta \sqrt{|\beta \mu|}} \left[ \frac{Y^{\frac{1}{2}}}{k - m + \frac{1}{2}}e^{\frac{x(Y)}{2}}W_{k + \frac{1}{2}, m - \frac{1}{2}}(x(Y)) \right]_{Y=A}^{Y=B} \] (B-12)
\[ -\frac{KY^{\frac{1}{2}}}{k + m + \frac{1}{2}}e^{-\frac{3x(Y)}{2}}W_{k + \frac{1}{2}, m + \frac{1}{2}}(x(Y)) \] (B-13)
B.

\[ J_\lambda(K, A, B) = \int_A^B (Y - K) \phi_\lambda(Y) m(Y) dY \]

B - 1. \( J_\lambda(K, A, B) = \int_A^B (Y - K) \phi_\lambda(Y) m(Y) dY \), with \( \mu \neq 0 \)

The closed-form solution for the integral \( J_\lambda(K, A, B) \) is obtained in the same way as for \( I_\lambda(K, A, B) \), with \( \mu \neq 0 \)

B - 1.1 \[ \text{If } \mu > 0 \text{ and } \beta < 0 \]

\[ J_\lambda(K, A, B) = \frac{1}{\delta \sqrt{||\beta \mu||}} \left[ Y^{\frac{1}{2}} e^{\frac{-x(Y)}{2}} W_{k+\frac{1}{2}, m+\frac{1}{2}}(x(Y)) - \frac{KY^{-\frac{1}{2}}}{k - m + \frac{1}{2}} e^{\frac{x(Y)}{2}} W_{k+\frac{1}{2}, m-\frac{1}{2}}(x(Y)) \right]_{Y=B}^{Y=A} \]

B - 1.2 \[ \text{If } \mu < 0 \text{ and } \beta < 0 \]

\[ J_\lambda(K, A, B) = \frac{1}{\delta \sqrt{||\beta \mu||}} \left[ Y^{\frac{1}{2}} e^{\frac{-x(Y)}{2}} W_{k-\frac{1}{2}, m+\frac{1}{2}}(x(Y)) + KY^{-\frac{1}{2}} e^{\frac{x(Y)}{2}} W_{k-\frac{1}{2}, m-\frac{1}{2}}(x(Y)) \right]_{Y=B}^{Y=A} \]

B - 1.3 \[ \text{If } \mu > 0 \text{ and } \beta > 0 \]

\[ J_\lambda(K, A, B) = \frac{1}{\delta \sqrt{||\beta \mu||}} \left[ \frac{2mY^{\frac{1}{2}}}{k - m + \frac{1}{2}} e^{\frac{x(Y)}{2}} W_{k-\frac{1}{2}, m-\frac{1}{2}}(x(Y)) + \frac{KY^{-\frac{1}{2}}}{k + m + \frac{1}{2}} e^{\frac{x(Y)}{2}} W_{k+\frac{1}{2}, m+\frac{1}{2}}(x(Y)) \right]_{Y=B}^{Y=A} \]

B - 1.4 \[ \text{If } \mu < 0 \text{ and } \beta > 0 \]

\[ J_\lambda(K, A, B) = \frac{1}{\delta \sqrt{||\beta \mu||}} \left[ \frac{2mY^{\frac{1}{2}}}{k - m + \frac{1}{2}} e^{\frac{x(Y)}{2}} W_{k+\frac{1}{2}, m-\frac{1}{2}}(x(Y)) + \frac{KY^{-\frac{1}{2}}}{k + m + \frac{1}{2}} e^{\frac{x(Y)}{2}} W_{k+\frac{1}{2}, m+\frac{1}{2}}(x(Y)) \right]_{Y=B}^{Y=A} \]

2. \[ \text{Case } \mu = 0 \]

Considering \( \mu = 0 \) in equation (A-18), then \( g(Y) = \exp(0) = 1 \). So, for this case,

\[ m(Y) = 2\delta^{-2}Y^{-2\beta-2} \] (B-18)
\[ I_\lambda(K, A, B) = \int_A^B (Y - K) \psi_\lambda(Y) m(Y) dY, \text{ with } \mu = 0 \]

\section*{A - 2.1 If } \beta < 0

\( m(Y) = 2\delta^{-2}Y^{-2\beta-2}, \)

\[ \psi_\lambda(Y) = Y^{\frac{1}{2}} I_\nu(\sqrt{2\lambda}z(Y)), \]

\[ \nu = -\frac{1}{2\beta} \]

\[ z(Y) = -\frac{1}{\delta \beta} Y^{-\beta}, \]

\( m = -\frac{1}{4\beta}, \) where \( m(Y), \nu, z, m \) and \( \psi_\lambda \) are defined in \((B-18), (22), (18), (20), \) and \((15),\) respectively. So,

\[ I_\lambda(K, A, B) = \int_A^B (Y - K)Y^{\frac{1}{2}} I_\nu(\sqrt{2\lambda}z(Y))2\delta^{-2}Y^{-2\beta-2}dY \quad (B-19) \]

\[ = 2\delta^{-2} \int_A^B Y^{-2\beta-\frac{1}{2}} I_\nu(\sqrt{2\lambda}z(Y))dY - 2\delta^{-2} K \int_A^B Y^{-2\beta-\frac{1}{2}} I_\nu(\sqrt{2\lambda}z(Y))dY \]

Let us consider the first integral \( 2\delta^{-2} \int_A^B Y^{-2\beta-\frac{1}{2}} I_\nu(\sqrt{2\lambda}z(Y))dY. \) Making the change of variable

\[ w = \sqrt{2\lambda}z(Y) \Rightarrow w = \sqrt{2\lambda}(-\frac{1}{\delta \beta} Y^{-\beta}), \]

then,

\[ Y = \left(-\frac{\delta \beta}{\sqrt{2\lambda}} w\right)^{-\frac{1}{\beta}}, \]

and

\[ \frac{\partial Y}{\partial w} = \frac{\delta}{\sqrt{2\lambda} \sqrt{2\lambda}} \left(-\frac{\delta \beta}{\sqrt{2\lambda}} w\right)^{-\frac{1}{\beta}-1}. \]

So,

\[ 2\delta^{-2} \int_A^B Y^{-2\beta-\frac{1}{2}} I_\nu(\sqrt{2\lambda}z(Y))dY \]

\[ = 2\delta^{-2} \int_{w(A)}^{w(B)} \left[ \left(-\frac{\delta \beta}{\sqrt{2\lambda}} w\right)^{-\frac{1}{\beta}} \right]^{-2\beta-\frac{1}{2}} I_\nu(w) \frac{\delta}{\sqrt{2\lambda}} \left(-\frac{\delta \beta}{\sqrt{2\lambda}} w\right)^{-\frac{1}{\beta}-1}dw \]

\[ = \frac{2}{\delta \sqrt{2\lambda}} \left(-\frac{\delta \beta}{\sqrt{2\lambda}}\right)^{1-\frac{1}{\beta}} \int_{w(A)}^{w(B)} w^{1+\nu} I_\nu(w)dw \]
Now, applying equation 5 of Prudnikov et al (1986, p 46) to the last equation, then:

\[
2\delta^{-2} \int_{A}^{B} Y^{-2\beta - \frac{1}{2}} I_{\nu}(\sqrt{2\lambda}z(Y))dY \tag{B-20}
\]

\[
= \frac{2}{\delta \sqrt{2\lambda}} \left(-\frac{\delta \beta}{\sqrt{2\lambda}}\right)^{1-\frac{1}{2\beta}} \left[\left(-\frac{1}{\delta \beta}\right) Y^{-\beta}\right]^{1-\frac{1}{2\beta}} I_{\nu+1}(w) \right]^{w(B)}_{w(A)}
\]

\[
= \frac{2}{\delta \sqrt{2\lambda}} \left[Y^{-\beta + \frac{1}{2}} I_{\nu+1}(\sqrt{2\lambda}z(Y))\right]^{Y=B}_{Y=A}
\]

For the second integral \(2\delta^{-2} K \int_{A}^{B} Y^{-2\beta - \frac{3}{2}} I_{\nu}(\sqrt{2\lambda}z(Y))dY\), and making the same change of variable as in the first integral:

\[
2\delta^{-2} K \int_{A}^{B} Y^{-2\beta - \frac{3}{2}} I_{\nu}(\sqrt{2\lambda}z(Y))dY
\]

\[
= 2\delta^{-2} K \int_{w(A)}^{w(B)} \left[\left(-\frac{\delta \beta}{\sqrt{2\lambda}} w\right)^{1-\frac{1}{2\beta}} I_{\nu}(w) \frac{\delta}{\sqrt{2\lambda}} \left(-\frac{\delta \beta}{\sqrt{2\lambda}} w\right)^{-\frac{1}{2\beta}-1} dw
\]

\[
= \frac{2K}{\delta \sqrt{2\lambda}} \left(-\frac{\delta \beta}{\sqrt{2\lambda}}\right)^{1+\frac{1}{2\beta}} \int_{w(A)}^{w(B)} w^{1-\nu} I_{\nu}(w)dw
\]

Applying equation 5 of Prudnikov et al (1986, p 46) to the last equation:

\[
2\delta^{-2} K \int_{A}^{B} Y^{-2\beta - \frac{3}{2}} I_{\nu}(\sqrt{2\lambda}z(Y))dY = \tag{B-21}
\]

\[
2 \delta \sqrt{2\lambda} K \left[\left(-\frac{\delta \beta}{\sqrt{2\lambda}} w\right)^{1-\nu} I_{\nu-1}(w)\right]^{w(B)}_{w(A)} = \frac{2}{\delta \sqrt{2\lambda}} \left[K Y^{-\beta - \frac{1}{2}} I_{\nu-1}(\sqrt{2\lambda}z(Y))\right]^{Y=B}_{Y=A}
\]

Replacing (B-20) and (B-21) in (B-19):

\[
I_{\lambda}(K, A, B) = \frac{2}{\delta \sqrt{2\lambda}} \left[Y^{-\beta + \frac{1}{2}} I_{\nu+1}(\sqrt{2\lambda}z(Y)) - K Y^{-\beta - \frac{1}{2}} I_{\nu-1}(\sqrt{2\lambda}z(Y))\right]^{Y=B}_{Y=A} \tag{B-22}
\]

