Contents

Thanks .................................................................................................................................................. 2
Introduction ......................................................................................................................................... 3

A - Electromagnetic field and geometry .......................................................................................... 7
  A.1 Axiomatic of electromagnetism ................................................................................................. 7
    A.1.1 The axiomatic structure of electromagnetism (based on integration theory) and its relation to
gauge theory ........................................................................................................................................ 7
    A.1.2 Electrodynamics in differential forms (axiomatic approach) ............................................. 17

B – General relativity, gravitoelectromagnetism and the GP-B experiment ..................................... 34
  B.1 Einstein’s gravitational field equations and related topics ...................................................... 34
  B.2 Gravitoelectromagnetism and GP-B experiment .................................................................... 46
    B.2.1 Gravitoelectromagnetism, Linear form of the field equations and gauge invariance. ....... 46
    B.2.2 Gravitoelectromagnetism and the GP-B experiment ....................................................... 54
  B.3 Coupling gravity and the electromagnetic field ....................................................................... 60
    B.3.1 Introduction ....................................................................................................................... 61
    B.3.2 Electromagnetic waves as source of gravitational waves ................................................ 61

C – Gravity and geometry: space-time torsion ............................................................................... 67
  C.1 Introduction to extended theories of gravity .......................................................................... 67
  C.2 Fundamentals of differential geometry and torsion ............................................................... 70
  C.3 Different torsion theories ........................................................................................................ 81
    C.3.1 Riemann-Cartan space-time and Einstein-Cartan theory .................................................. 81
    C.3.2 Teleparallel equivalent of general relativity .................................................................... 87
    C.3.3 New teleparallel gravity .................................................................................................... 94
    C.3.5 Discussion ....................................................................................................................... 95
  C.4 Cartan’s structure equations .................................................................................................... 96
  C.5 Analogies with electromagnetism ............................................................................................ 99
  C.7 Einstein-Cartan and Gauge theories of gravity in differential forms ..................................... 104
  C.8 On the coupling and unification of electromagnetism and gravity ....................................... 106
  C.9 Cosmology with torsion – some examples ............................................................................ 111
  C.10 Testing space-time torsion ..................................................................................................... 117

D - Open questions and final remarks. Physics and geometry .................................................... 125

References ......................................................................................................................................... 130

Bibliography ...................................................................................................................................... 132
Thanks

First of all I wish to thank all teachers that I have encountered in my path since childhood until this present moment. They were crucial in a very personal and deep sense to my learning process. This includes primary school (“A Torre”) which was fundamental to strengthen my link to science as well as to the development of some lines of force that guided my relation with Nature. It also includes all the teachers from secondary school, and naturally all the teachers from the department of physics of the Faculty of Sciences of the University of Lisbon. I wish to give a very warm thank to CAAUL (Center for Astronomy and Astrophysics of the University of Lisbon) and OAL (Astronomical Observatory of Lisbon) and also to CFCUL (Center for Philosophy of Sciences of the University of Lisbon). My special thanks go to Prof. Dr. Paulo Crawford that has accompanied my evolution and has greatly motivated my interests and researches and nurtured them with numerous rich conversations and of course with his lectures. I thank him for his own unique way as a teacher, a supervisor and a friend. More special thanks are to all my friends and most of all to my father, my mother and my brother (and Ché).

For all beings in all times who truly believe and work for the place of education, science, art, philosophy, interculturality, universal rights and peace within advanced and mature civilizations, I give my sincere respect and gratitude.

Francisco Cabral

CAAUL - Center for Astronomy and Astrophysics of the University of Lisbon
Introduction

Physicists believe that there is an underlying simplicity and unity in Nature that can be expressed by mathematical language. Although every physical phenomenon is presented to us as a rather complex pattern of interrelated “events”, the human mind is somehow able to recognize within Nature’s manifestations its underlying mathematical order. From the early thinkers that devoted their reasoning to the nature of motion, and the related conceptions of space and time, to the XXth century theories of gravitation, geometry has always been an essential aspect of this “epistemological bridge” between Nature and scientific knowledge. Although many of the physical and philosophical questions related to matter, motion, space and time remain in some sense open, we strongly believe that there is a deep relation between the geometry of space-time and physical fields! The discovery that the geometry of space-time has a dynamical physical existence is remarkable evidence that the world of geometry penetrates the nature of the physical world in a profound way. Einstein’s theory of general relativity opened the door for a dynamical role of space-time geometry in physics.

As we will see, within the framework of general relativity, the coupling between electromagnetism and space-time geometry (gravity) is done in a very natural manner since the energy-momentum tensor of the electromagnetic field acts as a source of curvature and the metric tensor is implicit in Maxwell’s equations in arbitrary manifolds. In spite of this, an interaction between two fields does not necessarily correspond to a unifying picture, coming from a single mathematical formalism. Kaluza and Klein however found that the gravitational field equations in 5-dimensional “vacuum” can give in a single gesture the 4-d Einstein equations and Maxwell’s equations (under certain assumptions). In this approach, electromagnetic phenomena as well as mass and charge are nothing but manifestations of 5-d geometry (or properties of the 5-d “vacuum”). Although it didn’t introduce any new prediction it showed the power of geometrical methods for “unifying descriptions” and introduced the mechanism of compactification of extra dimensions, a fundamental aspect of string theories. Nevertheless, even in the framework of classical electromagnetism expressed through differential forms and integration theory there is a deep relation between the (geometrical) properties of space-time and the electromagnetic field, via the so called (space-time) constitutive relations. From these relations, one can also show that the light cone is a derived concept and therefore, the causal (conformal) structure of space-time (Riemann or Minkowsky) is derivable from electrodynamics.

There are other ways to explore a single formalism that addresses different physical interactions besides raising the number of space-time dimensions. One possibility is to include other geometrical entities previously unconsidered. In fact, the exploration of geometries beyond the (pseudo) Riemannian allows us to introduce torsion and search for its physical relevance in the construction of alternative theories of gravity and unified modern field theories. Mathematically, this kind of procedure is essentially rooted on the fact that one can arbitrarily add tensors to the Einstein or Christoffel connection (and obtain a general affine connection), that may be regarded as (gauge) fields describing the degrees of freedom of different physical interactions. The issue of torsion will be deeply developed in this work.

In the standard model of physics + gravity there are different types of fields such as spinors (representing fermions), vectors (representing bosons) and tensors (representing gravity – the space-time metric field). There’s also the Higgs mechanism that uses scalar fields in processes of symmetry breaking of unified interactions in the context of cosmological evolution, and correspondingly there are scalar fields as possible candidates to explain an inflation period and the present apparent acceleration of the universe. Now, if we believe that at some high energy scale there is a mathematical description that unifies different sets of phenomena (governed by different dynamical interactions below that energy scale), the usual procedure when we don’t have such theory is to expect some signatures at some low energy limit of the theory. Normally one expects new symmetries and/or new fields! These predictions might be tested experimentally with increasingly more energetic laboratory conditions or in extreme astrophysical conditions. This approach usually includes the prediction of new particles, but it is equally valid to introduce new geometrical structures. So, at this stage one can maintain the common ideas about space-time, and postulate new particles or explore more deeply its geometrical nature. Historically, new ideas about space-time gave us new insights into physics by revealing new phenomena and/or improving our geometrical methods crucial for modern physics. It is also in this context that extended theories of gravity with torsion might be relevant.

The dynamics of elementary particles is successfully explained through gauge theories. These are field theories in which the Lagrangian is invariant under a continuous group of local transformations. In the appendix I address with more detail the basics on gauge theory, but it is worthwhile to briefly review its importance in modern physics and the importance of geometrical methods in establishing a common framework to deal with different interactions. The transformations - gauge transformations - form a Lie group which is the symmetry group or the gauge group of the theory. Associated with any Lie group is the Lie algebra of group generators and for each group generator there is a corresponding vector field called the gauge field. Gauge fields (or gauge potentials) included in the Lagrangian ensure its invariance under the local group of transformations - gauge
invariance – and represent the interaction under study. If the symmetry group is non-commutative, it is referred to as non-Abelian, like in the Yang–Mills theory that is briefly addressed in the appendix. When the theory is quantized, the quanta of the gauge fields are called gauge bosons. Quantum electrodynamics is an Abelian gauge theory with the symmetry group U(1) (one parameter, unitary group) and has one gauge field - the electromagnetic 4-potential (a vector field), with the photon being the gauge boson. The Standard Model is a non-Abelian gauge theory with the symmetry group U(1)×SU(2)×SU(3) and have a total of twelve gauge bosons: the photon, three weak bosons and eight gluons.

When Lagrangians are invariant under a transformation identically performed at every point in the space in which the physical processes occur, they are said to have a global symmetry (global invariance). The requirement of local symmetry, the cornerstone of gauge theories, is a stricter constraint. Local gauge symmetries can be viewed as analogues of the equivalence principle of general relativity in which at each point in space-time is allowed a choice of a local reference (coordinate) frame - local Lorenz invariance. Both symmetries reflect a redundancy in the description of a system.

Historically, these ideas were first stated in the context of classical electromagnetism and later in general relativity, however the modern importance of gauge symmetries appeared first in quantum electrodynamics. Today, gauge theories are useful in condensed matter, nuclear and high energy physics, in different models of gravity, unifying theories and pure mathematics.

Given the importance of the gauge formalism in the description of electromagnetic and (generalized) gravitational classical theories, as well as in clarifying their common formal structures, a natural motivation may arise to identify the properties of space-time geometry as the main corner-stones for the theoretical and experimental analogies between these physical interactions. It is convenient to express electromagnetism in a general manifold exploring the link with space-time geometry, and on the other hand, to explore within general relativity and other theories of gravity analogies with the electromagnetic formalism. Nevertheless, the gauge approach to electrodynamics deals with properties of gauge fields which represent the electromagnetic field, and with matter fields, it does not reflect properties of space-time. Contrastingly, space-time properties are somehow reflected in the constitutive relations between the field’s strengths (E, B) and the excitations (D, H) as we will see. Even in the electromagnetic gauge theory these relations must be postulated [1].

One can ask the following question: Is it possible to translate the electromagnetic gauge invariance in terms of space-time symmetries, as Weyl intended? This would immediately put electromagnetism and gravity on equal footings. The original interpretation of gauge invariance from Weyl changed with the development of quantum theory and Yang-Mills theory. Many aspects of the initial formalism remained but it was clear that the gauge symmetries found in quantum theory didn’t require new space-time symmetries but rather internal symmetries in internal local spaces. In fact, later developments on gauge theory after Weyl’s first contributions, made clear that there is a common geometrical approach within gauge theories and this is related to the idea of identifying the gauge potential as a connection relating quantities in internal spaces at different space-time points. The (gauge) symmetries of these local internal spaces are reflected on the dynamics due to the fact that the connection acts as the generators of the symmetry group and with the connection one builds a covariant derivative appearing in the dynamical equations. In this way the local gauge symmetries are linked to the properties of the physical interaction. Gauge theories link symmetries in physics (group theory) and the dynamics of physical interactions (field equations) via geometrical concepts.

Although as it appears electromagnetism does not rely on space-time symmetries (in contrast to gravity) there are strong analogies between gravitation and electromagnetism. These are clear in the classical descriptions of electrostatics and Newtonian gravitation and also as a result that Einstein’s description of gravity can be explored in the linear regime where we recover equations in almost complete analogy with Maxwell’s equations and the Lorenz force: the so called gravitoelectromagnetism!

The fact that gravity was geomterized can motivate the search for a similar geometrical model for electromagnetism. One problem that naturally arises in any attempt to describe the motion of charges in electromagnetic fields in a similar way as it was done for test particles in a gravitational field (geometrically), is the fact that in electromagnetism, in contrast to gravity, we do not have any sort of (weak) equivalence principle. First of all, charge is not a universal property (there are neutral charges) and in the standard interpretation of this physical quantity, it does not have anything to do with inertia. As far as we know, classically the motion of a charge in an electromagnetic field depends not only on the initial conditions but also on the charge itself. We cannot cut the charge out of the equations of motion.

If we could in fact conclude that the motion should be independent of the inherent properties of the test body and if this was confirmed by experiments, then we could very naturally introduce a geometrical (space-time) explanation for such independence. Alternatively one could take the point of view that a charged “test particle” is in fact moving in a field that results from the external field and the one that itself creates, meaning that it is possible in principle to reconcile the idea of electromagnetism as some sort of space-time (geometrical) deformation with the fact the different charges have different trajectories in the same external field. One can also speculate about the nature of charge itself searching for a completely
geometrical approach: for example, there have been some suggestions that electric charge might be a manifestation of a “topological aspect” of space-time geometry [2].

It might be relevant to explore if there is some deep relation between charge and mass in non-point like elementary charges (like electrons). What is the charge distribution within the electron and how is it related to the mass distribution? Are they coupled? If we assume a continuous distribution throughout a finite volume of a 3d (classical) “particle”, this issue also puts into evidence the problem of continuity of a physical substance (and of space-time): What is the fundamental nature of this physical property that we call charge (or mass)? Is it continuously distributed within the “elementary” particle? What is the meaning of “an infinitesimal charge” (or mass)? What is it made of? Is it made of more fundamental or “elementary” objects? Is it essentially space-time, and if so what is space-time? Is there a low-limit threshold for space-time distances?

Modern physics is moving through very deep questions regarding the nature of the physical realm. These profound questions about the nature of matter, space-time and energy are also within the field of physical ontology. Nevertheless, it seems to me that these questions can only be appropriately formulated if we take into account the challenging and still unresolved “problems” of quantum physics regarding the wave-particle duality and the quantum state reduction as well as the quantum gravity issue. In fact, the nature of charge (and mass) is a central question in modern physical theories and some attempts to unify the four known interactions, such as string theories, admit that the properties of the so called elementary particles may have ultimately a geometrical nature. It should also be underlined that the hypothetical Higgs boson may have recently been detected in the Large Hadron Collider at CERN and this important result strengthens the idea that the different masses of the particles can result from (different) interactions with the Higgs field. It provides a universal mechanism to explain the inertia of particles and it is relevant for the processes of symmetry breaking in the early universe. Although the question about the nature of mass is transferred to the Higgs field itself, it seems to me that this mechanism for providing the inertia of “elementary particles” together with the weak equivalence principle may also give a useful mechanism to the elaboration of a consistent theory of quantum gravity. Is the Higgs “all there is”? Can the quantum nature of the Higgs field provide the quantum structure of space-time? Can gravity, electromagnetic fields, nuclear fields and all the elementary particles be seen as “organized” manifestations of this fundamental field? If so, what role for geometric structures such as curvature and torsion? These are issues that I find very interesting to explore.

The questions about the nature of space-time, matter and energy fields are strongly intertwined to the search for a deeper understanding of the fundamental relation between nature and geometry. Some light on the meaning of the existence of a mathematical structure in physical knowledge might be hidden in this fundamental relation.

There is a natural tendency in physics for searching a simplified and unifying picture of nature and in the history of physics analogical though has revealed to be very useful (for instance in Schrodinger’s search for a quantum wave equation). In this sense, the strong analogies between gravity and electromagnetism should be explored using the appropriate geometrical methods. Progress in this topic might bring interesting theoretical and technological applications. It is also relevant for relativistic astrophysics (cosmology) and possibly for the issue of finding a consistent quantum theory of gravity.

An exploration of analogies between different sets of phenomena may include new physical hypothesis. These must be formulated in a clear and consistent mathematical (quantitative) model and tested experimentally. The new model must include the old one within some range of validity, and may predict potentially new, unknown phenomena.

Part A of this work begins with the axiomatic structure of electromagnetism expressed in differential forms and integration theory and its relation to gauge theory. Some considerations regarding the issue of connecting electromagnetism and space-time geometry are briefly outlined. It is enhanced the fact that the conformal structure of space-time comes out of this formalism under some hypothesis.

Part B is devoted to General relativity (GR), gravitoelectromagnetism, the GP-B experiment and the coupling of gravitational and electromagnetic waves. GR is introduced, some related issues are discussed and the field equations are derived from a variational principle (including the Palatini approach). The framework of gravitoelectromagnetism and the physical importance of the Gravity probe B experiment, which detected the gravitomagnetic field of Earth with high precision, are exposed. Still in this section, the issue of coupling gravity and electromagnetism is explored with special attention to an interesting coupling between electromagnetic and gravitational waves (via gravitoelectromagnetism).

Part C is devoted to gravity with torsion. It starts with an introduction to the reasons for searching alternative/extended theories of gravity and continues with several issues in differential geometry such as tetrads, connections, curvature and torsion. Several theories with torsion are briefly exposed and compared, also focusing different interpretations for the role of torsion. The differences and equivalence between the teleparallal equivalent of general relativity and Einstein’s theory are also outlined. Cartan’s structure equations are derived and some analogies between gravity and electromagnetism are explored.
using also the *Einstein-Cartan theory*. Finally, some *cosmological applications* and *experimental tests of gravity with torsion* are presented and discussed.

The last chapter is devoted to some final considerations and open questions concerning space-time and physics. It presents some ideas on the role of space-time structures in physics motivated from the present study of gravity and electromagnetism. Some open questions regarding the unification and geometrization of physics with its connection to space-time physicalism are analysed.
A - Electromagnetic field and geometry

A.1 Axiomatic of electromagnetism

Electrodynamics relies on conservation laws and symmetry principles (also known from elementary particle physics). In the classical framework, I present the work of Friedrich W. Hehl and collaborators [1,3] - an axiomatic approach that use the calculus of differential forms, integration theory, Poincaré lemma and Stokes theorem in the context of tensor analysis in 3-d space. I will present two alternative ways of deriving Maxwell’s theory, the first is based on integration theory and the second on the exterior calculus of differential forms. These approaches make clearer the geometrical significance of some electromagnetic quantities and most of the expressions obtained are in fact metric independent (and coordinate independent). The starting point for a formal derivation of Maxwell’s theory comes in the form of the following 4 main axioms (postulates):

• Charge conservation (axiom 1);
• Lorentz force (axiom 2);
• Magnetic flux conservation (axiom 3);
• Linear “space-time relations” or constitutive relations - (axiom 4).

These axioms (which will be explored in this work) allow us to obtain the principal aspects of the theory such as: 1) Maxwell equations; 2) Constitutive relations; 3) Lorentz force. Two additional axioms, related to the energy-momentum distribution of the electromagnetic field, are required for a macroscopic description of electromagnetism (in matter). These are the following:

• Specification of the Energy-momentum distribution of electromagnetic field by means of the energy-momentum tensor (axiom 5);
  (From this one obtains the energy density and the energy flux density - Poynting vector).
• Splitting of the total electric charge and currents in a bound or material component which is conserved and a free or external component - (axiom 6).

I will start with the mathematical preliminaries that are based on the following topics:

I - Integration theory  II - Poincaré lemma  III - Stokes theorem

A.1.1 The axiomatic structure of electromagnetism (based on integration theory) and its relation to gauge theory

I Basics on Integration theory

Integration is an operation that yields coordinate independent values and it requires an integration measure. We are interested in integration over curves, surfaces and three dimensional volumes that are embedded in the three dimensional space. We have to define line, surface and volume elements as integration measures. After, we can search for “natural” objects as integrands that yield through integration coordinate independent physical quantities.

1.1 Integration over a curve and covectors (1-forms) as line integrands

Consider some curve in 3d space,

\[ c(t) = (c^1(t), c^2(t), c^3(t)) \]

With line element given by

\[ dc(t) = \left( \frac{\partial c^1(t)}{\partial t}, \frac{\partial c^2(t)}{\partial t}, \frac{\partial c^3(t)}{\partial t} \right) dt \]

An arbitrary coordinate transformation, \( y^\mu = y^\mu(x^k) \), implies the line element transformation:

\[ dc^\mu = \frac{\partial y^\mu}{\partial x^k} dc^k \]

Now, consider integrating some covector field along the given curve:
\[ \int \alpha_k \, dc^k = \int \left( \alpha_1 \frac{\partial c^1(t)}{\partial t} + \alpha_2 \frac{\partial c^2(t)}{\partial t} + \alpha_3 \frac{\partial c^3(t)}{\partial t} \right) \, dt \]

This is an invariant quantity, since the transformation rule for 1-forms (covectors),

\[ \alpha'_\mu \equiv \frac{\partial x^k}{\partial y^\mu} \alpha_k \]

Ensures the invariance:

\[ \alpha'_\mu \, dc^\mu = \alpha_k \, dc^k \]

Therefore we conclude that:

**Covectors (1-forms) are natural line integrands!**

### 1.2 Integration over a 2d-surface and (contravariant) vector densities as surface integrands

Consider some surface in 3d space \( a(r, s) = (a^1(r, s), a^2(r, s), a^3(r, s)) \). The area and orientation of the infinitesimal surface is characterized by a covariant vector:

\[ da(r, s) = (da_1(r, s), da_2(r, s), da_3(r, s)) \]

This is obtained by the vector product of the 2 “edges” of the surface element:

\[ \left( \frac{\partial a^1(r, s)}{\partial r}, \frac{\partial a^2(r, s)}{\partial r}, \frac{\partial a^3(r, s)}{\partial r} \right) \, dr \times \left( \frac{\partial a^1(r, s)}{\partial s}, \frac{\partial a^2(r, s)}{\partial s}, \frac{\partial a^3(r, s)}{\partial s} \right) \, ds \]

\[ \Rightarrow \quad da_i = \epsilon_{ijk} \frac{\partial a^j}{\partial r} \frac{\partial a^k}{\partial s} \, ds \, dr \]

Consider an arbitrary coordinate transformation \( \{x^\mu\} \rightarrow \{y^\mu\} \). The Levi-Civita components are by definition equal to 1, 0 or -1 in any coordinate system, therefore in general, the Levi-Civita components \( \epsilon_{ijk} \) do not transform according to the usual transformation rule for (0,3) tensors:

\[ \epsilon_{ijk'} \neq \frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} \epsilon_{ijk} \]

This comes from the fact that the determinant of the transformation matrix is, in general, not equal to one:

\[ \left| \begin{array}{ccc} \frac{\partial x^1}{\partial y^1} & \frac{\partial x^1}{\partial y^2} & \frac{\partial x^1}{\partial y^3} \\ \frac{\partial x^2}{\partial y^1} & \frac{\partial x^2}{\partial y^2} & \frac{\partial x^2}{\partial y^3} \\ \frac{\partial x^3}{\partial y^1} & \frac{\partial x^3}{\partial y^2} & \frac{\partial x^3}{\partial y^3} \end{array} \right| = \det \left( \frac{\partial x}{\partial y} \right) = \frac{\partial x^1}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} \epsilon_{ijk} \neq 1 \]

We see that the required transformation is given by:

\[ \epsilon_{ijk'} = \frac{1}{\det \left( \frac{\partial x}{\partial y} \right)} \frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} \epsilon_{ijk} = \frac{1}{\det \left( \frac{\partial y}{\partial x} \right)} \frac{\partial y^i}{\partial x^i} \frac{\partial y^j}{\partial x^j} \frac{\partial y^k}{\partial x^k} \epsilon_{ijk} \]

Now I can compute the correct transformation rule for the surface element in order to find an appropriate surface integrand:

\[ da_i = \epsilon_{ijk'} \frac{\partial a^j}{\partial s} \frac{\partial a^k}{\partial r} \, ds \, dr \]

\[ = \left| \frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} \epsilon_{ijk} \right| \frac{\partial a^j}{\partial s} \frac{\partial a^k}{\partial r} \, ds \, dr \]

\[ = \left| \frac{\partial a^m}{\partial y^{i'}} \frac{\partial a^m}{\partial y^{j'}} \frac{\partial a^m}{\partial y^{k'}} \epsilon_{ijk} \right| \frac{\partial a^m}{\partial s} \frac{\partial a^m}{\partial r} \, ds \, dr \]
Now we can introduce a suitable integrand with 3 independent components that should transform according to:

$$\beta^h r = \frac{1}{|J|} \frac{\partial y^h}{\partial x^i} \beta^i$$

Transformation rules that involve the determinant of the transformation matrix characterize the so-called (tensor) densities. In this case, we have a vector density. The following surface integration

$$\int \beta^i da_i = \int \int \beta^i \epsilon_{ijk} \frac{\partial a^j}{\partial s} \frac{\partial a^k}{\partial r} dr ds$$

Yields an invariant quantity since

$$\beta^i da_i = \beta^k da_k$$

Therefore we conclude that:

Vector densities are natural surface integrands!

### I.3 Integration over a 3d-volume and scalar densities as volume integrands

Consider a parameterization of a given volume

$$\nu(t,s,r) = (v^1(t,s,r), v^2(t,s,r), v^3(t,s,r))$$

An elementary volume is characterized by three edges:

$$\begin{vmatrix}
\frac{\partial v^1(t,s,r)}{\partial t} & \frac{\partial v^1(t,s,r)}{\partial r} & \frac{\partial v^1(t,s,r)}{\partial s} \\
\frac{\partial v^2(t,s,r)}{\partial t} & \frac{\partial v^2(t,s,r)}{\partial r} & \frac{\partial v^2(t,s,r)}{\partial s} \\
\frac{\partial v^3(t,s,r)}{\partial t} & \frac{\partial v^3(t,s,r)}{\partial r} & \frac{\partial v^3(t,s,r)}{\partial s}
\end{vmatrix} dt dr ds$$

This infinitesimal volume is given by the determinant:

$$dv = \det \left( \frac{\partial v^1}{\partial t}, \frac{\partial v^1}{\partial r}, \frac{\partial v^1}{\partial s}, \frac{\partial v^2}{\partial t}, \frac{\partial v^2}{\partial r}, \frac{\partial v^2}{\partial s}, \frac{\partial v^3}{\partial t}, \frac{\partial v^3}{\partial r}, \frac{\partial v^3}{\partial s} \right) dt dr ds$$

Now, given the fact that, for a general coordinate transformation,

$$\epsilon_{ij\kappa\nu} = \frac{1}{|J|} \frac{\partial x^i}{\partial y^i} \frac{\partial x^j}{\partial y^j} \frac{\partial x^k}{\partial y^k} \frac{\partial x^\kappa}{\partial y^\kappa}$$

We see that:

$$dv' = \epsilon_{ij\kappa\nu} \frac{\partial v^i}{\partial t} \frac{\partial v^j}{\partial s} \frac{\partial v^k}{\partial r} dt dr ds = \int \int \int \frac{\partial x^i}{\partial y^i} \frac{\partial x^j}{\partial y^j} \frac{\partial x^k}{\partial y^k} \frac{\partial y^i}{\partial x^i} \frac{\partial y^j}{\partial x^j} \frac{\partial y^k}{\partial x^k} \frac{\partial v^i}{\partial t} \frac{\partial v^j}{\partial s} \frac{\partial v^k}{\partial r} dt dr ds$$

$$\Rightarrow dv' = \frac{1}{\det \left( \frac{\partial y^i}{\partial x^j} \right)} dv$$

We therefore conclude that a natural volume integrand has one component and transforms according to

$$y' = \frac{1}{\det \left( \frac{\partial y^i}{\partial x^j} \right)} y$$

In this way an integration over the given volume,
\int 
\gamma d\nu = \int \gamma^i j k \frac{\partial v^i}{\partial t} \frac{\partial v^j}{\partial s} \frac{\partial v^k}{\partial r} dt dr ds

Gives an invariant quantity since

\gamma' d\nu' = \gamma d\nu

Scalar densities are natural volume integrands!