A - 2.2 \(\beta > 0\)

\[
m(Y) = 2\delta^{-2} Y^{-2\beta - 2},
\]

\[
\psi_{\lambda}(Y) = Y^{\frac{1}{2}} K_{\nu}(\sqrt{2\lambda}z(Y)),
\]

\[
\nu = \frac{1}{2\beta},
\]

\[
z(Y) = \frac{1}{\delta \beta} Y^{-\beta},
\]
\[ m = \frac{1}{\delta \beta}, \text{ where } m(Y), \nu, z, m, \text{ and } \psi_\lambda \text{ are defined in (B-18), (22), (18), (20) and (15), respectively. So,} \]

\[ I_\lambda(K, A, B) = \int_A^B (Y - K) Y^{\frac{1}{2}} K_\nu(\sqrt{2\lambda z(Y)}) 2\delta^{-2} Y^{-2\beta - 2} dY \quad \text{(B-23)} \]

\[ = 2\delta^{-2} \int_A^B Y^{-2\beta - \frac{1}{2}} K_\nu(\sqrt{2\lambda z(Y)}) dY \]

\[ - 2\delta^{-2} K \int_A^B Y^{-2\beta - \frac{3}{2}} K_\nu(\sqrt{2\lambda z(Y)}) dY \]

Let us consider the first integral \( 2\delta^{-2} \int_A^B Y^{-2\beta - \frac{1}{2}} K_\nu(\sqrt{2\lambda z(Y)}) dY \). Making the change of variable

\[ w = \sqrt{2\lambda z(Y)} \Rightarrow w = \sqrt{2\lambda}(\frac{1}{\delta \beta} Y^{-\beta}), \]

then,

\[ Y = (\frac{\delta \beta}{\sqrt{2\lambda}} w)^{-\frac{1}{\beta}} \]

and,

\[ \frac{\partial Y}{\partial w} = - \frac{\delta}{\sqrt{2\lambda}} (\frac{\delta \beta}{\sqrt{2\lambda}} w)^{-\frac{1}{\beta} - 1}. \]

So,

\[ 2\delta^{-2} \int_{w(A)}^{w(B)} \left[ \left( \frac{\delta \beta}{\sqrt{2\lambda}} w \right)^{-\frac{1}{\beta}} \right]^{-2\beta - \frac{1}{2}} K_\nu(w) \left( - \frac{\delta}{\sqrt{2\lambda}} \left( \frac{\delta \beta}{\sqrt{2\lambda}} w \right)^{-\frac{1}{\beta} - 1} \right) dw \]

\[ = - \frac{2}{\delta \sqrt{2\lambda}} (\frac{\delta \beta}{\sqrt{2\lambda}})^{1 - \frac{1}{\beta}} \int_{w(A)}^{w(B)} w^{1 - \nu} K_\nu(w) dw \]

Applying equation 5 of Prudnikov et al (1986, p 47) to the last equation:

\[ 2\delta^{-2} \int_A^B Y^{-2\beta - \frac{1}{2}} K_\nu(\sqrt{2\lambda z(Y)}) dY \]

\[ = - \frac{2}{\delta \sqrt{2\lambda}} (\frac{\delta \beta}{\sqrt{2\lambda}})^{1 - \frac{1}{\beta}} \left[ - w^{1 - \nu} K_{1 - \nu}(w) \right]_{w(A)}^{w(B)} = \frac{2}{\delta \sqrt{2\lambda}} \left[ Y^{-\beta + \frac{1}{2}} K_{1 - \nu}(\sqrt{2\lambda z(Y)}) \right]_{Y = A}^{Y = B} \]

Using, in last equation, the result 9.6.6 from Abramowitz and Stegun (1972, p 375), then

\[ 2\delta^{-2} \int_A^B Y^{-2\beta - \frac{1}{2}} K_\nu(\sqrt{2\lambda z(Y)}) dY = \frac{2}{\delta \sqrt{2\lambda}} \left[ Y^{-\beta + \frac{1}{2}} K_{\nu-1}(\sqrt{2\lambda z(Y)}) \right]_{Y = A}^{Y = B} \quad \text{(B-24)} \]
For the second integral $2\delta^{-2}K \int_A^B Y^{-2\beta - \frac{3}{2}} K_\nu (\sqrt{2\lambda} z(Y)) dY$, and making the same change of variable as in the first integral, then

$$2\delta^{-2}K \int_A^B Y^{-2\beta - \frac{3}{2}} K_\nu (\sqrt{2\lambda} z(Y)) dY$$

$$= 2\delta^{-2}K \int_{w(A)}^{w(B)} \left[ \left( \frac{\delta \beta}{\sqrt{2\lambda}} w \right)^{-\frac{1}{\delta}} \right]^{-2\beta - \frac{3}{2}} K_\nu (w) \left( -\frac{\delta}{\sqrt{2\lambda}} \right) \left( \frac{\delta \beta}{\sqrt{2\lambda}} w \right)^{-\frac{3}{2} - 1} dw$$

$$= -\frac{2K}{\sqrt{2\lambda}} \left( \frac{\delta \beta}{\sqrt{2\lambda}} \right)^{1+\frac{1}{\delta}} \int_{w(A)}^{w(B)} w^{1+\nu} K_\nu (w) dw$$

Applying equation 5 of Prudnikov et al (1986, p 47) to the last equation:

$$2\delta^{-2}K \int_A^B Y^{-2\beta - \frac{3}{2}} K_\nu (\sqrt{2\lambda} z(Y)) dY \quad \text{(B-25)}$$

$$= -\frac{2}{\sqrt{2\lambda}} \left( \frac{\delta \beta}{\sqrt{2\lambda}} \right)^{1+\frac{1}{\delta}} \left[ -w^{1+\nu} K_{1+\nu} (w) \right]_{w(A)}^{w(B)} = \frac{2}{\sqrt{2\lambda}} \left[ K Y^{-\beta - \frac{\nu-1}{2}} K_{\nu+1} (\sqrt{2\lambda} z(Y)) \right]_{Y=A}^{Y=B}$$

Replacing (B-24) and (B-25) in (B-23):

$$I_\lambda (K, A, B) = \frac{2}{\sqrt{2\lambda}} \left[ Y^{-\beta + \frac{\nu}{2}} K_{\nu-1} (\sqrt{2\lambda} z(Y)) - K Y^{-\beta - \frac{\nu}{2}} K_{\nu+1} (\sqrt{2\lambda} z(Y)) \right]_{Y=A}^{Y=B} \quad \text{(B-26)}$$

B - 2. $J_\lambda (K, A, B) = \int_A^B (Y - K) \phi_\lambda (Y) m(Y) dY$, with $\mu = 0$

The expressions for integrals $J_\lambda (K, A, B)$, with $\mu = 0$, are obtained in the same way as for $I_\lambda (K, A, B)$, with $\mu = 0$. Hence, we just summarise the formulae:

B - 2.1 $\beta < 0$

$$J_\lambda (K, A, B) = \frac{2}{\delta \sqrt{2\lambda}} \left[ -Y^{-\beta + \frac{1}{2}} K_{\nu+1} (\sqrt{2\lambda} z(Y)) + K Y^{-\beta - \frac{1}{2}} K_{\nu-1} (\sqrt{2\lambda} z(Y)) \right]_{Y=A}^{Y=B} \quad \text{(B-27)}$$

B - 2.2 $\beta > 0$

$$J_\lambda (K, A, B) = \frac{2}{\delta \sqrt{2\lambda}} \left[ -Y^{-\beta + \frac{1}{2}} I_{\nu-1} (\sqrt{2\lambda} z(Y)) + K Y^{-\beta - \frac{1}{2}} I_{\nu+1} (\sqrt{2\lambda} z(Y)) \right]_{Y=A}^{Y=B} \quad \text{(B-28)}$$
All the various solutions can be compiled into

\[ I_\lambda(K, A, B) = \begin{cases} 
(B-4) & \text{if } \beta < 0 \land \mu > 0 \\
(B-9) & \text{if } \beta < 0 \land \mu < 0 \\
(B-11) & \text{if } \beta > 0 \land \mu > 0 \\
(B-12) & \text{if } \beta > 0 \land \mu < 0 \\
(B-22) & \text{if } \beta > 0 \land \mu = 0 \\
(B-26) & \text{if } \beta < 0 \land \mu = 0
\end{cases} \] (B-29)

\[ J_\lambda(K, A, B) = \begin{cases} 
(B-14) & \text{if } \beta < 0 \land \mu > 0 \\
(B-15) & \text{if } \beta < 0 \land \mu < 0 \\
(B-16) & \text{if } \beta > 0 \land \mu > 0 \\
(B-17) & \text{if } \beta > 0 \land \mu < 0 \\
(B-27) & \text{if } \beta > 0 \land \mu = 0 \\
(B-28) & \text{if } \beta < 0 \land \mu = 0
\end{cases} \] (B-30)

Note that these close-form solutions for integrals \( I_\lambda(K, A, B) \) and \( J_\lambda(K, A, B) \) are different from the solutions in Davidov and Linetsky (2001, p 964) for the case \( \mu = 0 \) and \( \beta > 0 \). This difference is in the order of the Bessel function for the cases referred.
C Appendix - Proof of Proposition 5

For equation (38):

Using (10)

\[
\frac{1}{\lambda} E^Q \left[ e^{-(r+\lambda)S} \right] = \frac{1}{\lambda} \phi_\alpha(S) \frac{1}{\phi_\alpha(L)}, \quad (C-1)
\]

with \( \alpha \) defined in (41).

Now the different cases must be considered:

1. IF \( \mu < 0, \beta < 0 \)

In this case:

\[ \epsilon = 1, \]

\[ x(S) = \frac{\mu}{\beta^2} S^{-2\beta}, \]

\[ m = -\frac{1}{4\beta}, \]

where \( \epsilon, x(S) \) and \( m \) are defined respectively in (19), (17) and (20).

Using (16),

\[
\frac{1}{\lambda} \phi_\alpha(S) = \frac{1}{\lambda} \frac{S^{\beta+\frac{1}{2}} e^{\frac{1}{2} x(S)} W_{k_\alpha,m}(x(S))}{L^{\beta+\frac{1}{2}} e^{\frac{1}{2} x(L)} W_{k_\alpha,m}(x(L))},
\]

where

\[ k_\alpha = \epsilon \left( \frac{1}{2} + \frac{1}{4\beta} \right) - \frac{\alpha}{2|\mu\beta|}, \quad (C-2) \]

and \( \alpha \) is defined in (41).

Using equation 13.1.33 of Abramowitz and Stegun (1972, p 505), and (17) the last equation is rewritten as
\[
\frac{1}{\lambda} \phi_\alpha(S) = \frac{1}{\lambda} \phi_\alpha(L) = \frac{1}{\lambda} S^{\beta + \frac{1}{2}} e^{\frac{1}{2} x(S)} e^{-\frac{1}{2} x(S)} x(S)^{m + \frac{1}{2}} U \left( m - k_\alpha + \frac{1}{2}, 1 + 2m, x(S) \right) \\
\frac{1}{\lambda} L^{\beta + \frac{1}{2}} e^{\frac{1}{2} x(L)} e^{-\frac{1}{2} x(L)} x(L)^{m + \frac{1}{2}} U \left( m - k_\alpha + \frac{1}{2}, 1 + 2m, x(L) \right) \\
= \frac{1}{\lambda} S^{\beta + \frac{1}{2}} \left( \frac{\mu}{\beta^2} S^{-2\beta} \right)^{\frac{1}{2} - \frac{1}{4\beta}} U \left( m - k_\alpha + \frac{1}{2}, 1 + 2m, x(S) \right) \\
\frac{1}{\lambda} L^{\beta + \frac{1}{2}} \left( \frac{\mu}{\beta^2} L^{-2\beta} \right)^{\frac{1}{2} - \frac{1}{4\beta}} U \left( m - k_\alpha + \frac{1}{2}, 1 + 2m, x(L) \right) \\
= \frac{1}{\lambda} S \times U \left( m - k_\alpha + \frac{1}{2}, 1 + 2m, x(S) \right) \\
\frac{1}{\lambda} L \times U \left( m - k_\alpha + \frac{1}{2}, 1 + 2m, x(L) \right).
\]

Applying equation 13.1.29 of Abramowitz and Stegun (1972, p 505), and using (20), (17) and (C-2), then

\[
\frac{1}{\lambda} \phi_\alpha(S) = \frac{1}{\lambda} S \times x(S)^{1 - 1 + \frac{1}{2\beta}} U \left( 1 + m - k_\alpha + \frac{1}{2} - 1 - 2m, 1 - 2m, x(S) \right) \\
\frac{1}{\lambda} L \times x(L)^{1 - 1 + \frac{1}{2\beta}} U \left( 1 + m - k_\alpha + \frac{1}{2} - 1 - 2m, 1 - 2m, x(S) \right) \\
= \frac{1}{\lambda} U \left( \frac{\alpha}{2\mu^2}, 1 + \frac{1}{2\beta}, x(S) \right) \\
\frac{1}{\lambda} \frac{\alpha}{2\beta} \left( 1 + \frac{1}{2\beta}, x(L) \right)
\]

2. \( \mu \leq 0, \beta > 0 \)

In this case:

\( \epsilon = -1, \)

\( x(S) = -\frac{\mu}{\beta^2} S^{-2\beta}, \)

\( m = \frac{1}{2\beta}, \)

where \( \epsilon, x(S) \) and \( m \) are defined respectively in (19), (17) and (20).

Using (16),

\[
\frac{1}{\lambda} \phi_\alpha(S) = \frac{1}{\lambda} S^{\beta + \frac{1}{2}} e^{\frac{1}{2} x(S)} M_{k_\alpha, m}(x(S)) \\
\frac{1}{\lambda} L^{\beta + \frac{1}{2}} e^{\frac{1}{2} x(L)} M_{k_\alpha, m}(x(L))
\]

where \( k_\alpha \) and \( \alpha \) are defined in (C-2) and (41). Using equation 13.1.32 of Abramowitz and Stegun (1972, p 505), as well as (17) and (C-2), the last equation becomes
\begin{align*}
\frac{1}{\lambda} \frac{\phi_\alpha(S)}{\phi_\alpha(L)} &= \frac{1}{\lambda} \frac{S^{\beta + \frac{1}{2}} e^{-\frac{x(S)}{2}} e^{-\frac{x(\lambda)}{2}} x(S) \frac{1}{2} + m}{L^{\beta + \frac{1}{2}} e^{-\frac{x(L)}{2}} e^{-\frac{x(\lambda)}{2}} x(L) \frac{1}{2} + m} \frac{M(m - k_\alpha + \frac{1}{2}, 1 + 2m, x(S))}{x(S)} \\
&= \frac{1}{\lambda} \frac{e^{-x(S)} M \left( \frac{1}{2 \beta^2} - \frac{1}{2} + \frac{1}{4 \beta^2} \frac{1}{2} + 2 \times \frac{1}{4 \beta^2} \frac{1}{2} + 1 + 2 \times \frac{1}{4 \beta^2} \frac{1}{2}, x(S) \right)}{e^{-x(L)} M \left( \frac{1}{2 \beta^2} - \frac{1}{2} + \frac{1}{4 \beta^2} \frac{1}{2} + 2 \times \frac{1}{4 \beta^2} \frac{1}{2}, x(L) \right)} \\
&= \frac{1}{\lambda} e^{-x(S)} M \left( \frac{1}{2 \beta^2} - \frac{1}{2} + \frac{1}{4 \beta^2} \frac{1}{2} + 2 \times \frac{1}{4 \beta^2} \frac{1}{2}, x(S) \right)
\end{align*}

3. \( \mu > 0, \beta < 0 \)

In this case:

\( \epsilon = -1, \)

\( x(S) = -\frac{\mu}{\beta^2} S^{-2\beta}, \)

\( m = -\frac{1}{4 \beta}, \)

where \( \epsilon, x(S) \) and \( m \) are defined in (19), (17) and (20).

Using (16) and (21):

\begin{align*}
\frac{1}{\lambda} \frac{\phi_\lambda(S)}{\phi_\lambda(L)} &= \frac{e^{-r\lambda} S^{\beta + \frac{1}{2}} e^{-\frac{1}{2} x(S)} W_{k_\alpha, m}(x(S))}{\lambda L^{\beta + \frac{1}{2}} e^{-\frac{1}{2} x(L)} W_{k_\alpha, m}(x(L))}
\end{align*}

where \( k_\alpha \) and \( \alpha \) are defined in (C-2) and (41). Using equation 13.1.33 of Abramowitz and Stegun (1972, p. 505), and (17), the last equation becomes

\begin{align*}
\frac{1}{\lambda} \frac{\phi_\lambda(S)}{\phi_\lambda(L)} &= \frac{1}{\lambda} \frac{S^{\beta + \frac{1}{2}} e^{-\frac{1}{2} x(S)} e^{-\frac{1}{2} x(S)} x(S)^{m + \frac{1}{2}} U \left( m - k_\alpha + \frac{1}{2}, 1 + 2m, x(S) \right)}{L^{\beta + \frac{1}{2}} e^{-\frac{1}{2} x(L)} e^{-\frac{1}{2} x(L)} x(L)^{m + \frac{1}{2}} U \left( m - k_\alpha + \frac{1}{2}, 1 + 2m, x(L) \right)} \\
&= \frac{1}{\lambda} \frac{S^{\beta + \frac{1}{2}} \left( -\frac{\mu}{\beta^2} S^{-2\beta} \right)^{\frac{1}{2} - \frac{1}{2} \beta^2} U \left( m - k_\alpha + \frac{1}{2}, 1 + 2m, x(S) \right)}{L^{\beta + \frac{1}{2}} \left( -\frac{\mu}{\beta^2} L^{-2\beta} \right)^{\frac{1}{2} - \frac{1}{2} \beta^2} U \left( m - k_\alpha + \frac{1}{2}, 1 + 2m, x(L) \right)} \\
&= \frac{1}{\lambda} e^{-x(S)} S \times U \left( m - k_\alpha + \frac{1}{2}, 1 + 2m, x(S) \right) \\
&= \frac{1}{\lambda} e^{-x(L)} L \times U \left( m - k_\alpha + \frac{1}{2}, 1 + 2m, x(L) \right).
\end{align*}
Applying equation 13.1.29 of Abramowitz and Stegun (1972, p 505), and using (20) and (C-2), then

\[
\frac{1}{\lambda} \frac{\phi_\lambda(S)}{\phi_\lambda(L)} = \frac{1}{\lambda} \frac{e^{-x(S)S}}{e^{-x(L)L}} \times \left( -\frac{\mu}{\beta \lambda^2} S^{-2\beta} \right)^{\frac{1}{2\beta}} U \left( 1 + \frac{1}{2\beta} - \frac{\alpha}{2\beta}, 1 + \frac{1}{2\beta}, x(S) \right)
\]

\[
\times \left( -\frac{\mu}{\beta \lambda^2} L^{-2\beta} \right)^{\frac{1}{2\beta}} U \left( 1 + \frac{1}{2\beta} - \frac{\alpha}{2\beta}, 1 + \frac{1}{2\beta}, x(S) \right)
\]

\[
= \frac{1}{\lambda} \frac{e^{-x(S)U}}{e^{-x(L)U}} \left( 1 + \frac{1}{2\beta} - \frac{\alpha}{2\beta}, 1 + \frac{1}{2\beta}, x(L) \right)
\]

4. \( \mu > 0, \beta > 0 \)

In this case:

\[ \epsilon = 1, \]

\[ x(S) = \frac{\mu}{\beta \lambda^2} S^{-2\beta}, \]

\[ m = \frac{1}{4\beta}, \]

where \( \epsilon, x(S) \) and \( m \) are defined respetively in (19), (17) and (20).

Using (16),

\[
\frac{1}{\lambda} \frac{\phi_\lambda(S)}{\phi_\lambda(L)} = \frac{1}{\lambda} \frac{S^{\beta+\frac{1}{2}} e^{\frac{1}{2} x(S) M_{k_\alpha, m}}} {\frac{1}{2} e^{\frac{1}{2} x(L) M_{k_\alpha, m}}} \]

where \( k_\alpha \) and \( \alpha \) are defined in (C-2) and (41).