Densities are sensitive toward changes of the scale of elementary volumes. In physics they represent additive quantities – extensities - describing how much of a quantity is distributed within a volume or over the surface of a volume. Extensities (represented by densities) are in contrast to intensities which (for our physical purposes) will represent the strength of a physical field.

II Poincare lemma

In what follows I use the following notation:

\begin{align*}
\partial_i f & \quad \text{Gradient of a function} \\
e^{ijk} \partial_j v_k & \quad \text{Curl of a covector} \\
\partial_i v^i & \quad \text{Divergence of a vector (density)}
\end{align*}

Poincare lemma states under which conditions certain mathematical objects can be expressed in terms of derivatives of other objects (potentials). Consider integrands \( \alpha_k, \beta^i, \gamma \) of line, surface and volume integrals, respectively. Let us assume that they are defined in open connected region of three dimensional space.

1. If \( \alpha_k \) is curl free, it can be written as the gradient of a scalar function \( f \),

\[ e^{ijk} \partial_j \alpha_k = 0 \quad \Rightarrow \quad \alpha_i = \partial_i f \]

2. If \( \beta^i \) is divergence free, it can be written as the curl of the integrand \( \alpha_i \) of a line integral,

\[ \partial_i \beta^i = 0 \quad \Rightarrow \quad \beta^i = e^{ijk} \partial_j \alpha_k \]

3. The integrand \( \gamma \) of a volume integral can be written as the divergence of an integrand \( \beta^i \) of a surface integral:

\[ \gamma \text{ is a volume integrand} \quad \Rightarrow \quad \gamma = \partial_i \beta^i \]

III Stokes Theorem

Applied to line integrands \( \alpha_k \) and surface integrands \( \beta^i \), stokes theorem yields the following identities:

\[ \int \nabla \beta^i d\nu = \oint \beta^i d\alpha_i \]

\[ \int \epsilon^{ijk} \partial_j \alpha_k d\nu = \oint \alpha_i d\epsilon^i \]

Now I will proceed with the axiomatic of electrodynamics!
The 4 axioms for electrodynamics:

**Axiom 1 – Charge conservation**

The total charge within a certain volume should give an invariant quantity

\[ Q = \int_V \rho dv \]

\[ \text{Dim}[Q] = \text{As} = \text{C} \quad \text{Dim}[\rho] = \text{As/m}^3 = \text{C/m}^3 \]

The charge density \( \rho \) is a *scalar density* and Poincare lemma implies that:

\[ \rho = \partial_t D^i \]

**Gauss law**

Therefore,

\[ D^i \text{ is a surface integrand – vector density (extensity)} \]

On the other hand electric current density may be implicitly defined through the expression:

\[ I = \int_S j^i da_i \]

\[ \text{Dim}[I] = \text{A} \quad \text{Dim}[J^i] = \text{A/m}^2 \]

\[ J^i \text{ is a surface integrand – vector density (extensity)} \]

Now applying charge conservation one requires that

\[ \int_V \frac{\partial \rho}{\partial t} dv = - \int_V \frac{\partial j^i}{\partial a_i} dv = 0 \]

Applying Stokes theorem we get

\[ \int_V \left( \frac{\partial \rho}{\partial t} + \partial_j j^i \right) dv = 0 \]

\[ \Rightarrow \partial_t \rho + \partial_j j^i = 0 \quad \Leftrightarrow \quad \partial_i (\partial_t D^i + J^i) = 0 \]

Since the quantity \( (\partial_t D^i + J^i) \) is a surface integrand and its divergence vanishes, Poincare lemma states that it should be identical to the curl of a line integrand (a covector):

\[ \partial_t D^i + J^i = \epsilon^{ijk} \partial_j H_k \]

**Maxwell-Ampère law**

We therefore arrive at the following definitions:

\[ D^i - \text{Electric Excitation: vector density (extensity) – natural surface integrand} \]

\[ H_k - \text{Magnetic Excitation: covector (intensity) – natural line integrand} \]
From *charge conservation* alone we obtained the inhomogeneous Maxwell equations:

- **Gauss law** \( \rho = \partial_i D^i \)
- **Maxwell-Ampère law** \( \partial_i D^i + J^i = \varepsilon^{ijk} \partial_j H_k \)

We have obtained \( D^i \) and \( H_k \) from charge conservation and Poincaré lemma without introducing the concept of force. Notice that since charge conservation is valid on microscopic physics, the same is true for the inhomogeneous equations and for the quantities derived. Charge conservation has long ago been empirically established (~1750, Franklin) and today we can catch single electrons and protons and can count them individually [1].

\( D^i, H_k \) are also microscopically valid quantities!

### Axiom 2 – Lorentz force

Taking into account the fact that the work done by some force is a line integral we postulate that our Lorentz force law should be expressed in terms of natural line integrands:

- \( f_i = q(E_i + \varepsilon_{ijk}u^j B^k) \)

As we will see \( B^k \) is a natural surface integrand and therefore a vector density field, while \( E_i \) is a 1-form field. The electric field and magnetic fields are not independent and their components are related through Lorentz transformations between inertial frames. Suppose that in the proper frame of the charge \( q \) there is a magnetic field \( B^k \). We conclude that there is no Lorentz force and therefore no acceleration. This result should be invariant! Recall that in relativity we could write:

\[
\frac{dp}{d\tau} = f \quad \text{invariant equation}
\]

\[
f = \gamma_\nu \left( \frac{1}{c} \vec{f}, \vec{u}, \vec{f} \right) = \gamma_\nu \frac{d}{d\tau} \left( \vec{E}, \vec{B} \right)
\]

In the proper frame we see that

\[
\frac{dp}{d\tau} = f' = 0
\]

Relativity requires that in any inertial system with a relative velocity \( u \) (see figure), there should be no Lorentz force either. In fact an observer in this frame should detect an electric field given by \( E_i = -\varepsilon_{ijk}u^j B^k \). To see that this is true we can compute the square of the 4-force in this inertial frame

\[
\eta(f, f) = \left( \frac{qE^2 \vec{u}}{c} \right)^2 - \left( q\vec{E} + q\vec{u} \times \vec{B} \right)^2 = 0 \quad \Rightarrow
\]

\[
\Rightarrow \frac{q^2}{c^2} (\vec{E} \cdot \vec{u})^2 - q^2 \vec{E}^2 - (2q\vec{E} + q(\vec{u} \times \vec{B})), q(\vec{u} \times \vec{B}) = 0
\]

\[
\Rightarrow \frac{q^2}{c^2} (\vec{E} \cdot \vec{u})(\vec{E} \cdot \vec{u}) - q^2 \vec{E}^2 - 2q^2 \vec{E} \cdot (\vec{u} \times \vec{B}) = 0
\]

This equation is true if \( \vec{E} = -\vec{u} \times \vec{B} \) as can be shown by substitution:

\[
\frac{q^2}{c^2} ((\vec{u} \times \vec{B}), \vec{u})^2 - q^2 (\vec{u} \times \vec{B})^2 + 2q^2(\vec{u} \times \vec{B})^2 - q^2 (\vec{u} \times \vec{B})^2 = 0 \quad \Rightarrow
\]
We have introduced the four electromagnetic field quantities $D^i$, $H^i$ and $E_i$, $B^i$ which are interrelated by physical and mathematical properties [1].

**Axiom 3 – Magnetic flux conservation**

Helmholtz works on hydrodynamics enabled to conclude that vortex lines are conserved. Vortex lines that pierce through a 2-d surface can be integrated over to originate a scalar called circulation. Circulation in a perfect fluid is constant provided the loop enclosing the surface moves with fluid [1]. There is some analogy between the vortex line scenario in hydrodynamics and magnetic flux lines. At microscopic level magnetic flux occurs in quanta and the corresponding magnetic flux unit is called flux quantum or fluxon. One fluxon carries $\Phi_0 = \frac{h}{2e} \approx 2,07 \times 10^{-15}$ $Wb$ ($h$ is Planck’s constant and $e$ is the elementary electric charge), [1]. Single quantized magnetic flux lines have been observed in the interior of type II superconductors if exposed to sufficiently strong magnetic field and they can be counted! Therefore there is good experimental evidence for the conservation of magnetic flux [1]. Let us assume that magnetic flux:

$$\Phi \equiv \oint S B^i d a_i$$

obeys, in analogy with charge conservation, to a continuity equation

$$\int_S \frac{\partial B^i}{\partial t} d a_i = - \oint \Phi d c^i = 0 \quad \frac{\partial \Phi}{\partial t} + \oint \Phi d c^i = 0$$

We therefore define a magnetic flux current $J^\Phi_i$ which is a natural line integrand – a covector or 1-form.

- **$B^i$ – Magnetic field:** vector density (extensity) - natural surface integrand
- **$J^\Phi_i$ – Magnetic flux current:** convector (intensity) –natural line integrand
Applying Stokes theorem

\[ \oint \int \frac{\partial J^\Phi}{\partial t} \, da_i = \int \epsilon^{ijk} \partial_j J_k^\Phi \, da_i = \int \frac{\partial B^i}{\partial t} \, da_i = -\int \epsilon^{ijk} \partial_j J_k^\Phi \, da_i = \partial_i B^i + \epsilon^{ijk} \partial_j J_k^\Phi = 0 \]

Applying the divergence to the last equation (remembering that the divergence of a curl is zero) and taking into account Poincare lemma with regard to the divergence of a vector density one gets:

\[ \partial_i (\partial_i B^i) = 0 \Rightarrow \partial_i B^i = \rho_{mag} \Rightarrow \partial_t \rho_{mag} = 0 \]

The last conclusion is due to the fact that partial derivatives commute. Here, \( \rho_{mag} \) represents the magnetic charge density – it is a scalar density. Now, suppose we make a general coordinate transformation \( \{x^i, t\} \to \{x'^i, t'\} \)

In general, \( \partial_t \rho_{mag} \neq 0 \), so we require that

\[ \rho_{mag} = 0 \]

Therefore, from magnetic flux conservation we derived:

\[ \partial_i B^i = 0 \]

Gauss law for magnetism

Note that \( J_k^\Phi \) and the electric field strength have the same geometrical properties (their both covectors, natural line integrands) and the same physical dimension (The S.I. units, V/m, correspond to magnetic flux/(time*length)). It is plausible to make (in accordance with Lenz rule): \( J_k^\Phi = E_k \). Therefore Faraday’s induction law reflects magnetic flux conservation!

\[ \partial_i B^i + \epsilon^{ijk} \partial_j E_k = 0 \]

Faraday law

Notice that in the rest frame of a magnetic flux line \( B^{i'} \) we have \( J_{kr}^\Phi = 0 \). Suppose that in this frame \( E_{kr} = 0 \) then, \( f_{i'} = 0 \)

A Lorentz transformation together with \( \partial_i B^i = 0 \) imply that in the lab frame we have:

\[ J_i^\Phi = -\epsilon_{ijk} u^j B^k \]

And therefore

\[ E_i = -\epsilon_{ijk} u^j B^k \Rightarrow J_k^\Phi = E_k \]

Axiom 4 - Linear “space-time relations” (constitutive relations)

So far we obtained \( 1+3+1+3 = 8 \) partial differential equations for the 12 unknowns \( D^i, E_i, H_i, B^i \). In fact only 6 are dynamical equations! The other 2 are “constraints” in the sense that they are fulfilled at all times (by virtue of the truly dynamical equations) if fulfilled at one time. To make Maxwell equations a determined set of partial differential equations we need the so called constitutive relations between the excitations \( D^i, H_i \) and the field strengths \( E_i, B^i \). We will consider the simplest case for the constitutive relations for fields in vacuum:

- Invariant under translation and rotation;
- Local and linear;
- Should not mix electric and magnetic properties.

These features characterize the “vacuum” and not the electromagnetic field itself ([11]).

Now, consider the fact that \( E_i \) and \( H_i \) are natural line integrands so they transform according to the expression:

\[ E_{\mu} = \frac{\partial x^k}{\partial y^\mu} E_k \quad H_{\mu} = \frac{\partial x^k}{\partial y^\mu} H_k \]

On the other hand \( D^i \) and \( B^i \) are vector densities, natural surface integrands, transforming in accordance with:
We introduce the symmetric metric field $g_{ab}$ for the 3d hypersurfaces that determines spatial distances and can provide a notion of orthogonality. Its determinant is denoted by $g$ and it follows that $g^{ij}\sqrt{|g|}$ transforms like a density and maps a covector into a (contravariant) vector density.