Using equation 13.1.32 of Abramowitz and Stegun (1972, p 505), as well as (17) and (20) the last equation becomes

\[
\frac{1}{\lambda} \frac{\phi_\lambda(S)}{\phi_\lambda(L)} = \frac{1}{\lambda} S^{\beta+\frac{1}{2}} e^{\frac{x(S)}{2}} e^{-\frac{x(S)}{2}} x(S)^{\frac{1}{2} + m} M (m - k_\alpha + \frac{1}{2}, 1 + 2m, x(S))
\]

\[
\times \frac{1}{\lambda} \frac{L^{\beta+\frac{1}{2}} e^{\frac{x(L)}{2}} e^{-\frac{x(L)}{2}} x(L)^{\frac{1}{2} + m}} {x(L)^{\frac{1}{2} + m} M (m - k_\alpha + \frac{1}{2}, 1 + 2m, x(L))}
\]

\[
= \frac{1}{\lambda} S^{\beta+\frac{1}{2}} \left( -\frac{\mu}{\beta \lambda^2} S^{-2\beta} \right)^{\frac{1}{2\beta} + \frac{1}{4\beta}} \left( \frac{\mu}{\beta \lambda^2} L^{-2\beta} \right)^{\frac{1}{2\beta} + \frac{1}{4\beta}} \frac{M (m - k_\alpha + \frac{1}{2}, 1 + 2m, x(S))}{M (m - k_\alpha + \frac{1}{2}, 1 + 2m, x(L))}
\]

\[
= \frac{1}{\lambda} M \left( \alpha + \frac{1}{2\beta}, x(S) \right)
\]

\[
\times \frac{1}{\lambda} M \left( \alpha + \frac{1}{2\beta}, x(L) \right).
\]
5. If $\mu = 0, \beta < 0$

Using (16),

\[
\frac{1}{\lambda} \frac{\phi_\lambda(S)}{\phi_\lambda(L)} = \frac{S^{1/2} K_\nu}{\bar{L}^{1/2}} \left( \frac{\sqrt{2\lambda Z(S)}}{\sqrt{2\lambda Z(L)}} \right)
\]

where $\nu$ and $Z(S)$ are defined in (22) and (18).

6. If $\mu = 0, \beta > 0$

Using (16),

\[
\frac{1}{\lambda} \frac{\phi_\lambda(S)}{\phi_\lambda(L)} = \frac{S^{1/2} I_\nu}{\bar{L}^{1/2}} \left( \frac{\sqrt{2\lambda Z(S)}}{\sqrt{2\lambda Z(L)}} \right)
\]

where $\nu$ and $Z(S)$ is defined in (22) and (18).

The proof for equations (39) and (40) is analogous to the proof above for (38).
D Appendix - C++ Program

//Pricing Knock Out Barrier Call Under CEV Diffusion

#include <iostream>
#include <iomanip>
#include <math.h>
#include <vector>
#include <complex>
#include <algorithm>
#include <fstream>

using namespace std;

typedef long double ldouble;

typedef vector<ldouble> fvector;

typedef complex<ldouble> ldcomplex;

typedef complex<ldouble> dcomplex;

typedef vector<ldouble> fvector;

typedef ldouble Z;

const ldouble PI = 3.141592653589793238462643;

const ldouble machineepsilon = 5E-16;

const long double ld1 = 1.0;

//Definition of sign function

ldouble sign(ldouble a, ldouble b)
D.1 The Gamma Function

// Gamma function for reals

ldouble gamma(ldouble x)
{
    int i,k,m;
    ldouble ga,gr,r,z;
    static ldouble g[] = {1.0,
        0.5772156649015329,
        -0.6558780715202538,
        -0.420026350340952e-1,
        0.1665386113822915,
        -0.42197734555443e-1,
        -0.9621971527877e-2,
        0.7218943246663e-2,
        54};
-0.11651675918591e-2,
-0.2152416741149e-3,
0.1280502823882e-3,
-0.201348547807e-4,
-0.12504934821e-5,
0.1133027232e-5,
-0.2056338417e-6,
0.6116095e-8,
0.50020075e-8,
-0.11812746e-8,
0.1043427e-9,
0.77823e-11,
-0.36968e-11,
0.51e-12,
-0.206e-13,
-0.54e-14,
0.14e-14};

if (x > 171.0) return 1e308; // This value is an overflow flag.

if (x == (int)x) {
    if (x > 0.0) {
        ga = 1.0; // use factorial
        for (i=2;i<x;i++) {

55
ga *= i;

else

if (fabs(x) > 1.0) {
    z = fabs(x);
    m = (int)z;
    r = 1.0;
    for (k=1;k<=m;k++) {
        r *= (z-k);
    }
    z -= m;
}
else
    z = x;

gr = g[24];
for (k=23;k>=0;k--) {
    gr = gr*z+g[k];
}
ga = 1.0/(gr*z);

if (fabs(x) > 1.0) {
    ga *= r;
    if (x < 0.0) {
        ga = -M_PI/(x*ga*sin(M_PI*x));
    }
}
}
return ga;

// Complex Gamma and LnGamma function

complex<ldouble> cgamma(complex<ldouble> z, int OPT)
{
    // OPT = 0 ==> Complex Gamma; OPT = 1 ==> Complex ln gamma
    complex<ldouble> g,z0,z1;
    double x0,q1,q2,x,y,th,th1,th2,g0,gi,gr1,gi1;
    double na,t,x1,y1,sr,si;
    int i,j,k;
    static double a[] = {
        8.333333333333333e-02,
        -2.777777777777778e-03,
        7.936507936507937e-04,
        // More entries...
    };
x = real(z);
y = imag(z);
if (x > 171) return complex<double>(1e308,0);
if ((y == 0.0) && (x == (int)x) && (x <= 0.0))
    return complex<double>(1e308,0);
else if (x < 0.0) {
x1 = x;
y1 = y;
x = -x;
y = -y;
}
x0 = x;
if (x <= 7.0) {
    na = (int)(7.0-x);
x0 = x+na;
```c
q1 = sqrt(x0*x0+y*y);

th = atan(y/x0);

gr = (x0-0.5)*log(q1)-th*y-x0+0.5*log(2.0*M_PI);

gi = th*(x0-0.5)+y*log(q1)-y;

for (k=0;k<10;k++){
    t = pow(q1,-1.0-2.0*k);
    gr += (a[k]*t*cos((2.0*k+1.0)*th));
    gi -= (a[k]*t*sin((2.0*k+1.0)*th));
}

if (x <= 7.0) {
    gr1 = 0.0;
    gi1 = 0.0;
    for (j=0;j<na;j++) {
        gr1 += (0.5*log((x+j)*(x+j)+y*y));
        gi1 += atan(y/(x+j));
    }
    gr -= gr1;
    gi -= gi1;
}

if (x1 <= 0.0) {
    q1 = sqrt(x*x+y*y);
```
th1 = atan(y/x);

sr = -sin(M_PI*x)*cosh(M_PI*y);

si = -cos(M_PI*x)*sinh(M_PI*y);

q2 = sqrt(sr*sr+si*si);

th2 = atan(si/sr);

if (sr < 0.0) th2 += M_PI;

gr = log(M_PI/(q1*q2))-gr;

gi = -th1-th2-gi;

x = x1;

y = y1;

}

if (OPT == 0) {

    complex<ldouble> cc=pow(z/exp(ldouble(1)),z)*sqrt(ldouble(2*PI)/z)*pow(ldouble(1)+ldouble(1)/(ldouble(15)*pow(z,ldouble(2)))),ldouble(1.25)*z);//using paper
    if((x<=-10) && x! = int(x) && (y<=-10)) return cc;

    else{
        g0 = exp(gr);
        gr = g0*cos(gi);
        gi = g0*sin(gi);
    }
}

g = complex<double>(gr,gi);
D.2 Whittaker Functions

const ldouble double_exact=0.5e-14;

bool operator ==(dcomplex a, dcomplex b)
{
  bool P;
  P= (real(a)==real(b) && imag(a)==imag(b));
  return P;
}

bool operator !=(dcomplex a, dcomplex b)
{
  bool P;
  P= (real(a)!=real(b) || imag(a)!=imag(b));
  return P;
}

dcomplex operator +(int a, dcomplex b)
{
  dcomplex c,P;
}
real(P) = real(b) + a;
imag(P) = imag(b);
return P;
}
dcomplex operator -(int a, dcomplex b)
{
dcomplex c, P;
real(P) = real(b) - a;
imag(P) = imag(b);
return P;
}
dcomplex operator -(dcomplex b, int a)
{
dcomplex c, P;
real(P) = real(b) - a;
imag(P) = imag(b);
return P;
}
dcomplex operator +(dcomplex b, int a)
{
dcomplex P, c;
real(P) = real(b) + a;
imag(P)=imag(b);
return P;
}
dcomplex operator /(dcomplex b, int a)
{
dcomplex P;
real(P)=real(b)/a;
imag(P)=imag(b)/a;
return P;
}
dcomplex operator *(int a, dcomplex b)
{
dcomplex P, c;
real(P)=real(b)*a;
imag(P)=imag(b)*a;
return P;
}
dcomplex deviation(dcomplex num)
{
return(dcomplex(abs(real(num)*double_exact), abs(imag(num)*double_exact)));
}
dcomplex abs_reim(dcomplex z){return(dcomplex(abs(real(z)),abs(imag(z))));}
dcomplex KummerM_sum_max(dcomplex a, ldouble b, ldouble z)
{
    long n;

dcomplex sn, dsn, apn, bpn, fct, dsum, sum, last_cdres,
dz, dapn, dbpn, res_min_es;
ldouble np1, dnp1, ed, es, es_min, last_derr;
bool last_corr;

if(a==(0,0)||b==0) last_cdres=1.0;
else if(b==0||b==-int(abs(b)))last_cdres=1e308;
else
{
    sum = 1.0; sn = 1.0; dsn = deviation(sn);
    dsum = 0.0; dz = abs(z*double_exact); es_min = 1.0;
    n=0;
    for(;;){
        apn = a+n; bpn = b+n; np1=n+1.0; fct = apn/bpn*z/np1;
        dbpn = deviation(bpn);
        dapn = deviation(apn);
        dnp1 = abs(np1*double_exact);
        dsn = abs_reim(dsn * fct) +
        abs_reim(sn * dapn*bpn/(bpn*bpn)*z/np1)+
        abs_reim(sn * dbpn*apn/(bpn*bpn)*z/np1)+
    }
abs_reim(sn * apn/bpn*dz*np1/(np1*np1)) +
abs_reim(sn * apn/bpn*z*dpn1/(np1*np1));

sn = sn * fct;

dsum = dsum + dsn; sum = sum + sn;

es = max(abs(real(sn)/real(sum)) , abs(imag(sn)/imag(sum)));

ed = max(abs(real(dsum)/real(sum)), abs(imag(dsum)/imag(sum)));

if (es<es_min) {
    res_min_es = sum; es_min=es; }

if (ed>es) break; n=n+1; }

last_cdres = sum; last_derr = max(es, ed);

last_corr = last_derr < 0.1;

if (!last_corr ) last_cdres = res_min_es;

}

return(last_cdres);

////////////////////////////////////////////////////////////////////////

dcomplex Kummer_U(dcomplex kk, ldouble mm, ldouble zz)
{

dcomplex U;

    U = (PI/sin(mm*PI)) * ((KummerM_sum_max(kk,mm, zz) / (cgamma(ldouble(1.0)-
        mm+kk,0) * gamma(mm))) - (pow(zz,1-mm) * KummerM_sum_max(ldouble(1.0)+kk-
        mm,2-mm, zz) / (cgamma(kk,0) * gamma(2.0-mm))));
return U;

}

dcomplex Whittaker_M(dcomplex kk, ldouble mm, ldouble zz)
{

dcomplex WM;

WM=exp(-zz*0.5)*pow(zz,0.5+mm)
*KummerM_sum_max(ldouble(0.5)-kk+mm,1.0+2*mm, zz);

return WM;

}

/////////////////////////////////////////////////////////////////////////
//Definition of Whittaker Wk,m(x) function for k complex and m,x double

dcomplex Whittaker_W(ldcomplex kk, ldouble mm, ldouble zz)
{

dcomplex ww1;

ww1=exp(-0.5*zz)*pow(zz,mm+0.5)*Kummer_U(ldouble(0.5)+mm-kk,1+2*mm,zz);

return ww1;

}

D.3 Modified Bessel Functions

// Definition of the Bessel Function I and K for double order and complex argument

#define eps 1e-15

#define el 0.5772156649015329
static complex<ldouble> cii(0.0,1.0);
static complex<ldouble> czero(0.0,0.0);
static complex<ldouble> cone(1.0,0.0);

int msta1(double x,int mp)
{
    double a0,f0,f1,f;
    double n0, n1,nn;
    a0 = fabs(x);
    n0 = (int)(1.1*a0)+1;
    f0 = 0.5*log10(6.28*n0)-n0*log10(1.36*a0/n0)-mp;
    n1 = n0+5;
    f1 = 0.5*log10(6.28*n1)-n1*log10(1.36*a0/n1)-mp;
    for (int i=0;i<20;i++) {
        nn = n1-(n1-n0)/(1.0-f0/f1);
        f = 0.5*log10(6.28*nn)-nn*log10(1.36*a0/nn)-mp;
        if (abs(nn-n1) < 1) break;
        n0 = n1;
        f0 = f1;
        n1 = nn;
        f1 = f;
    }
    return int(nn);
int msta2(double x, int n, int mp)
{
double a0, ejn, hmp, f0, f1, f, obj;
double n0, n1;
double nn;
a0 = fabs(x);
hmp = 0.5 * mp;
ejn = 0.5 * log10(6.28 * n) - n * log10(1.36 * a0 / n);
if (ejn <= hmp) {
    obj = mp;
    n0 = (int)(1.1 * a0);
    if (n0 < 1) n0 = 1;
}
else {
    obj = hmp + ejn;
    n0 = n;
}
f0 = 0.5 * log10(6.28 * n0) - n0 * log10(1.36 * a0 / n0) - obj;
n1 = n0 + 5;
f1 = 0.5 * log10(6.28 * n1) - n1 * log10(1.36 * a0 / n1) - obj;
for (int i = 0; i < 20; i++) {

nn = n1-(n1-n0)/(1.0-f0/f1);

f = 0.5*log10(6.28*nn)-nn*log10(1.36*a0/nn)-obj;

if (abs(nn-n1) < 1) break;

n0 = n1;

f0 = f1;

n1 = nn;

f1 = f;

} return int(nn)+10;

} // BesselFunction Iv(z)

dcomplex cbessIv(ldouble v,complex<ldouble>z)
{
complex<ldouble> z1,z2,ca1,ca,cs,cr,ci0,cbi0,cf,cf1,cf2, CBIv;
complex<ldouble> ct,cp,cbk0,ca2,cr1,cr2,csu,cws,cb;
complex<ldouble> cg0, cg1, cgk, cbk1, cvk;
ldouble a0,v0,v0p,v0n,vt,w0,piv,gap,gan;
int m,n,k, kz;
vector<complex<ldouble>> civ(500), ckv(500);
a0 = abs(z);

z1 = z;

z2 = z*z;
n = (int)v;

v0 = v-n;

piv = M_PI*v0;

vt = 4.0*v0*v0;

if (int (v) == 0)
{
    if (n == 0) n = 1;
    if (a0 < 1e-100)
    {
        for (k=0;k<n;k++)
        {
            civ[k] = czero;
            ckv[k] = complex<ldouble>(-1e308,0);
        }
        if (v0 == 0.0)
        {
            civ[0] = cone;
        }
        CBIv=civ[0];
    }
    if (v0 == 0.0)
    {
        civ[0] = cone;
    }
    CBIv=civ[0];
}

if (a0 >= 50.0) kz = 8;

else if (a0 >= 35.0) kz = 10;
else kz = 14;

if (real(z) <= 0.0) z1 = -z;

if (a0 < 18.0)
{
    if (v0 == 0.0)
    {
        ca1 = cone;
    }
    else
    {
        v0p = 1.0 + v0;
        gap = gamma(v0p);
        ca1 = pow(ldouble(0.5)*z1,v0)/gap;
    }
    ci0 = cone;
    cr = cone;

    for (k=1;k<=50;k++)
    {
        cr *= ldouble(0.25)*z2/(k*(k+v0));
        ci0 += cr;
        if (abs(cr/ci0) < eps) break;
    }
}
cbi0 = ci0*ca1;
CBIv=cbi0;
}
else {
ca = exp(z1)/sqrt(ldouble(2.0)*M_PI*z1);

for (k=1;k<=kz;k++) {
    cr *= -0.125*(vt-pow(2.0*k-1.0,2.0))/((ldouble)k*z1);
    cs += cr;
}

for (k=0;k<=n;k++)
if (a0 < 1e-100)
    
for (k=0;k<=n;k++)
{ 
civ[k] = czero;
ckv[k] = complex<double>(-1e308,0);
}
if (v0 == 0.0)
{

civ[0] = cone;
}
CBIv=civ[0];
}
if (a0 >= 50.0) kz = 8;
else if (a0 >= 35.0) kz = 10;
else kz = 14;
if (real(z) <= 0.0) z1 = -z;
if (a0 < 18.0)
{
if (v0 == 0.0)
{
ca1 = cone;
}
else
{

v0p = 1.0+v0;

gap = gamma(v0p);

cal = pow(ldouble(0.5)*z1,v0)/gap;

}  
ci0 = cone;

cr = cone;

for (k=1;k<=50;k++)
{

cr *= ldouble(0.25)*z2/(k*(k+v0));

ci0 += cr;

if (abs(cr/ci0) < eps) break;
}

cbi0 = ci0*cal;

CBIv=cbi0;

}

else {

ca = exp(z1)/sqrt(ldouble(2.0)*M_PI*z1);

cs = cone;

cr = cone;

for (k=1;k<=kz;k++)
{

cr *= -0.125*(vt-pow(2.0*k-1.0,2.0))/((ldouble)k*z1);

cs += cr;
}

}  
cbi0 = ca*cs;  
CBIv=cbi0;  
}  
// Iv using backward recurrence  
m = msta1(a0,200);  
if (m < n) n = m;  
else m = msta2(a0,n,15);  

 cf2 = czero;  
 cf1 = complex<ldouble>(1.0e-100,0.0);  
for (k=m;k>=0;k--) 
{  
  cf = 2.0*(v0+k+1.0)*cf1/z1+cf2;  
  if (k <= n) civ[k] = cf;  
  cf2 = cf1;  
  cf1 = cf;  
}  

 cs = cbi0/cf;  
for (k=0;k<=n;k++) 
{  
  civ[k] *= cs;  
}
CBIv=civ[n];
return CBIv;
}
}

// BesselFunction Kv(z)
dcomplex cbessKv(ldouble v, complex<ldouble> z)
{
complex<ldouble> z1,z2,ca1,ca,cs,cr,ci0,cbi0,cf,cf1,cf2, CBIv,CBKv;
complex<ldouble> ct,cp,cbk0,ca2,cr1,cr2,csu,cws,cb;
complex<ldouble> cg0,cg1,cgk,cbk1,cvk;
ldouble a0,v0,v0p,v0n,vt,w0,piv,gap,gan;
int m,n,k, kz;
vector < complex<ldouble > > civ(500), ckv(500);
a0 = abs(z);
z1 = z;
z2 = z*z;
n = (int)v;
v0 = v-n;
piv = M_PI*v0;
vt = 4.0*v0*v0;
if (int (v) == 0) {

if (n == 0) n = 1;

if (a0 < 1e-100)
{
    for (k=0;k<=n;k++)
    {
        civ[k] = czero;
        ckv[k] = complex<ldouble>(-1e308,0);
    }
}

if (v0 == 0.0)
{
    civ[0] = cone;
}

CBIv = civ[0];

if (a0 >= 50.0) kz = 8;
else if (a0 >= 35.0) kz = 10;
else kz = 14;

if (real(z) <= 0.0) z1 = -z;

if (a0 < 18.0)
{
    if (v0 == 0.0)
    {
        
    }
ca1 = cone;
}
else
{

v0p = 1.0+v0;
gap = gamma(v0p);
ca1 = pow(ldouble(0.5)*z1,v0)/gap;
}
ci0 = cone;
cr = cone;
for (k=1;k<=50;k++)
{

cr *= ldouble(0.25)*z2/(k*(k+v0));


i0 += cr;
if (abs(cr/ci0) < eps) break;
}
cbi0 = ci0*ca1;
CBIv=cbi0;
}
else {

c = exp(z1)/sqrt(ldouble(2.0)*M_PI*z1);
cs = cone;
cr = cone;

for (k=1;k<=kz;k++) {
    cr *= -0.125*(vt-pow(2.0*k-1.0,2.0))/((ldouble)k*z1);
    cs += cr;
}

cbi0 = ca*cs;