Therefore we have (axiom 4):

$$D^i = \varepsilon_0\sqrt{g}g^{ij}E_j$$

$$H_i = (\mu_0\sqrt{g})^{-1}g_{ij}B^j$$

**Space-time constitutive relations**

$$g^{ij}\sqrt{|g|} = \frac{1}{|J|} \frac{\partial x^i}{\partial y^j} \frac{\partial x^j}{\partial y^i} g^{ij}\sqrt{|g|}$$

$$= \frac{\partial x^i}{\partial y^j} = \varepsilon_0|J|\sqrt{|g|}g_{mn}\frac{\partial x^j}{\partial y^m} \frac{\partial y^i}{\partial y^m}$$

$$= \varepsilon_0|J|\sqrt{|g|}g_{mn}\frac{\partial x^j}{\partial y^m} \frac{\partial y^i}{\partial y^m} E_j = \varepsilon_0|J|\sqrt{|g|}g_{mn}\frac{\partial x^j}{\partial y^m} \frac{\partial y^i}{\partial y^m} E_j$$

$$\Rightarrow D^i = \varepsilon_0|J|\sqrt{|g|}g^{ij}E_j$$

In flat space and in Cartesian coordinates we recover the familiar vacuum relations between the field strengths $(E_j, B^j)$. The usual interpretation is that the electric and magnetic constants $\varepsilon_0$, $\mu_0$ characterize the vacuum. There dimensions are $\text{Dim} [\varepsilon_0] = \text{As}/\text{Vm}$, $\text{Dim} [\mu_0] = \text{Vs}/\text{Am}$.

The constitutive equations in matter are more complicated and it would be appropriate to derive them, using an averaging procedure, from a microscopic model of matter. This lies within the subject of solid state or plasma physics, for example. Hehl and Obukhov [3] have obtained the constitutive relations of a general linear magnetoelectric medium:

$$D^i = \left( (\varepsilon^{ij} - \varepsilon^{ijk}n_k) E_j + (\gamma^{ij} + s_{ij})B^j \right) + (\alpha - s)\delta^i_j B^j$$

$$H_i = \left( (\mu^{-1}_{ij} - \varepsilon^{ijk}m^k) E_j + (\gamma^{ij} + s_{ij})E_j \right) + (\alpha + s)\delta^i_j E_j$$

- The matrices $\varepsilon_{ij}$ and $\mu^{-1}_{ij}$ are symmetric and have 6 independent components each. They correspond to the permittivity tensor and permeability tensor (reciprocal permeability tensor).

The magnetoelectric cross-term $\gamma^{ij}$, which is trace-free ($\gamma^{ij} = 0$), has 8 independent components. It is related to the Fresnel-Fizeau effects [4]

- The 4-dimensional pseudo-scalar $\alpha$, called *axion*. It corresponds to the perfect electromagnetic conductor of Lindell & Sihvola [5], a Tellegen type structure [6].

- Until now, we have a total of $6+6+8+1=21$ independent components. We can have 15 more components related to dissipation (which cannot be derived from a Lagrangian) the so-called *skewon piece*, namely $3 + 3$ components of $n_k$ and $n^k$ (electric and magnetic Faraday effects), 8 components from the matrix $\tilde{s}^i_j$ (optical activity), which is traceless, and 1 component from the 3-dimensional scalar $s$ (spatially isotropic optical activity).

We end up with the general linear medium with $20 + 1 + 15 = 36$ independent components.

With the introduction of the constitutive relations, the axiomatic approach to classical electrodynamics is completed.
Relation between the axiomatic and the gauge approach

As already mentioned, electromagnetic interaction can be formulated as a gauge field based on corresponding gauge symmetry. Some basic topics in this context are:

- Physical matter fields (which represent electrons, for example) are described microscopically by complex wave functions;
- The arbitrariness of the absolute phase of these wave functions constitutes a one dimensional rotational type symmetry U(1) (the circle group). This is the (gauge) symmetry group of electromagnetic field theory;
- To derive observable quantities we need to define derivatives of the wave functions in a way that is invariant under the gauge symmetry;
- The construction of such “gauge” covariant derivatives requires the introduction of gauge potentials.

The scalar potential \( \phi \) defines a gauge covariant derivative with respect to time \( D_t^\phi \) and the vector potential defines a covariant derivative \( D_i^A \) with respect to the 3 independent directions of space,

\[
D_\mu \equiv \partial_\mu + \frac{iq}{\hbar} A_\mu, \quad A_\mu = (\phi, A_1, A_2, A_3) = (\phi, -A^1, -A^2, -A^3)
\]

The gauge potentials describe an electrodynamical non-trivial situation if their corresponding electric and magnetic field strengths

\[
E_i = -\partial_i \phi - \partial_i A_i, \quad B^i = \epsilon^{ijk} \partial_j A_k
\]

are non-vanishing! The axioms we used find a proper place within the gauge approach [1]. To see this, let us first see the relation between Noether theorem and the first axiom – electric charge conservation.

a) Noether theorem and electric charge conservation (axiom 1)

Laws of nature described by field theories can often be characterized by a Lagrangian density which in the standard case, is a function of the fields of the theory and their first derivatives \( \mathcal{L}(\Psi, \partial_\mu \Psi, \partial_t \Psi) \). There are symmetry and other guiding principles that tell us how to obtain an appropriate Lagrangian density for a given theory. From it one constructs the Lagrangian and the action:

\[
L = \int \mathcal{L}(\Psi, \partial_\mu \Psi, \partial_t \Psi) d\nu, \quad S = \int L dt
\]

Once we have an appropriate Lagrangian density the equations of motion which determine the dynamics of the fields follow from a variational principle applied to the action \( S \) with respect to variations of \( \Psi \) (and its derivatives):

\[
\delta_\Psi S = 0 \quad \Rightarrow \quad \text{Evolution equations for } \Psi
\]

Noether theorem connects the symmetries of a Lagrangian density to conserved quantities:

- Time translation \( \delta_t \mathcal{L} = 0 \) \( \Rightarrow \) energy conservation
- Spatial translations \( \delta_x \mathcal{L} = 0 \) \( \Rightarrow \) momentum conservation
- Spatial rotations \( \delta_w \mathcal{L} = 0 \) \( \Rightarrow \) angular momentum conservation

These symmetries of space-time are called external symmetries. As is well known, Noether theorem also works for internal symmetries – like gauge symmetries. Gauge invariance of the Lagrangian implies a conserved current with an associated charge. Let \( \delta_e \mathcal{L} = 0 \) denote this invariance,

Gauge transformation \( \delta_e \mathcal{L} = 0 \quad \Rightarrow \quad \text{charge conservation}

For electrodynamics, we have to specify in the Lagrangian density that part representing the matter fields corresponding to electrically charged particles. Invariance of this Lagrangian density under the gauge symmetry of electrodynamics yields the conservation of electric charge. We can arrive at electric charge conservation from gauge invariance via the Noether theorem.
b) Minimal coupling and the Lorentz force (axiom 2)

The Lagrangian density of the electrically charged particles has to be gauge invariant. If these are represented by their wave functions, the corresponding Lagrangian density will contain gauge covariant derivatives. If the charged particles are represented by point particles, we have to replace within the Lagrangian density the energy $\varepsilon$ and the momentum of each particle according to:

$$\varepsilon \to \varepsilon + q\phi \quad p_i \to p_i - qA_i$$

In this way we ensure gauge invariance of the Lagrangian density of electrically charged particles. This is the so called “minimal coupling”. Due to minimal coupling, we relate electrically charged particles and the electromagnetic field in a natural way that is dictated by the requirement of gauge invariance. Having ensured gauge invariance of the action $S$, we can derive equations of motion which contain the Lorentz force law. Therefore the Lorentz force is a consequence of the minimal coupling procedure which couples electrically charged particles to the electromagnetic potentials and makes the Lagrangian gauge invariant.

c) Bianchi identity and magnetic flux conservation (axiom 3)

The electromagnetic gauge potentials $\phi$ and $A_k$ are often introduced as mathematical tools to facilitate the integration of the Maxwell equations. Within the gauge approach the gauge potentials are fundamental physical quantities. The mathematical structure of the gauge potentials already implies the homogeneous Maxwell equations and, in turn, magnetic flux conservation!

Magnetic flux conservation, within the gauge approach, appears as the consequence of a geometric identity! The mathematical identity that is reflected in the homogeneous Maxwell equations is a special case of a “Bianchi identity”. Bianchi identities are the result of differentiating a potential twice! For example in electrostatics the familiar result,

$$E_i = \partial_i \phi \implies \varepsilon^{ijk} \partial_j E_k = \varepsilon^{ijk} \partial_j \partial_k \phi = 0$$

is mathematically speaking expressing a Bianchi identity.

d) Gauge approach and constitutive relations (axiom 4)

The gauge approach towards electrodynamics does not reflect properties of space-time. The constitutive equations do reflect properties of space-time. In spite of this, in the gauge approach, the constitutive equations have to be postulated as an axiom in some way. Note that the gauge potentials are directly related to the field strengths $E_i$ and $B_i$, and the excitations $D_i$ and $H_i$ are part of the inhomogeneous Maxwell equations which, within the gauge approach, are derived as equations of motion from an action principle.

Since the action itself involves the gauge potentials, we obtain equations of motion for the excitations rather than for the field strengths because during the construction of the action from the gauge potentials the constitutive equations are already used, at least implicitly.

A.1.2 Electrodynamics in differential forms (axiomatic approach)

The basic postulates are exactly the same [7]:

- Charge conservation
- Magnetic flux conservation
- Lorentz force
- Linear “space-time relations”

The mathematical formalism will now rely on the exterior calculus of differential forms. I will begin with some basics on this topic. One advantage of this formalism is that it shows clearly some results which are completely metric independent.
Mathematical preliminaries – Calculus of differential forms

Suppose a 3-dimensional manifold with a given metric and consider a set of local coordinates \( x^a = \{ x^1, x^2, x^3 \} \). Consider also a coordinate basis for the tangent vector space \( e_a = \{ \partial_1, \partial_2, \partial_3 \} \) and the corresponding dual basis of 1-forms \( dx^a = \{ dx^1, dx^2, dx^3 \} ; dx^a(\partial_j) = \delta^a_j \). A general \( k \)-form on a \( d \)-dimensional manifold has a total of \( \mathcal{C}_{d}^{k} = \frac{d!}{k!(d-k)!} \) independent components which are the components of a completely antisymmetric tensor of the type \((0, k)\). The following expressions are valid for arbitrary 1, 2 and 3 forms in a 3 dimensional space:

<table>
<thead>
<tr>
<th>Form</th>
<th>Mathematical expression</th>
<th>Independent components</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-form</td>
<td>( \varphi = \varphi_1 dx^a = \varphi_1 dx^1 + \varphi_2 dx^2 + \varphi_3 dx^3 )</td>
<td>3</td>
</tr>
<tr>
<td>2-form</td>
<td>( w = \frac{1}{2} w_{ab} dx^a \wedge dx^b = w_{12} dx^1 \wedge dx^2 + w_{23} dx^2 \wedge dx^3 + w_{31} dx^3 \wedge dx^1 )</td>
<td>3</td>
</tr>
<tr>
<td>3-form</td>
<td>( \eta = \eta_{123} dx^1 \wedge dx^2 \wedge dx^3 )</td>
<td>1</td>
</tr>
</tbody>
</table>

Let us take a brief look on some of the important operations over differential forms:

| Tensor (direct) product | \( F \otimes D = H \Rightarrow H_{ijk...}^{abc...} = F_{ijk...}^{abc...} D_{ijk...}^{abc...} \) |
| Exterior product (wedge) | \( F \wedge D \equiv F \otimes D - D \otimes F \) |

Strictly speaking, the Wedge (exterior) product is not a generalization of the vector product. The vector product is a superposition of the wedge product and the Hodge duality operation (which requires the metric on the manifold), [7]. Wedge product is a pre-metric operation and is related to the so called Grassmann Algebra. Other important operations are the exterior differentiation, the interior product and the already mentioned Hodge dual operator.

- **Exterior differentiation** \( d \)
  - Increases the rank of the form by 1;
  - Generalizes the “grad”;
  - Represents a pre-metric extension of the “curl” operator;
  - Is nilpotent: \( dd = 0 \)

- **Interior product of a vector with a k-form**
  - Decreases the rank of the form by 1. It corresponds to a simple contraction, for example:
    \[
    \mathbf{v} \cdot dx^a = v^a \quad (I \text{ will use a dot as notation for interior product})
    \]

- **Hodge dual operator** \( * \)
  - Maps k-forms into \((3-k)\)-forms. Its introduction necessarily requires the metric
  The metric introduces a natural volume 3-form \( \eta \equiv \sqrt{\det(g_{ab})} dx^1 \wedge dx^2 \wedge dx^3 \)
  which underlies the definition of the Hodge operator \( * \).
  Example:
  \[
  * dx^a = \sqrt{\det(g_{ab})} [g^{a1} dx^2 \wedge dx^3 + g^{a2} dx^3 \wedge dx^1 + g^{a3} dx^1 \wedge dx^2]
  \]

The following table shows 3 important operations over a general k-form, namely exterior differentiation, interior product (with a vector) and the Hodge star.
The notions of odd and even forms are related to the orientation of the manifold. They are distinguished by the fact that under a reflection (i.e., a change of orientation) an even form does not change sign whereas an odd form does! This aspect is relevant in the context of integration theory.

• For a k-form an integral over a k-dimensional subspace is defined. For example, a 1-form can be integrated over a curve, a 2-form over a 2-surface, and a volume 3-form over the whole 3-dimensional space.

• Stokes’s theorem in this formalism can be expressed in the following way:

\[ \int_{\partial C} w = \int_{C} dw \]

Here \( w \) is an arbitrary k-form and \( C \) is an arbitrary (k+1) dimensional hyper surface with the boundary \( \partial C \). Now follows a derivation of electrodynamics using the 4 axioms expressed in differential forms.

**POSTULATE 1 - Charge conservation and the inhomogeneous Maxwell equations.**
Charge and current densities can be expressed as forms defined in a 3 dimensional manifold. We have the following definitions:

• **Electric charge and charge density**

\[ Q = \iiint \rho \]

We see that \( \rho \) is necessarily a 3-form and therefore it has \( C^3_3 = 1 \) independent component

\[ \rho = \rho_{123} dx_1 \wedge dx_2 \wedge dx_3 \]

\( \text{Dim} [\rho] = \text{dim} [\rho] = Q \Rightarrow \text{S.I unit: C} \quad \text{Dim} [\rho_{abc}] = \text{Q} \cdot \text{L}^3 \)

• **Electric current and current density**

\[ i = \iint J \]

From this follows that \( J \) is a 2-form which has \( C^3_2 = 3 \) independent components

\[ J = \frac{1}{2} j_{ab} dx^a \wedge dx^b = j_{12} dx^1 \wedge dx^2 + j_{23} dx^2 \wedge dx^3 + j_{31} dx^3 \wedge dx^1 \]

\( \text{Dim} [j] = \text{dim} [J] = QT^{-1} \Rightarrow \text{S.I unit: A=Cs}^2 \quad \text{Dim} [j_{ab}] = QT^{-1} \cdot \text{L}^2 \)

Global and local charge conservation can be expressed by the equations:
Now, since $\rho$ is a 3-form it can be derived from a 2-form $D$, we have:

\[ dD = \rho \]

$D$ is a 2-form with 3 independent components:

\[ D = D_{12} dx^1 \wedge dx^2 + D_{23} dx^2 \wedge dx^3 + D_{31} dx^3 \wedge dx^1 \]

\[ \text{Dim} [D] = \text{dim} [\rho] = Q \Rightarrow \text{S.I unit: C} \quad \text{Dim} [D_{ab}] = Q \cdot L^2 \Rightarrow \text{S.I unit: Cm}^2 \]

On the other hand, this definition and charge conservation implicitly implies the other non-homogeneous electromagnetic equation:

\[ \begin{cases} 
    dD = \rho \\
    \text{Charge conservation} \Rightarrow d(\partial_1 D + J) = 0
\end{cases} \]

\[ \text{Maxwell – Ampère law} \]

\[ \partial_1 D + J = dH \]

So we introduce the magnetic excitation $H$ which is a 1-form with 3 independent components:

\[ H = H_a dx^a = H_1 dx^1 + H_2 dx^2 + H_3 dx^3 \]

\[ \text{Dim} [H] = \text{dim} [J] = QT^{-1} \Rightarrow \text{S.I unit: A = Csec} \quad \text{Dim} [H_a] = QT^{-1} L^{-1} \Rightarrow \text{S.I unit: A} = \text{Am}^{-1} \]

**POSTULATE 2 - Lorentz force and the electric and magnetic field strengths**

Assuming that force is a form field defined in a 3 dimensional manifold, the work-energy relation allows us to conclude that $f$ is a 1-form:

\[ w = \int f \]

$f$ is a 1-form $\Rightarrow$ it as $C_3^1 = 3$ independent components

\[ f = f_a dx^a = f_1 dx^1 + f_2 dx^2 + f_3 dx^3 \]

\[ \text{dim} [f] = \text{ML}^2 T^{-2} \Rightarrow \text{S.I unit: j} = \text{kg.m}^2 \cdot \text{s}^{-2} \quad \text{dim} [f_a] = \text{MLT}^2 \Rightarrow \text{S.I unit: j.m}^1 = \text{kg.m.s}^{-2} \]

**Electromagnetic force 1-form**

We can construct the expression for the Lorentz force on an elementary charge which can be used to define the electric and magnetic field intensities:
Here \( \mathbf{v} \) is a 3-velocity vector, \( \mathbf{v} = v/e_j = v/\partial_j \) of the elementary charge, its absolute dimension is \( T^{-1} \). We therefore can conclude the following regarding the electromagnetic field intensities: \( E \) is a 1-form \( \Rightarrow \) it as \( C_1^3 \) = 3 independent components

\[
E = E_0dx^a = E_1dx^1 + E_2dx^2 + E_3dx^3
\]

\[
\text{Dim } [E] = \text{dim}[e] = \text{QLT}^2 \Rightarrow \text{S.I. unit: j.c.} = (\text{j.s}).\text{c.s}^{-1} \quad (\text{h/tq, h-action})
\]

\[
\text{Dim } [E_0] = \text{QLT}^2 \Rightarrow \text{S.I. unit: j.c.m}^{-1} = (\text{j.s}).\text{c.m}^{-1}\text{s}^{-1} \quad (\text{h/tql})
\]

\( \mathbf{v}.\mathbf{B} \) is a 1-form \( \Rightarrow \) \( \mathbf{B} \) is a 2-form: it as \( C_2^3 \) = 3 independent components

\[
\mathbf{B} = B_{12}dx^1\wedge dx^2 + B_{23}dx^2\wedge dx^3 + B_{31}dx^3\wedge dx^1
\]

\[
\text{Dim } [e(\mathbf{v}.\mathbf{B})] = \text{dim}[f] = \text{MLT}^2 \Rightarrow \text{dim}[\mathbf{B}] = \text{QLT}^2 \Rightarrow \text{S.I. unit: j.s.c}^{-1} \quad (\text{h/q})
\]

\[
\text{Dim } [B_{ab}] = \text{QLMT}^{-1} \Rightarrow \text{S.I. unit: j.s.m}^{-2}.\text{c}^{-1} \quad (\text{h/q}^2)
\]

Lorentz force presupposes charge conservation and should not be seen as a standalone pillar of electrodynamics.

**POSTULATE 3 - Magnetic flux conservation and the homogenous Maxwell equations**

Taking into account the rank of the forms representing the field strengths, the only integrals we can build up from \( E \) and \( \mathbf{B} \) are line and surface integrals respectively. From a dimensional point a view it then seems plausible to postulate the following conservation equation:

\[
\frac{\partial}{\partial t} \int_S \mathbf{B} = -\oint \mathbf{E}
\]

Using Stokes theorem

\[
\oint \mathbf{E} = \int \int d\mathbf{E}
\]

We arrive via magnetic flux conservation at the following:

\[
d\mathbf{E} + \partial_t \mathbf{B} = 0 \quad (3 \text{ independent eq.)}
\]

This equation in turn implies that

\[
d\mathbf{E} + \partial_t \mathbf{B} = 0 \quad \Rightarrow \quad d(d\mathbf{E} + \partial_t \mathbf{B}) = 0 \quad \Rightarrow \quad \partial_t(d\mathbf{B}) = 0
\]

Since an integration constant other than zero is meaningless (recall the relativity principle) we obtain the last homogeneous Maxwell equation:

\[
d\mathbf{B} = 0 \quad (1 \text{ independent eq.)}
\]

Magnetic flux conservation gains evidence from the dynamics of an Abrikosov flux line lattice in a superconductor. There the quantized flux lines can be counted, they do not vanish nor are created from nothing, rather they move in and out crossing the boundary \( \partial S \) of the surface \( S \) under consideration.
As a consequence of magnetic flux conservation, the electromagnetic field is conservative and we can introduce 0-form potential \( \Lambda \)

\[
\oint S \, df = 0 \quad \text{(valid for any arbitrary surface) } \Rightarrow \, df = 0
\]

\[
df = 0 \iff dE + d(\mathbf{v} \cdot \mathbf{B}) = 0 \iff d(\mathbf{E} + (\mathbf{v} \cdot \mathbf{B})) = 0 \Rightarrow \mathbf{E} + (\mathbf{v} \cdot \mathbf{B}) = -d\Lambda
\]

Using \( df = 0 \) and magnetic flux conservation (\( d\mathbf{E} + \partial_t \mathbf{B} = 0 \)) we see that:

\[
\partial_t \mathbf{B} - d(\mathbf{v} \cdot \mathbf{B}) = 0
\]

Therefore, the conservative nature of the electromagnetic force and magnetic flux conservation express the same principle which allow us to introduce the electromagnetic potentials \( \phi \) (0-form) and \( \mathbf{A} \) (1-form):

\[
df = 0 \iff d\mathbf{E} + d(\mathbf{v} \cdot \mathbf{B}) = 0 \iff d\mathbf{E} + \partial_t \mathbf{B} = 0 \iff d\mathbf{E} + \partial_t (d \mathbf{A}) = 0 \iff d(\mathbf{E} + \partial_t \mathbf{A}) = 0
\]

\[
\Rightarrow \quad \mathbf{E} = -d\phi - \partial_t \mathbf{A}
\]

By substitution on the Lorentz force equation we get the following relation \( d\Lambda = (\mathbf{v} \cdot d\mathbf{A}) - d\phi - \partial_t \mathbf{A} \). In summary, we found the following potentials:

\[
\text{Potentials:} \quad \begin{align*}
\Lambda & \quad \text{0-form} \\
\phi & \quad \text{0-form} \\
\mathbf{A} & \quad \text{1-form}
\end{align*}
\quad \begin{align*}
\mathbf{E} & = -d\phi - \partial_t \mathbf{A} \\
\mathbf{B} & = \mathbf{dA} \\
f & = ed\Lambda = e((\mathbf{v} \cdot d\mathbf{A}) - d\phi - \partial_t \mathbf{A})
\end{align*}
\]

Now follows a derivation of the component form for the Lorentz force, expressed in terms of the potentials, on a general manifold with metric.

\[
f_i = e\left(\mathbf{E} + (\mathbf{v} \cdot \mathbf{B})\right) = e\left((-d\phi - \partial_t \mathbf{A}) + (\mathbf{v} \cdot \mathbf{dA})\right)
\]

\[
= e\left(\begin{array}{c}
\left(-\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} + A_m \frac{\partial x^a}{\partial t} \Gamma^m_{ai} + (\mathbf{v} \cdot \mathbf{dA})_i\right) = e\left(\mathbf{E} - \left(-\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} + A_m \frac{\partial x^a}{\partial t} \Gamma^m_{ai} + (\mathbf{v} \cdot \mathbf{dA})_i\right)\right)
\end{array}\right)
\]

\[
= e\left(\begin{array}{c}
\left(-\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} + A_m \frac{\partial x^a}{\partial t} \Gamma^m_{ai} + \left(\frac{1}{2} \frac{\partial A_k}{\partial x^i} - \frac{\partial A_l}{\partial x^k}\right)\mathbf{e}^l \wedge \mathbf{e}^k\right) + (\mathbf{v} \cdot \mathbf{e}^l)\wedge \mathbf{e}^k + (\mathbf{v} \cdot \mathbf{e}^k)\wedge \mathbf{e}^l\right)
\end{array}\right)
\]

\[
= e\left(\begin{array}{c}
\left(-\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} + A_m \frac{\partial x^a}{\partial t} \Gamma^m_{ai} + \left(\frac{1}{2} \frac{\partial A_k}{\partial x^i} - \frac{\partial A_l}{\partial x^k}\right)\mathbf{e}^l \wedge \mathbf{e}^k\right)
\end{array}\right)
\]

\[
f_i = e\left(\begin{array}{c}
\left(-\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} + A_m \frac{\partial x^a}{\partial t} \Gamma^m_{ai} + \frac{1}{2} \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_l}{\partial x^k}\right)\mathbf{e}^l \wedge \mathbf{e}^k\right)
\end{array}\right)
\]

\[
= e\left(\begin{array}{c}
\left(-\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} + A_m \frac{\partial x^a}{\partial t} \Gamma^m_{ai} + \frac{1}{2} \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_l}{\partial x^k}\right)\mathbf{e}^l \wedge \mathbf{e}^k\right)
\end{array}\right)
\]

\[
= e\left(\begin{array}{c}
\left(-\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} + A_m \frac{\partial x^a}{\partial t} \Gamma^m_{ai} + \frac{1}{2} \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_l}{\partial x^k}\right)\mathbf{e}^l \wedge \mathbf{e}^k\right)
\end{array}\right)
\]

\[
= e\left(\begin{array}{c}
\left(-\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} + A_m \frac{\partial x^a}{\partial t} \Gamma^m_{ai} + \frac{1}{2} \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_l}{\partial x^k}\right)\mathbf{e}^l \wedge \mathbf{e}^k\right)
\end{array}\right)
\]

\[
= e\left(\begin{array}{c}
\left(-\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} + A_m \frac{\partial x^a}{\partial t} \Gamma^m_{ai} + \frac{1}{2} \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_l}{\partial x^k}\right)\mathbf{e}^l \wedge \mathbf{e}^k\right)
\end{array}\right)
\]

\[
= e\left(\begin{array}{c}
\left(-\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} + A_m \frac{\partial x^a}{\partial t} \Gamma^m_{ai} + \frac{1}{2} \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_l}{\partial x^k}\right)\mathbf{e}^l \wedge \mathbf{e}^k\right)
\end{array}\right)
\]

\[
= e\left(\begin{array}{c}
\left(-\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} + A_m \frac{\partial x^a}{\partial t} \Gamma^m_{ai} + \frac{1}{2} \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_l}{\partial x^k}\right)\mathbf{e}^l \wedge \mathbf{e}^k\right)
\end{array}\right)
\]

\[
= e\left(\begin{array}{c}
\left(-\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} + A_m \frac{\partial x^a}{\partial t} \Gamma^m_{ai} + \frac{1}{2} \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_l}{\partial x^k}\right)\mathbf{e}^l \wedge \mathbf{e}^k\right)
\end{array}\right)
\]