CBIV=cbi0;

}{

else{
{

if (n == 0) n = 1;

if (a0 < 1e-100)
{
    for (k=0;k<=n;k++)
    {
        civ[k] = czero;
        ckv[k] = complex<ldouble>(-1e308,0);
    }
}

if (v0 == 0.0)
{
    civ[0] = cone;
}
CBIv=civ[0];

if (a0 >= 50.0) kz = 8;
else if (a0 >= 35.0) kz = 10;
else kz = 14;
if (real(z) <= 0.0) z1 = -z;
if (a0 < 18.0)
{
if (v0 == 0.0)
{
ca1 = cone;
}
else
{
  v0p = 1.0+v0;
  gap = gamma(v0p);
  ca1 = pow(ldouble(0.5)*z1,v0)/gap;
}
cli0 = cone;

cr = cone;
for (k=1;k<=50;k++)
{ 
  cr *= ldouble(0.25)*z2/(k*(k+v0));
  ci0 += cr;
  if (abs(cr/ci0) < eps) break;
}

cbi0 = ci0*ca1;
CBIv=cbi0;
}

else {
  ca = exp(z1)/sqrt(ldouble(2.0)*M_PI*z1);
  cs = cone;
  cr = cone;
  for (k=1;k<=kz;k++) {
    cr *= -0.125*(vt-pow(2.0*k-1.0,2.0))/((ldouble)k*z1);
    cs += cr;
  }
  cbi0 = ca*cs;
  CBIv=cbi0;
}

// Iv using backward recurrence

m = msta1(a0,200);
if (m < n) n = m;
else m = msta2(a0,n,15);

cf2 = czero;

cf1 = complex<ldouble>(1.0e-100,0.0);

for (k=m;k>=0;k– )
{
    cf = 2.0*(v0+k+1.0)*cf1/z1+cf2;
    if (k <= n) civ[k] = cf;
    cf2 = cf1;
    cf1 = cf;
}

cs = cbi0/cf;

for (k=0;k<=n;k++)
{
    civ[k] *= cs;
}

CBIv=civ[n];

if(int (v)==0)
{
    if (a0 <= 9.0)
    {
        if (v0 == 0.0)
ct = -log(ldouble(0.5)*z1)-ldouble(0.5772156649015329);

cs = czero;

w0 = 0.0;

cr = cone;

for (k=1;k<=50;k++)
{
    w0 += 1.0/k;
    cr *= ldouble(0.25)*z2/(ldouble)(k*k);
    cp = cr*(w0+ct);
    cs += cp;
    if ((k >= 10) && (abs(cp/cs) < eps)) break;
}

cbk0 = ct+cs;

}

else {
    v0n = 1.0-v0;
    gan = gamma(v0n);
    ca2 = ldouble(1.0)/(gan*pow(ldouble(0.5)*z1,v0));
    ca1 = pow(ldouble(0.5)*z1,v0)/gap;
    csu = ca2-ca1;
    cr1 = cone;
cr2 = cone;

cws = czero;

for (k=1;k<=50;k++) {
    cr1 *= ldouble(0.25)*z2/(k*(k-v0));
    cr2 *= ldouble(0.25)*z2/(k*(k+v0));
    csu += ca2*cr1-ca1*cr2;
    if ((k >= 10) && (abs((cws-csu)/csu) < eps)) break;
    cws = csu;
}

cbk0 = csu*ldouble(M_PI_2)/sin(piv);
}
}

else {
    cb = exp(-z1)*sqrt(ldouble(M_PI_2)/z1);
    cs = cone;
    cr = cone;

    for (k=1;k<=kz;k++) {
        cr *= 0.125*(vt-pow(2.0*k-1.0,2.0))/((ldouble)k*z1);
        cs += cr;
    }
    cbk0 = cb*cs;
}
CBKv=cbk0;

return CBKv;
}

else
{
if (a0 <= 9.0)
{
if (v0 == 0.0)
{
ct = -log(ldouble(0.5)*z1)-ldouble(0.5772156649015329);
cs = czero;
w0 = 0.0;
cr = cone;
for (k=1;k<=50;k++)
{
  w0 += 1.0/k;
  cr *= ldouble(0.25)*z2/(ldouble)(k*k);
cp = cr*(w0+ct);
cs += cp;
  if ((k >= 10) && (abs(cp/cs) < eps)) break;
}
}
}
}

cbk0 = ct+cs;
} 
else {

v0n = 1.0-v0;

gan = gamma(v0n);

ga2 = ldouble(1.0)/(gan*pow(ldouble(0.5)*z1,v0));

ga1 = pow(ldouble(0.5)*z1,v0)/gap;

csu = ca2-ca1;

cr1 = cone;

cr2 = cone;

cws = czero;

for (k=1;k<50;k++) {
    cr1 *= ldouble(0.25)*z2/(k*(k-v0));
    cr2 *= ldouble(0.25)*z2/(k*(k+v0));
    csu += ca2*cr1-ca1*cr2;
    if ((k > 10) && (abs((cws-csu)/csu) < eps)) break;
    cws = csu;
}

ckb0 = csu*ldouble(M_PI_2)/sin(piv);
}
}
else {

cb = exp(-z1)*sqrt(ldouble(M_PI_2)/z1);
cs = cone;

cr = cone;

for (k=1;k<=kz;k++) {
    cr *= 0.125*(vt-pow(2.0*k-1.0,2.0))/((ldouble)k*z1);
    cs += cr;
}

cbk0 = cb*cs;
}

cbk1 = (ldouble(1.0)/z1-civ[1]*cbk0)/civ[0];

ckv[0] = cbk0;

ckv[1] = cbk1;

cg0 = cbk0;

cg1 = cbk1;

CBKv = cbk1;

for (k=2;k<=n;k++)
{
    cgk = 2.0*(v0+k-1.0)*cg1/z1+cg0;
    ckv[k] = cgk;
    cg0 = cg1;
    cg1 = cgk;
    CBKv=cgk;
}
if (real(z) < 0.0)
{
for (k=0;k<=n;k++)
{

cvk = exp((k+v0)*M_PI*cii);
if (imag(z) < 0.0)
{

ckv[k] = cvk*ckv[k]+ldouble(M_PI)*cii*civ[k];
civ[k] /= cvk;
}
else if (imag(z) > 0.0)
{

ckv[k] = cvk[k]/cvk-ldouble(M_PI)*cii*civ[k];
civ[k] *= cvk;
}
}

CBKv=civ[n];

return CBKv;;
}
D.4 Function for the solutions $\phi_\lambda$ and $\psi_\lambda$

//Implementing equation 15

ldcomplex pssilambda(ldcomplex la, ldouble ss, ldouble bb, ldouble sg, ldouble r1, ldouble q1, ldouble y)

//ss=S; bb=beta; sg=sigma;; r1=risk free rate; q1=dividend yield; la=lambda
{
    ldouble ld1=1.0;
    ldouble dd, ll;
    dcomplex qi;
    dcomplex kbm;
    ldouble x, m, epslon, aa, mb, miu;
    dcomplex k;
    ldouble z, v;
    dd = sg / pow(ss, bb); //delta
    miu = r1 - q1; //miu
    mb = miu * bb;
    x = (abs(miu) * pow(y, (-2.0 * bb))) / (pow(dd, ldouble(2.0)) * abs(bb));
    z = pow(y, -bb) / (dd * abs(bb));
    epslon = sign(miu, bb);
    m = 0.25 / abs(bb);
    ll = 2.0 * abs(mb);
    k = epslon * (ldouble(0.5) + (0.25 / bb) - (la / ll));
v = 0.5 / abs(bb);

kbm = sqrt(ldouble(2.0)*la)*z; // to use in bessel function

if (bb < 0 && miu != 0) qi = pow(y, (bb + ldouble(0.5))) * exp(epsilon*x/2) * Whittaker_M(k, m, x);
else if (bb > 0 && miu != 0) qi = pow(y, (bb + ldouble(0.5))) * exp(epsilon*x/2) * Whittaker_W(k, m, x);
else if (bb < 0 && miu == 0) qi = pow(y, ldouble(0.5)) * cbessIv(v, kbm);
else qi = pow(y, ldouble(0.5)) * cbess Kv(v, kbm);
return qi;
}

// Implementing equation 16
ldcomplex … lambda(ldcomplex la, ldouble ss, ldouble bb, ldouble sg, ldouble r1, ldouble q1, ldouble y)

// ss = S; bb = beta; sg = sigma; r1 = risk free rate; q1 = dividend yield; la = lambda
{
ldouble ld1 = 1.0;
ldouble dd, ll;
dcomplex fi;
dcomplex kbm;
ldouble x, m, epsilon, aa, mb, miu;
dcomplex k;
ldouble z, v;
dd=sg/pow(ss,bb);//delta  
miu=r1-q1;//miu  
mb=miu*bb;  
x=(abs(miu)*pow(y,(-2.0*bb)))/(pow(dd,ldouble(2.0))*abs(bb));  
z=pow(y,-bb)/(dd*abs(bb));  
epslon = sign(miu, bb);  
m=0.25/abs(bb);  
ll=2.0*abs(mb);  
k=epslon*(ldouble(0.5)+ (0.25/bb))-(la/ll);  
v=0.5/abs(bb);  
kbm = sqrt(ldouble(2.0)*la)*z; // to use in bessel function  
if (bb<0 && miu !=0) fi = pow(y,(bb+ldouble(0.5))) * exp(epslon*x/2) * Whittaker _W(k,m,x);  
else if (bb >0 && miu!=0) fi=pow(y,(bb+0.5)) * exp(epslon*x/2) * Whittaker _M(k,m,x);  
else if (bb <0 && miu==0) fi=pow(y,ldouble(0.5)) * cbessKv(v,kbm);  
else fi= pow(y,ldouble(0.5)) * cbessIv(v,kbm);  
return fi;  
}  