Maxwell equations nearly complete the construction of the theory. We find 6 evolution equations for the 12 components \((\mathbf{D}, \mathbf{H}; \mathbf{E}, \mathbf{B})\). The required reduction of the number of variables comes from axiom 4.
**POSTULATE 4 - Linear and isotropic space-time relations**

Because \( E \) and \( H \) are 1-forms and \( D \) and \( B \) are 2-forms, we will assume the following linear relations between the electromagnetic intensities and their excitations in vacuum:

\[
D = \varepsilon_0 \cdot E = \varepsilon_0 \varepsilon_0 E_a g^{ab} (e_b \cdot \eta) = \varepsilon_0 \varepsilon_0 E_a g^{ab} \left( e_b \cdot (\sqrt{g}e^1 \wedge e^2 \wedge e^3) \right)
\]

\[
D = \varepsilon_0 \sqrt{g} E_a g^{ab} \left( (e_b \cdot e^1) e^2 \wedge e^3 + (e_b \cdot e^2) e^1 \wedge e^3 + (e_b \cdot e^3) e^1 \wedge e^2 \right)
\]

\[
D = \varepsilon_0 \sqrt{g} E_a g^{ab} \left( g^{a1} e^2 \wedge e^3 + g^{a2} e^1 \wedge e^3 + g^{a3} e^1 \wedge e^2 \right)
\]

\[
\begin{align*}
D_{12} &= \varepsilon_0 \sqrt{g} E_a g^{a3} \\
D_{23} &= \varepsilon_0 \sqrt{g} E_a g^{a1} \\
D_{31} &= \varepsilon_0 \sqrt{g} E_a g^{a2}
\end{align*}
\]

\[
\Rightarrow D_{jk} = \varepsilon_0 \sqrt{g} \varepsilon_{jkl} E_a g^{ai}
\]

Analogously, we have:

\[
B = \mu_0 \cdot H = \mu_0 \sqrt{g} H_a (g^{a1} e^2 \wedge e^3 + g^{a2} e^1 \wedge e^3 + g^{a3} e^1 \wedge e^2)
\]

\[
\begin{align*}
B_{12} &= \mu_0 \sqrt{g} H_a g^{a3} \\
B_{23} &= \mu_0 \sqrt{g} H_a g^{a1} \\
B_{31} &= \mu_0 \sqrt{g} H_a g^{a2}
\end{align*}
\]

\[
\Rightarrow B_{jk} = \mu_0 \sqrt{g} \varepsilon_{jkl} H_a g^{ai}
\]

Remember that the Hodge dual operator maps \( k \)-forms into \((3-k)\)-forms. Its introduction necessarily requires the metric. The metric introduces a natural volume 3-form \( \eta = \sqrt{g} e^1 \wedge e^2 \wedge e^3 \) which underlies the definition of the Hodge operator.

---

**Observation:**

Remember that considering \( D \) and \( B \) as natural surface integrands, they must transform as contravariant vector densities, while \( E \) and \( H \) transform as covariant vectors being natural line integrands. Because \( \sqrt{\det g} g^{ab} \) transforms like a density and maps a covariant vector into a contra variant vector density, the postulate for linear and isotropic constitutive relations for vacuum follows. We therefore have, in this notation:

\[
\begin{align*}
D^i &= \varepsilon_0 \sqrt{\det g} E^i g^{ij} \\
H_i &= (\mu_0 \sqrt{\det g})^{-1} g_{ij} B^j \quad ([1])
\end{align*}
\]

\[
( \mu_0 \sqrt{\det g} g^{ik} H_i = \delta^k_j B^j = B^k )
\]

---

**Inhomogeneous equations for the electromagnetic potentials, on arbitrary manifolds and arbitrary coordinates**

Taking into account the relations between the electromagnetic field strengths \( B = dA, \ E = -d\Phi - \partial_t A \), and the relation between the field excitations and their “sources”, \( \partial_t D + J = dH, \ dD = \rho \), we may find, via the constitutive relations, expressions relating the charge and current distributions, the metric and the potentials. Using Gauss law and the relation between \( E \) and \( D \):

\[
\varepsilon_{ijk}(\partial_i D_{23} + \partial_2 D_{31} + \partial_3 D_{21}) = \rho_{ij}
\]

\[
\partial_k \left( \varepsilon_0 \sqrt{g} \varepsilon_{i} j k E_a g^{a1} \right) = \varepsilon_0 \sqrt{g} \varepsilon_{i} j k \partial_s (E_a g^{a1}) + \varepsilon_0 \varepsilon_{i} j k E_a g^{a1} \partial_s (\sqrt{g})
\]

\[
= \varepsilon_0 \sqrt{g} \varepsilon_{i} j k \left( \varepsilon_s E_a g^{a1} + E_a (\partial_s g^{a1}) + E_a g^{a1} \left( \frac{1}{2} g^{mn} \partial_s g_{mn} \right) \right)
\]

\[
( \partial_k g = g^{ij} \partial_k g_{ij} )
\]

\[
\varepsilon_{ijk} \varepsilon_0 \sqrt{g} \left( (\partial_i E_a) g^{a1} + E_a (\partial_i g^{a1}) + \frac{1}{2} E_a g^{a1} g^{mn} (\partial_i g_{mn}) \right) \epsilon_{i23} + \text{perm} = \rho_{ij}
\]
This expression reduces to a familiar expression \((\varepsilon_{ijk} \varepsilon_0 \varepsilon g \left( \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial \phi}{\partial t} \right) g^{al} (\partial_t g^{al} + \frac{1}{2} (\partial_a \phi - \partial_t A_a) g^{al} g^{mn} (\partial_t g_{mn}) \right) \varepsilon_{i23} + \text{perm}) = -\rho_{sijk}
\)

One can also invert the relation between \(B\) and \(H\):

\[
B_{ijk} = \mu_0 \sqrt{g} \varepsilon_{ijk} H_a g^{ai} \Rightarrow \varepsilon_{ijkl} H_b g^{bil} (\mu_0 \sqrt{g})^{-1} B_{jk} = 2 \delta_i^a H_a g^{ai} = 2 \delta_i^a \varepsilon_{ijk} H_a g^{ai}
\]

\[
= \varepsilon_{ijk} (\mu_0 \sqrt{g})^{-1} B_{jk} \Rightarrow 2 H_a g^{ai} g_{hs} = 2 H_a \delta^a_i = \varepsilon_{ijk} H_{hs} (\mu_0 \sqrt{g})^{-1} B_{jk}
\]

Here I used the fact that \(\varepsilon_{ijk} \varepsilon_{jkl} = 2 \delta_i^l\). We can express the components of \(H\) in terms of the potential \(A\):

\[
H_{(2 \mu_0 \sqrt{g})^{-1} \varepsilon_{ijk} H_{hs} (\partial_i A_k - \partial_k A_i)}
\]

Now, let us develop the other inhomogeneous equation \((\partial_t D + J = dH)\):

\[
(\partial_t (\varepsilon_0 \star E))_{jk} + J_{jk} = (dH)_{jk} \Rightarrow \partial_t (\varepsilon_0 \sqrt{g} e_{ijk} E_a g^{al}) + J_{jk} = \partial_t H_k - \partial_k H_j
\]

\[
= \partial_t (\varepsilon_0 \sqrt{g} e_{ijk} (-\partial_a \phi - \partial_t A_a) g^{al}) + J_{jk} = \partial_t H_k - \partial_k H_j = \partial_t \left( (2 \mu_0 \sqrt{g})^{-1} \varepsilon_{ijkl} g_{kl} B_{mn} \right) - \partial_k \left( (2 \mu_0 \sqrt{g})^{-1} \varepsilon_{ijkl} g_{kl} B_{mn} \right)
\]

\[
\Rightarrow \partial_t (\varepsilon_0 \sqrt{g} e_{ijk} (-\partial_a \phi - \partial_t A_a) g^{al}) + J_{jk} = \partial_t \left( (2 \mu_0 \sqrt{g})^{-1} \varepsilon_{ijkl} g_{kl} B_{mn} \right) - \partial_k \left( (2 \mu_0 \sqrt{g})^{-1} \varepsilon_{ijkl} g_{kl} B_{mn} \right)
\]

\[
\Rightarrow \partial_t (\varepsilon_0 \sqrt{g} e_{ijk} (-\partial_a \phi - \partial_t A_a) g^{al}) + J_{jk} = \varepsilon_{ijkl} g_{ns} (\partial_t g^{al} - \partial_n A_l)
\]

...
Here I used again the result \( \partial_k g = g^{ij} \partial_k g_{ij} \). We therefore get

\[
\varepsilon_0 \varepsilon_{ijk} \sqrt{g} \left( g^{ai} \left( -\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 A_a}{\partial t^2} - \left( \frac{\partial \phi}{\partial x^a} + \frac{\partial A_a}{\partial t} \right) \frac{1}{2} g^{-1} (\partial_i g) g^{ai} \right) - \frac{\varepsilon^mn g^{-1/2}}{2\mu_0} \left( 2 \frac{\partial A_n}{\partial x^m} - \frac{\partial A_m}{\partial x^n} \right) \right) = -f_{jk}
\]

In a flat and unchanging manifold we have

\[
-\frac{\varepsilon_{ijk}}{c^2} \sqrt{g} \left( g^{ai} \left( -\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 A_a}{\partial t^2} - \frac{\partial A_a}{\partial x^a} \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} g^{-1} (\partial_i g) g^{ai} \right) \right) - \frac{\varepsilon^{mn} g^{-1/2}}{2\mu_0} \left( 2 \frac{\partial A_n}{\partial x^m} - \frac{\partial A_m}{\partial x^n} \right) \right) = -\mu_0 f_{jk}
\]

So we have obtained the generalized coupled inhomogeneous Maxwell equations for the potentials in component form, which on a flat manifold \((g_{ks}, g = \text{const.})\), on arbitrary coordinates, are given by:

<table>
<thead>
<tr>
<th>Field</th>
<th>Mathematical object</th>
<th>Independent components</th>
<th>Related to</th>
<th>Reflection</th>
<th>Components dimensions</th>
<th>Absolute dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electric excitation ( D )</td>
<td>Odd 2-form</td>
<td>3</td>
<td>Area</td>
<td>(-D)</td>
<td>(q/l^2)</td>
<td>(q \equiv \text{charge})</td>
</tr>
<tr>
<td>Magnetic excitation ( H )</td>
<td>Odd 1-form</td>
<td>3</td>
<td>Line</td>
<td>(-H)</td>
<td>(q/tl)</td>
<td>(q/t)</td>
</tr>
<tr>
<td>Electric field strength ( E )</td>
<td>Even 1-form</td>
<td>3</td>
<td>Line</td>
<td>(E)</td>
<td>(\phi_m/tl)</td>
<td>(\phi_m/t)</td>
</tr>
<tr>
<td>Magnetic field strength ( B )</td>
<td>Even 2-form</td>
<td>3</td>
<td>Area</td>
<td>(B)</td>
<td>(\phi_m/l^2)</td>
<td>(\phi_m \equiv \text{mag flux})</td>
</tr>
</tbody>
</table>

Table 1 – Geometrical electromagnetic field quantities and their physical dimensions ([7])
The Maxwell equations together with the Maxwell-Lorentz space-time (or aether) relations, constitute the foundations of classical electrodynamics. These laws, in the classical domain, are assumed to be of universal validity. Only if vacuum polarization effects of quantum electrodynamics are taken care of or hypothetical nonlocal terms should emerge from huge accelerations, Axiom 4 can pick up corrections yielding a nonlinear law (Heisenberg-Euler electrodynamics, [8]) or a nonlocal law (Volterra-Mashhoon electrodynamics, [9]) respectively. In this sense, the Maxwell equations are “more universal” than the Maxwell-Lorentz space-time relations. The latter ones are not completely untouchable. We may consider them as constitutive relations for space-time itself.

**SI-Units and summary of the relevant equations**

The fundamental dimensions in the SI-system for mechanics and electrodynamics are (t, t,M, q/t), with M as mass. And for each of those a unit was defined. However, since action – we denote its dimension by h – is a relativistic invariant quantity and since the electric charge is more fundamental than the electric current, one can choose as the basic units (t, h, q). Thus, instead of the kilogram and the ampere, we have joulexsecond (or weberxcoulomb) and the coulomb: (m, Wb×C,C). In the SI-system, we have μ0 = 4π × 10^{−7} Wb.s.C.m. and ε0 = 8.85 × 10^{−12} C s Wb.m.

The following table gives a resume of the relevant equations obtained in this formalism:

<table>
<thead>
<tr>
<th>Physical relations</th>
<th># eq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>charge conservation</td>
<td>1</td>
</tr>
<tr>
<td>Gauss law</td>
<td>1</td>
</tr>
<tr>
<td>Maxwell – Ampère law</td>
<td>3</td>
</tr>
<tr>
<td>Faraday law (magnetic flux conservation)</td>
<td>3</td>
</tr>
<tr>
<td>Gauss law (magnetism)</td>
<td>1</td>
</tr>
<tr>
<td>Lorentz force</td>
<td>3</td>
</tr>
<tr>
<td>Potentials and electric field</td>
<td>3</td>
</tr>
<tr>
<td>Vector potential and magnetic field</td>
<td>3</td>
</tr>
<tr>
<td>Constitutive relations</td>
<td>3</td>
</tr>
<tr>
<td>Coupled inhomogeneous Maxwell equations for the potentials with explicit dependence on spatial geometry</td>
<td>1</td>
</tr>
</tbody>
</table>

4-dimensional formalism using differential forms

Until now we have been defining the fields and sources in a 3 dimensional manifold and therefore the metric tensor that appears in the formulas corresponds to the spatial part of the 4d metric. In fact this corresponds to the procedure where space and time are locally separated to form a foliation with 3-dimentional spatial hyper-surfaces orthogonal to temporal coordinate lines. It turns out that it is possible to construct an equivalent 4 dimensional treatment of electromagnetism using forms consisting in the following definitions and equations:
• Faraday 2-form:

\[ F = \frac{1}{2} F_{\alpha \beta} dx^\alpha \wedge dx^\beta \]

This form is a special case of the curvature form on the U(1) principal fiber bundle (see section 8) on which both electromagnetism and general gauge theories may be described. Here \( F_{\alpha \beta} \) represent the components of Maxwell field tensor! This form has \( C_2^4 = 6 \) independent components.

• Current 3-form:

\[ J = \frac{1}{3!} \epsilon_{\alpha \beta \gamma} J^\alpha dx^\beta \wedge dx^\gamma \wedge dx^\delta \]

Here \( J^\alpha \) correspond to the components of the 4-current density! The current form has \( C_3^4 = 4 \) independent components!

• Maxwell 2-form (dual to Faraday 2-form):

\[
(G_{\alpha \beta}) = \begin{pmatrix}
0 & -E_1 & -E_2 & -E_3 \\
E_1 & 0 & -B_2 & B_3 \\
E_2 & B_2 & 0 & -B_1 \\
E_3 & B_3 & B_1 & 0
\end{pmatrix}
\]

\[
(G_{\alpha \beta}) = \begin{pmatrix}
0 & -H_1 & -H_2 & -H_3 \\
H_1 & 0 & -D_3 & D_2 \\
H_2 & D_3 & 0 & -D_1 \\
H_3 & D_2 & D_1 & 0
\end{pmatrix}
\]

• Maxwell equations in geometrized units:

These equations are:

- Coordinate free and don’t involve a metric;
- Invariant under arbitrary coordinate transformations (not just Poincaré group)

What restrict the invariance group of electromagnetism to the Poincaré group are the equations \( G = \star F \)

- The definition of Hodge star \( \star \) needs the metric;
- Assuming Minkowsky space, these equations are invariant under the isometries of Minkowsky space!

The inhomogeneous equations can be written in the following manner:

\[ d \star F = J \]

Now remembering that \( d^2 = 0 \) we immediately recover a (pre-metric) expression for

- charge conservation:

\[ dJ = 0 \]

On the other hand it is also easy to get the inhomogeneous equations in terms of the potential 1-form \( A \). We start by defining this electromagnetic potential through the expression:

\[ F = dA \]

\( A \) is a 1 form, determined by \( F \) up to a gauge transformation \( A \rightarrow A + d\lambda \).

Then, using the inhomogeneous equations we get:

\[ \star d \star F = J \quad \delta \equiv \star d \star \]

\[ \Rightarrow \delta dA = \star J \]

Now, choosing a gauge in which \( \delta A = 0 \), we see that

\[ (\delta d + d\delta)A = 0 \quad \Leftrightarrow \quad (d \star d \star + \star d \star d)A = \star J \]

This equation, which may be written as

\[ \Box A = -\star J \quad \Box \equiv -d \star d \star - \star d \star d \]

which can be derived from the action (4-form):
The associated Lagrangian is invariant under the local $U(1)$ transformations. Invariance under local $G$-transformations where $G$ can be an arbitrary Lie group, characterizes gauge theories in general. By investigating the geometry of gauge invariance one can interpret the requirement of local gauge invariance (related to the independence of the gauge fields at different space-time points) as expressing the absence of (instantaneous) action at a distance.

A.2. On the coupling between space-time geometry and electromagnetic fields

A.2.1 Space-time metric from local and linear electrodynamics.

The light cone (with Lorentzian signature) is invariant under the 15 parameter conformal group [10]:

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Parameters</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>Translations</td>
<td>4 param.</td>
<td>$x^\gamma \to x'^\gamma = x^\gamma + a^\gamma$</td>
</tr>
<tr>
<td>Lorentz transformations</td>
<td>6 param.</td>
<td>$x^\gamma \to x'^\gamma = N^\gamma_\mu x^\mu$</td>
</tr>
<tr>
<td>Dilatation</td>
<td>1 param.</td>
<td>$x^\gamma \to x'^\gamma = \rho x^\gamma$</td>
</tr>
<tr>
<td>Proper conformal transf.</td>
<td>4 param.</td>
<td>$x^\gamma \to x'^\gamma = \frac{x^\gamma}{1 + 2k^\mu x^\mu + k^\mu k^{\mu}x^2}$</td>
</tr>
</tbody>
</table>

Here, $a^\gamma$, $N^\gamma_\mu$, $\rho$, $k^\mu$ are 15 constant parameters and $x^2 = g_{\alpha\beta}(x^\alpha x^\beta)$. The Poincaré subgroup leaves the space-time interval $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ invariant! Dilatations and proper conformal transformations change the space-time interval by a scaling factor:

<table>
<thead>
<tr>
<th>Transformation</th>
<th>$ds^2 \to \rho^2 ds^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dilatation</td>
<td>$ds^2 = \sigma^2 ds^2$</td>
</tr>
<tr>
<td>Proper conformal transf.</td>
<td>$\sigma^{-1} = 1 + 2k^\mu x^\mu + k^\mu k^{\mu}x^2$</td>
</tr>
</tbody>
</table>

In all cases the light cone $ds^2 = 0$ is left invariant!!!

For massless particles, instead of the Poincaré group, the conformal or the Weyl group come under consideration, since massless particles move on the light cone. The Weyl subgroup has its corresponding Noether currents. It should be noticed that even though the light cone stays invariant under all transformations, two reference frames that are linked to each other by a proper conformal transformation do not stay inertial frames since their relative velocity is not constant. If we want to maintain the inertial character of the reference frames, we have to use the Weyl transformation and therefore to specialize to the case where $\kappa_i = 0$.

The conformal group in Minkowski space illustrates the importance of the light cone structure on a flat manifold. This is suggestive for the analysis of the light cone on an arbitrarily curved manifold!

Hehl and collaborators found a quartic Fresnel wave surface for light propagation from their axiomatization of electrodynamics [10]. In the case of vanishing birefringence in vacuum, the Fresnel wave surface degenerates and they recovered the light cone (determining 9 components of the metric tensor), thus obtaining the conformal and causal structure of
space-time and the Hodge star ** operator. In this framework, the conformal part of the metric emerges from the local and linear space-time relation (constitutive relations) as an electromagnetic construct. In this sense, the light cone is a derived concept!

![Pre-metric electrodynamics +
Space-time constitutive relation
(D,H) ←→ (E,B)
(Light propagation + vanishing birefringence)
Minkowsky (or pseudo-Riemann) Space-time geometry]

These authors also derived the Lorentzian (Minkowskian) signature has the “correct” one tracing it back to the sign on the Lenz rule which in turn is related to the positivity of electromagnetic energy [1].

“What seems to be conceptually important about the constitutive equations is that they not only provide relations between the excitations \( \mathbf{D}' \), \( \mathbf{H}' \), and the field strengths \( \mathbf{E}_\tau \), \( \mathbf{B}_\tau \), but also connect the electromagnetic field to the structure of space-time, which here is represented by the metric tensor \( g_{\mu \nu} \). The formulation of the first three axioms does not require information on this metric structure. The connection between the electromagnetic field and space-time, as expressed by the constitutive equations, indicates that physical fields and space-time are not independent of each other. The constitutive equations might suggest the point of view that the structure of space-time determines the structure of the electromagnetic field. However, one should be aware that the opposite conclusion has a better established validity: It can be shown that the propagation properties of the electromagnetic field determine the metric structure of space-time” (Hehl in [1]).

A.2.2 Considerations on the relation between geometry and electromagnetic fields

Coupling space-time geometry and electromagnetic fields

In this work I suggest the point of view in which the electric permittivity and magnetic susceptibility characterize the electromagnetic properties of space-time (instead of “vacuum”), being therefore related to geometrical properties (see section B.3). It fits along the line of thought in which space-time is seen as a (dynamical) physical entity (space-time physicalism).

The constitutive relations seem to suggest that \((\mu_0; \varepsilon_0)\) may characterize the geometrical properties of space-time associated to electromagnetic fields. In fact, a geometrical model of the electromagnetic interaction should be able to relate the sources and the fields in a way similar to the following relational diagram:

- curvature (& torsion?) ↔ \( R^a_{\beta\gamma\delta} ; \epsilon^a_{\mu\nu\rho} \) ↔ \( (E; B) \) ↔ \( F_{\mu\nu} \) ↔ \( G_{\mu\nu} \) ↔ \( (D; H) \) ↔ \( (\rho; j) \) ↔ \( T_{\alpha\beta} \) ↔ sources

( geometry ) (Constitutive relations (matter ) )

As already mentioned, the relation between geometrical quantities like curvature (and possibly torsion) and electromagnetic fields is naturally done in Einstein GR (or extended theories of gravity) at least from the point of view that these fields carry energy (and spin), affecting space-time geometry. Space-time geometry is also intrinsically contained within the constitutive relations! On the other hand, the relation between the electromagnetic excitations \((\mathbf{D}; \mathbf{H})\) and the so called “sources” is well established in the inhomogeneous Maxwell equations.

Another point that is worth to mention is the existence of some freedom in semantics underlying the physical relation between electric charges (and currents) and the fields. One can say that charge distributions give rise to fields or alternatively that the fields are the “sources”. Actually, mathematically it is the fields \((\mathbf{D}; \mathbf{H})\) that are “potentials” for the charge and currents densities. In this sense, it is wiser to say that there is a measurable physical relation between the electromagnetic and matter fields - we know empirically that charges and currents have fields associated with them. The casual relation may in fact be surprisingly “bi-directional”, in the sense that perturbations of charge distributions or field configurations have a mutual effect.

A consistent unification between the related concepts of “sources” and fields poses the following issues:
Regarding Maxwell fields as *force fields*, requires that a significant physical relation with the “sources” can rely on energetic terms. If they are force fields (non-virtual) they must carry energy and therefore, if one assumes some sort of causal relation between the matter and fields, it seems that this energy should be ultimately related to the energetic content of the matter source, in particular to the energetic content of charge (or of the associated mass) => *There should be an energetic content associated to charge*;

Regarding Maxwell fields as being ultimately related to *geometrical properties of space-time*, requires that a consistent physical relation between these fields and the matter fields must be geometrical in a complete and self-consistent way. It might be argued that this must imply that charge and mass, or generally speaking, *matter can be regarded as some sort of topological, i.e geometrical manifestation/configuration of space-time*.

In my opinion here lies a stimulating challenge to conjugate these “alternative” conceptions by pursuing a formalism that not only geometrizes electromagnetic fields and sources but also assumes a *physicalism of space-time*, in the sense that it possesses energy and possibly, microstructure. In this type of monism, energy, matter and space-time are somehow unified within a common physical ontology. I will return to this question on the last part of this work.

Following this line of thought one should generalize the linear space-time constitutive relations (in the so called “vacuum”). One starts from the point of view that the relevant or appropriate *space-time symmetries* are reflected on the permittivity and permeability tensors and these symmetries *are obtained in connection to the gravitational field equations* “in vacuum” for a given particular situation. Therefore, first we find out the metric or the “space-time geometry” and then, we are able to speculate about the symmetry properties of these tensor fields. If we happen to know the metric and these electromagnetic tensors, then we can solve the (generalize) linear constitutive relations (and Maxwell’s equations):

\[
B = * (\mu_0(H)) \quad \Rightarrow \quad B_{jk} = \sqrt{g} \varepsilon_{ijk} (\mu_0)_{mn} H^n g^{mi} = \sqrt{g} \varepsilon_{ijk} (\mu_0)_{mn} H^a g^{an} g^{mi}
\]

\[
D = * (\varepsilon_0(E)) \quad \Rightarrow \quad D_{jk} = \sqrt{g} \varepsilon_{ijk} (\varepsilon_0)_{mn} E^n g^{mi} = \sqrt{g} \varepsilon_{ijk} (\varepsilon_0)_{mn} E^a g^{an} g^{mi}
\]

Remember that there are 8 Maxwell equations for 12 variables and we can only solve them by introducing the 6 constitutive relations, so at the end, we have 2 degrees of freedom associated to the electromagnetic field. The problem is that in an arbitrary manifold we need to know the metric and the permittivity and permeability tensors that are only entering as “background parameters”. In fact, the metric is a *dynamical field* and is coupled to the electromagnetic field. It would be extremely convenient to unify all these fields into a unique field theory. On the other hand, the permittivity and permeability tensors might reflect the geometrical properties of space-time (homogeneity, isotropy) and, if so, there should be a formal dependence of the following type:

\[
\mu_0 = \mu_0(g, \Gamma) \quad \varepsilon_0 = \varepsilon_0(g, \Gamma)
\]

Going a little further on this speculation, if there is an *immanent and inherent physicalism of space-time* based on what could be called a *substantialism*, this might be manifested (or expressed) through a *mass-energy ontology* incorporated on the idea of a Higgs-aether-like field \(\Omega_{\text{aether-\|H\|gss}}\). According to this, not only could be possible to introduce a *quantum microstructure of space-time* and the geometrization of many physical quantities (possibly seen as epiphenomena of the fundamental Higgs-aether-like field), but also, using this “neo-materialistic” approach to space-time, the electric and permeability tensors could be expressed in terms of the properties of the microphysical structure of space-time.

\[
\mu_0 = \mu_0(\Omega_{\text{aether-\|H\|gss}}) \quad \varepsilon_0 = \varepsilon_0(\Omega_{\text{aether-\|H\|gss}}) \quad \Omega_{\text{aether-\|H\|gss}} = \Omega_{\text{aether-\|H\|gss}}(g, \Gamma)
\]

Now follows some speculations on the topic of connecting electromagnetic fields and space-time geometry.