Implementing Integrals $I_{\lambda}(K, A, B)$ and $J_{\lambda}(K, A, B)$  

//Implementig the Integral I - equation B-27  

dcomplex integrall(ldcomplex la,ldouble ss, ldouble kp, ldouble bb, ldouble sg, ldouble r1, ldouble q1, ldouble AA, ldouble BB)
//ss=S; bb=beta; sg=sigma; r1=risk free rate; q1=dividend yield; AA=down barrier; 
BB=up barrier; la = lambda

{
  ldouble m, ypsA, ypsB, ypsoA, ypsoB, cc, mb, cst, epslon;
  ldouble v, miu, ll;
  dcomplex IL;
  ldouble dd;
  ldouble ld1=1.0;
  dcomplex k, csto, kbmyA, kbmyB, kbmyoA, kbmyoB;
  miu=r1-q1;//miu
  dd=sg/pow(ss,bb);//delta
  m=0.25/abs(bb);
  mb=miu*bb;
  epslon=sign(miu,bb);
  ll=2.0*abs(mb);
  k=epslon*(ldouble(0.5)+ (0.25/bb))-(la/ll);
  ypsA= abs(miu)*pow(AA,(-2*bb))/(pow(dd,2)*abs(bb)); //definition of yA
  ypsB= abs(miu)*pow(BB,(-2*bb))/(pow(dd,2)*abs(bb)); //definition of yB
  cc=dd*sqrt(abs(mb)); // to use in constant in case miu!=0
  v=0.5/abs(bb);//order of Bessel function
  ypsoA=pow(AA,-bb)/(dd*abs(bb));
  ypsoB=pow(BB,-bb)/(dd*abs(bb));
cst=1/cc; // definition of the constant to multiplicate in case mu!=0
csto=ldouble(2.0)/(dd*sqrt(ldouble(2.0)*la)); // constant to multiplicate in case mu=0
kbmyoA=sqrt(ldouble(2.0)*la)*ypsoA; // to use in bessel with mu=0
kbmyoB=sqrt(ldouble(2.0)*la)*ypsoB; // to use in bessel with mu=0

if (bb < 0 && mu > 0) IL= cst*((pow(BB,ldouble(0.5)) * exp(ypsB/2) * Whittaker _M(k + ldouble(0.5),m+ldouble(0.5),ypsB) / (2*m + 1)) - (2*m*kp*pow(BB,ldouble(-0.5)) * exp(ypsB/2) * Whittaker _M(k+ldouble(0.5),m-0.5,ypsB) / (m-k -ldouble(0.5))) - ((pow( AA, ldouble(0.5)) * exp( ypsA/2) * Whittaker _M(k + ldouble(0.5),m+ldouble(0.5),ypsA) / (2*m+1)) - (2*m*kp*pow(AA,ldouble(-0.5)) * exp(ypsA/2) * Whittaker _M(k+ldouble(0.5), m-0.5,ypsA) / (m-k-ldouble(0.5)))));

else if (bb < 0 && mu < 0) IL= cst*((pow(BB,ldouble(0.5)) * exp(-ypsB/2) * Whittaker _M(k-ldouble(0.5),m+0.5,ypsB) / (2*m+1)) + (2*m *kp * pow( BB, ldouble(-0.5)) * exp(-ypsB /2) * Whittaker _M(k-ldouble(0.5),m-0.5,ypsB) / (m+k-ldouble(0.5))) - ((pow( AA, ldouble(0.5)) * exp(-ypsA/2) *Whittaker _M(k-ldouble(0.5),m+0.5,ypsA)/(2*m+1)) + (2*m *kp * pow( AA, ldouble(-0.5)) * exp(-ypsA/2) * Whittaker _M(k-ldouble(0.5),m-0.5,ypsA) / (m+k -ldouble(0.5)))));

else if (bb > 0 && mu > 0) IL= cst*((pow(BB,ldouble(0.5)) * exp(-ypsB/2) * Whittaker _W(k-ldouble(0.5),m-0.5,ypsB)) + (kp *pow( BB, ldouble(-0.5)) * exp(-ypsB/2) * Whittaker _W(k-ldouble(0.5),m+0.5,ypsB)) - ((pow( AA, ldouble(0.5)) * exp(-ypsA/2) * Whittaker _W(k-ldouble(0.5),m-0.5,ypsA)) + (kp*pow(AA, ldouble(-0.5)) * exp(-ypsA/2) * Whittaker _W(k-ldouble(0.5),m+0.5,ypsA)));

else if (bb > 0 && mu < 0) IL= cst*((pow(BB,ldouble(0.5)) * exp(ypsB/2) * Whittaker _W(k+ldouble(0.5),m-0.5,ypsB) / (k -m +ldouble(0.5))) - (kp*pow(BB, ldouble(-0.5)) * exp(ypsB/2) * Whittaker _W(k+ldouble(0.5),m+0.5,ypsB) / (m+k+ldouble(0.5))) - ((pow( AA, ldouble(0.5)) * exp(ypsA/2) * Whittaker _W(k+ldouble(0.5),m-0.5,ypsA) / (k-m+ldouble(0.5))) - (kp*pow(AA, ldouble(-0.5)) * exp(ypsA/2) * Whittaker _W(k + ldouble(0.5), m+0.5,ypsA) / (m+k+ldouble(0.5)))));

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else if (bb < 0 && miu == 0) IL = csto* ((pow(BB, 0.5-bb)*cbessIv(v+1, kbmyoB) - (kp * pow(BB,-0.5-bb)) * cbessIv( v-1, kbmyoB)) - (( pow( AA, 0.5-bb) * cbessIv(v+1,kbmyoA) - (kp*pow(AA,-0.5-bb)) * cbessIv( v-1, kbmyoA))));

else IL = csto* ((pow(BB,0.5-bb)*cbessKv(v-1, kbmyoB) - kp*pow(BB,-0.5-bb) * cbessKv( v+1, kbmyoB)) - (pow(AA, 0.5-bb) * cbessKv(v-1, kbmyoA) - kp * pow(AA,-0.5-bb) * cbessKv( v+1, kbmyoA))));

return IL;
}

//Implementig the Integral J - equation B-28
ldcomplex integralJ(ldcomplex la, ldouble ss, ldouble kp, ldouble bb, ldouble sg, ldouble r1, ldouble q1, ldouble AA, ldouble BB)

//ss=S; bb=beta; sg=sigma; r1=risk free rate; q1=dividend yield; AA=down barrier; BB=up barrier;la= lambda
{
ldouble m, ypsA, ypsB, ypsoA, ypsoB, cc, mb, cst, epsilon;
ldouble v, miu, ll;
dcomplex JL;
ldouble dd;
ldouble ld1=1.0;
dcomplex k, csto, kbmyA, kbmyB, kbmyoA, kbmyoB;
miu=r1-q1;//miu
dd=sg/pow(ss,bb);//delta
m=0.25/abs(bb);
mb = miu * bb;

epsilon = sign(miu, bb);

ll = 2.0 * abs(mb);

k = epsilon * (ldouble(0.5) + (0.25/bb)) - (la/ll);

ypsA = abs(miu) * pow(AA, (-2 * bb)) / (pow(dd, 2) * abs(bb)); // definition of yA

ypsB = abs(miu) * pow(BB, (-2 * bb)) / (pow(dd, 2) * abs(bb)); // definition of yB

cc = dd * sqrt(abs(mb)); // constant to use in case miu! = 0

v = 0.5 / abs(bb); // order of Bessel function

ypsoA = pow(AA, -bb) / (dd * abs(bb));

ypsoB = pow(BB, -bb) / (dd * abs(bb));

cst = 1 / cc; // definition of the constant to multiplicate in case miu! = 0

csto = ldouble(2.0) / (dd * sqrt(ldouble(2) * la)); // definition of the constant to multiplicate in case miu = 0

kbmyoA = sqrt(ldouble(2.0) * la) * ypsoA; // to use in bessel with miu = 0

kbmyoB = sqrt(ldouble(2.0) * la) * ypsoB; // to use in bessel with miu = 0

if (bb < 0 && miu > 0) JL = cst * ((pow(BB, ldouble(0.5)) * exp(ypsB / 2) * Whittaker _W(k + ldouble(0.5), m + 0.5, ypsB) / (k + m + ldouble(0.5))) - (kp * pow(BB, -ldouble(0.5)) * exp(ypsB / 2) * Whittaker _W(k + ldouble(0.5), m - 0.5, ypsB) / (k - m + ldouble(0.5))) - ((pow(AA, ldouble(0.5)) * exp(ypsA / 2) * Whittaker _W(k + ldouble(0.5), m + 0.5, ypsA) / (k + m + ldouble(0.5))) - (kp * pow(AA, -ldouble(0.5)) * exp(ypsA / 2) * Whittaker _W(k + ldouble(0.5), m - 0.5, ypsA) / (k - m + ldouble(0.5)))));

else if (bb < 0 && miu < 0) JL = cst * ((pow(BB, ldouble(0.5)) * exp(-ypsB / 2) * Whittaker _W(k - ldouble(0.5), m + 0.5, ypsB)) + (kp * pow(BB, ldouble(-0.5)) * exp(-ypsB / 2) * Whittaker _W(k - ldouble(0.5), m - 0.5, ypsB)) - ((pow(AA, ldouble(0.5)) * exp(-ypsA / 2) * Whittaker _W(k - ldouble(0.5), m + 0.5, ypsA)) + (kp * pow(AA, -ldouble(0.5)) * exp(-ypsA / 2) * Whittaker _W(k - ldouble(0.5), m - 0.5, ypsA)) - ((pow(BB, ldouble(0.5)) * exp(ypsB / 2) * Whittaker _W(k + ldouble(0.5), m + 0.5, ypsB)) - (kp * pow(BB, ldouble(-0.5)) * exp(ypsB / 2) * Whittaker _W(k + ldouble(0.5), m - 0.5, ypsB)) - ((pow(AA, ldouble(0.5)) * exp(ypsA / 2) * Whittaker _W(k + ldouble(0.5), m + 0.5, ypsA)) - (kp * pow(AA, -ldouble(0.5)) * exp(ypsA / 2) * Whittaker _W(k + ldouble(0.5), m - 0.5, ypsA))));

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Whittaker _W(k-ldouble(0.5),m+0.5,ypsA)) + (kp * pow(AA, ldouble(-0.5)) *exp(-ypsA/2) * Whittaker _W(k-ldouble(0.5),m-0.5,ypsA)));

    else if (bb >0 && miu >0) JL= cst*((2*m*pow(BB,ldouble(0.5)) * exp(-ypsB/2) *Whittaker _M(k-ldouble(0.5),m-0.5,ypsB) / (k +m - ldouble( 0.5))) + (kp*pow(BB, ldouble(-
0.5)) * exp(-ypsB/2) * Whittaker _M(k-ldouble(0.5),m+0.5,ypsB)/(2*m+1.0)) - ((2 * m * pow(AA, ldouble(0.5)) * exp(-ypsA/2) *Whittaker _M(k-ldouble(0.5),m-0.5,ypsA) /
(k+m-ldouble(0.5))) + (kp * pow(AA, ldouble(-0.5)) * exp(-ypsA/2) *Whittaker _M(k-ldouble(0.5),m+0.5,ypsA)/(2*m+1.0))));

    else if (bb >0 && miu <0) JL= cst*(((2 * m * pow(BB, ldouble(0.5)) * exp(ypsB/2) * Whittaker _M(k+ldouble(0.5),m-0.5,ypsB) / (k-m + ldouble( 0.5))) + (kp*pow(BB, ldouble(-0.5)) * exp(ypsB/2) * Whittaker _M(k+ldouble(0.5),m+0.5,ypsB) / (2*m+1.0)) -
((2*m*pow(AA, ldouble(0.5)) * exp(ypsA/2) * Whittaker _M(k+ldouble(0.5),m-0.5,ypsA) /
(k-m+ldouble(0.5))) + (kp * pow(AA, ldouble(-0.5)) *exp(ypsA/2) * Whittaker _M(k +
ldouble(0.5), m+0.5,ypsA) / (2*m+1.0)))));

    else if (bb <0 && miu ==0) JL= csto*((-pow(BB,0.5-bb) * cbessKv(v+1, kbmyoB)) + (kp*pow(BB,-0.5-bb) * cbessKv(v-1, kbmyoB)) -((-pow(AA,0.5-bb) * cbessKv(v+1, kb-
myoA)) + (kp*pow(AA,-0.5-bb) * cbessKv(v-1,kbmyoA))));

    else JL= csto*((-pow(BB,0.5-bb) * cbessIv(v-1, kbmyoB)) + (kp*pow(BB,-0.5-bb) * cbessIv(v+1, kbmyoB)) - ((-pow(AA, 0.5 -bb) *cbessIv(v-1,kbmyoA)) + (kp*pow(AA,-
0.5-bb) * cbessIv(v+1,kbmyoA))));

    return JL;

}

D.5 Function for the Wronskian and Δ(A, B)

//Implementig equation 24
ldcomplex wronskian( ldcomplex la, ldouble ss, ldouble bb, ldouble sg, ldouble r1, ldouble q1)

//ss=S; bb=beta; sg=sigma; r1=risk free rate; q1=dividend yield;
{
    ldouble dd,miu,m, mb,epslon, ll;
    ldouble ld1=1.0;
    dcomplex k, wrk;
    miu=r1-q1;//miu
    dd=sg/pow(ss,bb);//delta
    m=0.25/abs(bb);
    mb=miu*bb;
    epslon=sign(miu,bb);
    ll=2.0*abs(mb);
    k=epslon*(ldouble(0.5)+ (0.25/bb))-(la/ll);
    if (miu!=0) wrk=(2*abs(miu) * gamma(2.0*m+1)) / (pow(dd,2) * cgamma(m-k +ldouble(0.5),0));
    else wrk = abs(bb);
    return wrk;
}

//Implementing equation 13
ldcomplex DELTAAB( ldcomplex la, lddouble ss, lddouble bb, lddouble sg, lddouble r1, lddouble q1, lddouble A1, lddouble B1)
D.6 Laplace transform of the Double Knock Out Barrier option

// Implementing of Laplace transform in equation 25

ldcomplex laplac(ldcomplex lambda, ldouble SS, ldouble bet, ldouble K, ldouble sigm, ldouble rr, ldouble qq, ldouble lo, ldouble up)
{
    double miu, delt;
    ldcomplex cstt1, cstt2, lpl;
    ldouble ld1=1.0;
    miu=rr-qq; // definition of miu
    delt=sigm/pow(SS,bet); // delta
    ldcomplex deltalu = DELTAAB(lambda, SS, bet, sigm, rr, qq, lo, up); // equation (L,U)
ldcomplex deltals = DELTAAB(lambda, SS, bet, sigm, rr, qq, lo, SS);//delta(L,S)
ldcomplex deltasu = DELTAAB(lambda, SS, bet, sigm, rr, qq, SS, up);//delta(S,U)
cstt1 = deltals/(wronskian( lambda, SS, bet, sigm, rr, qq)*deltalu);
cstt2 = deltasu/(wronskian( lambda, SS, bet, sigm, rr, qq)*deltalu);

if (SS <= K) lpl=cstt1*(pssilambda(lambda, SS, bet, sigm, rr, qq,up) *integralJ(lambda, SS, K, bet, sigm, rr, qq, K, up) -filambda(lambda, SS, bet, sigm, rr, qq, up) * integralI(lambda, SS, K, bet, sigm, rr, qq, K, up));
else lpl=cstt1*(pssilambda(lambda, SS, bet, sigm, rr, qq, up) *integralJ(lambda, SS, K, bet, sigm, rr, qq, SS, up) -filambda(lambda, SS, bet, sigm, rr, qq, up) * integralI(lambda, SS, K, bet, sigm, rr, qq, SS, up)) +cstt2*(filambda(lambda, SS, bet, sigm, rr, qq, lo) *integralI(lambda, SS, K, bet, sigm, rr, qq, K, SS) -pssilambda(lambda, SS, bet, sigm, rr, qq, lo) *integralJ(lambda, SS, K, bet, sigm, rr, qq, K, SS));
return lpl;
}

D.7 Program for Euler Abate and Whitt method

//sum of binomial coeff

fvector vfSumOfBinomCoefficients(int m)
{
    int i;
    ldouble K=1<<m;
    fvector vBi(m+1,1.0);
    ldouble dBi=1;
for (i = 1; i < m/2 + 1; ++i)
{
    dBi = (dBi*(m-i+1))/i;
    vBi[i] = vBi[i-1] + dBi;
    vBi[m-i] = K - vBi[i-1];
}
for (i = 0; i < m; ++i)
{
    vBi[i] /= K;
}
sort(vBi.rbegin(),vBi.rend());
return vBi;
}
template < class FunctionObject>
ldouble dfAbateWhittEu(FunctionObject & gg, Z t, Z tol=1e-10, int n=35, int m=12)
{
    // Implement the Euler Abate and Witt method to invert the Laplace Transform
    const static complex<Z> Im1(0,1.0);
    const static Z ld1 = 1.0;
    fvector vWt = vfSumOfBinomCoefficients(m);
    double y1,y2,y3,y4,y5,y6,y7,y8;
    y1 = 100;  // Spot
    y2 = -3;  // Beta
\[ y_3 = 95; // \text{Strike} \]
\[ y_4 = 0.25; // \text{Volatility} \]
\[ y_5 = 0.1; // \text{r-risk free rate} \]
\[ y_6 = 0.1; // \text{q-dividend yield} \]
\[ y_7 = 90; // \text{L-lower barrier} \]
\[ y_8 = 600; // \text{U-upper barrier} \]

\[ \text{ldouble} \ \text{delta} = y_4 / \text{pow}(y_1, y_2); \]

\[ Z \ a, A, f = 0, f_1 = 0; \]
\[ A = -\log(t); \]
\[ a = \exp(A / 2.0) / t; \]
\[ \text{ldcomplex} \ aa(A / (2.0*t), 0.0); \]
\[ \text{ldcomplex} \ dd = (\pi / t) * \text{Im1}; \]
\[ f = \text{real}((\text{gg})(aa, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)) / 2.0; \]

// laplac(lambda, SS, bet, K, sigm, rr, qq, lo, up)
\[ \text{cout} << "\text{Spot} = \"y_1\" \text{endl;} \]
\[ \text{cout} << "\text{Beta} = \"y_2\" \text{endl;} \]
\[ \text{cout} << "\text{Strike} = \"y_3\" \text{endl;} \]
\[ \text{cout} << "\text{Volatility} = \"y_4\" \text{endl;} \]
\[ \text{cout} << "r = \"y_5\" \text{endl;} \]
\[ \text{cout} << "q = \"y_6\" \text{endl;} \]
cout<<"L = "<<y7<<endl;
cout<<"u = "<<y8<<endl;
cout<<"Delta = "<<delta<<endl;

int i,SIGN=1;

for(i=1;i<n+1;++i)
{
    SIGN*=-1;
    f+=(SIGN)*real((gg)(aa+=dd,y1,y2,y3,y4,y5,y6,y7,y8));
}
SIGN=1-2*((n+m+1)%2);
for(i=m;i>0;--i)
{
    f1+=(SIGN*=-1)*real((gg)(aa+i*ld1*dd,y1,y2,y3,y4,y5,y6,y7,y8))*vWt[i];
}
f+=f1;
return f*=a;

D.8 The main function

int main()
{

double ss, bb, mu, rr, qq, sg;
double up, down, kap, t;
dcomplex lamb;
t = 0.5; // Time to maturity

cout << "Pricing Knock Out Barrier Call Options Under The CEV Diffusion"
<< endl;
cout << "Using the Algorithm by Joseph Abate & Ward Whitt" << endl;
cout << "for Numerical inversion of Laplace Transforms" << endl;
cout << "time to expiration (in years) = " << t << endl;
ldouble xx = dfAbateWhittEu(laplac, t, ld1*1e-10, 35, 12);
cout << "AbateWhittEu for laplac = " << xx << endl;
cout << "CEV_VALUE = " << exp((-0.1)*t)*xx << endl;

system("pause");}
References