---

**I – Electromagnetic properties of vacuum as geometrical properties. The light cone and time asymmetry.**

Magnetic flux conservation allows us to define the magnetic flux current density \(f^\Phi_i\) introducing also a second rank tensor that relates this current to the magnetic field:

\[
f^\Phi_i = B^k v_{ki} = E_i
\]
\[ \int_{s} \frac{\partial B_{i}}{\partial t} \, da_{i} = - \oint_{s} f_{i}^{\rho} \, ds \quad \Rightarrow \quad \partial_{t} B^{m} + e^{mji} \partial_{j} (B^{r} \nu_{ri}) = 0 \]

\[ \Leftrightarrow \quad \partial_{t} B^{m} + e^{mji} \left( (\partial_{j} B^{r}) \nu_{ri} + B^{r} (\partial_{j} \nu_{ri}) \right) = 0 \]

Accordingly, this last equation (expressing magnetic flux conservation) is also compatible with Faraday law:

\[ e^{mji} \partial_{j} E_{i} = - \partial_{t} B^{m} \Rightarrow e^{mji} \partial_{j} (B^{r} \nu_{ri}) = e^{mji} \left( (\partial_{j} B^{r}) \nu_{ri} + B^{r} (\partial_{j} \nu_{ri}) \right) = - \partial_{t} B^{m} \]

We see that given the magnetic field components we have a set of 3 independent partial differential equations for the 9 unknowns of the tensor \((\nu_{mn})\). The set of algebraic equations \(B^{k} \nu_{kl} = E_{i}\) constitute 3 independent equations and since \(\nu_{kl} = \nu_{lk}\), out of the 6 independent components (for a symmetric tensor on a 3-dimensional manifold), only 3 are actually independent. This tensor represents the “magnetic flux velocity” and this “magnetic flow” is analogous to a fluid obeying a “continuity equation”, having some tensor characterizing its spatial velocities:

\[
(\nu_{ri}) = \begin{pmatrix}
\nu_{11} & \nu_{12} & \nu_{13} \\
\nu_{12} & \nu_{22} & \nu_{23} \\
\nu_{13} & \nu_{23} & \nu_{33}
\end{pmatrix}
\]

It has 3 independent components characterizing the electromagnetic energy flow along the three linearly independent directions! Using the appropriate coordinate system where this tensor is diagonal and we should be able to write the following, for “vacuum”:

\[
(\nu_{ri}) = \begin{pmatrix}
\nu_{11} & 0 & 0 \\
0 & \nu_{22} & 0 \\
0 & 0 & \nu_{33}
\end{pmatrix} = \begin{pmatrix}
1/\mu_{011} \varepsilon_{011} & 0 & 0 \\
0 & 1/\mu_{022} \varepsilon_{033} & 0 \\
0 & 0 & 1/\mu_{033} \varepsilon_{033}
\end{pmatrix}
\]

In the most general case we should have spatial and time dependence \(\mu_{0ij}(t, \vec{x})\), \(\varepsilon_{0ij}(t, \vec{x})\). As it is well known, normally one assumes that the speed of light in “vacuum” is given by its electromagnetic properties which are assumed to be isotropic, homogeneous and static \(c = \frac{1}{\sqrt{\mu_{0} \varepsilon_{0}}}\). This is a rather natural consequence of a geometric reasoning, namely the assumption that space is isotropic and homogeneous and time is homogeneous!

We see that the so called electromagnetic properties of “vacuum” in some sense should reflect geometrical properties (symmetries) of the space-time manifold. This assumption of homogeneity and isotropy is also supported by empirical observations. In fact, the assumption of homogeneity and isotropy is a natural starting point for highly symmetric spaces, but in the line of thought of general relativity, we must first solve Einstein-like field equations in order to get the metric structure of the manifold. The “light cone” becomes a local concept and in fact, generally there is a certain (local) deformation from the light cone of special relativity if we take into account gravity. Besides, one can try to obtain the most general self-consistent picture of the so called local conformal structure of space-time by taking into consideration the coupling between gravity and the electromagnetic field, recalling the fact that the latter contributes for Einstein equations via the energy-momentum tensor. Once again, the symmetry properties of space-time affect the propagation of electromagnetic-fields and these affect the geometrical properties of space-time.

In the wave description, the components \(v_{11}, v_{22}, v_{33}\), characterize the velocities of the propagating fields along the three independent directions. In order to deal with the light cone one needs to take into account the geometrical optics limit. We must then generalize the tensor \((\nu_{ri})\) by inserting factors \((\alpha_{1}, \alpha_{2}, \alpha_{3})\) that allow the projection of the ray direction into the three axes:

\[
(\nu_{ri}) = \begin{pmatrix}
\alpha_{1} \sqrt{1/\mu_{011} \varepsilon_{011}} & 0 & 0 \\
0 & \alpha_{2} \sqrt{1/\mu_{022} \varepsilon_{033}} & 0 \\
0 & 0 & \alpha_{3} \sqrt{1/\mu_{033} \varepsilon_{033}}
\end{pmatrix}
\]
In this way, the expressions for the elementary distances that light has traveled along the three spatial directions during an elementary time \(dt\) are

\[
v_1 dt = \frac{1}{\sqrt{\mu_{011}\epsilon_{011}}} dt = \sqrt{g_{11}} dx^1 \quad v_2 dt = \frac{1}{\sqrt{\mu_{022}\epsilon_{022}}} dt = \sqrt{g_{22}} dx^2 \quad v_3 dt = \frac{1}{\sqrt{\mu_{033}\epsilon_{033}}} dt = \sqrt{g_{33}} dx^3
\]

The light cone is locally defined by the expression

\[
g_{00}(dx^0)^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 = \left(\frac{(\alpha_1)^2}{\mu_{011}\epsilon_{011}} + \frac{(\alpha_2)^2}{\mu_{022}\epsilon_{022}} + \frac{(\alpha_3)^2}{\mu_{033}\epsilon_{033}}\right)(t,\vec{x}) dt^2 - \left[g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2\right] = 0
\]

Defining \(x^0 \equiv ct\), we have

\[
\left(\frac{(\alpha_1)^2}{\mu_{011}\epsilon_{011}} + \frac{(\alpha_2)^2}{\mu_{022}\epsilon_{022}} + \frac{(\alpha_3)^2}{\mu_{033}\epsilon_{033}}\right)(t,\vec{x}) = \ddot{v}_0(t,\vec{x}) = g_{00}c^2
\]

Usually one assumes spatial homogeneity and isotropy:

\[
(v_1)^2 + (v_2)^2 + (v_3)^2 = \frac{1}{\mu_0 \epsilon_0}((\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2)(t,\vec{x}) = \frac{1}{\mu_0 \epsilon_0} = c^2
\]

Notice that in the standard theory the values of \(\mu\) and \(\epsilon\) are the lowest for vacuum - in matter, the speed of light is always smaller than \(c\)! I postulate that these quantities not only characterize what could be called a certain “resistance” for the flow of the electromagnetic energy through space-time (or matter), but also represent geometrical properties of space-time. In fact, it is interesting to explore a geometrical explanation for past-future causality from the point of view of the light cone based on this idea.

The tensor \(v_{ik}(t,\vec{x})\) implicitly characterizes the electromagnetic properties of the spatial 3d-hipersurfaces (in a certain foliation of space-time related to some gauge choice). Its invariance under the action of a certain group of spatial transformations should reflect the spatial symmetries of the manifold. It should be possible to construct a similar space-time tensor containing the previous one and giving additionally some information regarding the motion of light in space-time through the fourth physical (time-like) direction.

\[
\left(v_{\alpha\beta}(t,\vec{x})\right) = \begin{bmatrix}
\frac{1}{\sqrt{\mu_{000}\epsilon_{000}}} & 0 & 0 \\
0 & \frac{1}{\sqrt{\mu_{011}\epsilon_{011}}} & 0 \\
0 & 0 & \frac{1}{\sqrt{\mu_{022}\epsilon_{022}}} \\
0 & 0 & 0 & \frac{1}{\sqrt{\mu_{033}\epsilon_{033}}} \end{bmatrix}_{(t,\vec{x})}
\]

This tensor should not be invariant under temporal inversions if we wish to introduce *time asymmetry* (and neglect the advanced potentials) through “geometrical” and electromagnetic reasoning. The minimum requirement is to consider that under (infinitesimal) time inversions, \(\delta t \rightarrow -\delta t\), we should have:
When a fluid propagates within a certain medium, typically it moves easily along the directions with smaller “resistivity”. Analogously time asymmetry in light propagation could result from the requirement that
\[
\frac{\partial}{\partial t} \left( (\mu_0)_{00}(\varepsilon_0)_{00}(t) \right) < 0
\]

Dimensional analysis and geometrization of electromagnetism

In part B the gravitoelectromagnetic potentials will be introduced but at this stage it is interesting to show explicitly their dimensions and compare with the electromagnetic potentials.

\[
\text{dim}[\phi_g] = L^2T^{-2} \quad \text{dim}[\phi] = L^2T^{-2}MC^{-1}
\]
\[
\text{dim}[\vec{A}_g] = LT^{-1} \quad \text{dim}[\vec{A}] = LT^{-1}MC^{-1}
\]

We immediately see that if charge had the dimension of M the electromagnetic field would only depend on geometrical dimensions (L,T) and the potentials for gravity and electromagnetism would define the same physical quantities! Notice also that in the linear gravitoelectromagnetic approach to Gravitational Waves, based on equations similar to Maxwell equations, we can define the gravitoelectric permittivity and gravitomagnetic permeability for vacuum, such that:

\[
c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = \frac{1}{\sqrt{\varepsilon_{0g}\mu_{0g}}} \Rightarrow \varepsilon_{0g}\mu_{0g} = \varepsilon_0\mu_0
\]

In the first order approximation, \(\varepsilon_{0g}\mu_{0g}\) characterize the background space-time which is fixed, but in higher order approaches these quantities change and gravity is no longer seen as a field in Minkowsky space but as a dynamical “background” geometry. Therefore, the previous equality may be hiding 2 interesting things:

1) Through analogy, in a non-linear approach we can view \((\varepsilon_0)_{ij}\), \((\mu_0)_{ij}\) as changing quantities. Now, since this change in the electromagnetic properties is linked to the (changing) quantities \((\varepsilon_{0g})_{ij}\), \((\mu_{0g})_{ij}\), it is related to the changing geometry of the “dynamical background metric”;

2) This equality may be a manifestation of the fact that not only electromagnetic fields create gravity, but also that gravity creates electromagnetic fields! In vacuum, gravitational fields are always associated to changing geometry, and changing geometry in vacuum should give different \(\varepsilon_0,\mu_0\) (expressed as function of \(\varepsilon_{0g},\mu_{0g}\)). This link or coupling between space-time geometry and electromagnetic fields has already been explored in the constitutive relations between the field intensities (\(\vec{E}\) and \(\vec{B}\)) and extensities (\(\vec{D}\) and \(\vec{H}\)).
B – General relativity, gravitoelectromagnetism and the GP-B experiment

B.1 Einstein’s gravitational field equations and related topics

Puzzled with the question of how to generalize the principle of relativity to all kinds of observers (not only inertial ones) Einstein constructed a geometrical theory for gravity – General Relativity (GR). It has been successfully tested in many experiments (see [12,13]) such as the so-called classical tests (bending of light, 1919; precession of Mercury’s perihelium; light signals delay or Shapiro effect, 1970; gravitational redshift – gravity prob A, 1956) and “modern tests” (binary pulsars – indirect measurement of gravitational waves; gravitational lensing; cosmological tests; gravitomagnetic tests: Laser Lunar Ranging LLR, LAser GEOdynamical Sattelites LAGEOS, Gravity Probe-B). Through GR, the study of gravity and space-time extends beyond Euclidean concepts and physics embraces different (pseudo) Riemann geometries. The previous works by Riemann and many others were fundamental, setting the fertile background for the intuitive genius and remarkable persistence of Einstein. The principal ideas leading to the theory are:

- The effects due to accelerated frames (inertial forces expressed by non vanishing connections) are locally equivalent to effects due to gravitational fields – equivalence principle. Gravity is locally absent in freely falling frames!
- Test particles in a gravitational field travel freely along the geodesics of space-time geometry. It reinforces the hypothesis of universality of free fall!
- Space-time curvature should depend on the (local) content of mass-energy as expressed by the energy-momentum tensor – dynamical character of space-time geometry;
- All physical laws should be mathematically equivalent for every observer and in all coordinate systems – principle of relativity and general covariance;
- For weak gravitational fields the theory should recover the Newtonian predictions (and the respective Poisson equation).

Einstein found the required dynamical equations for gravity constructing a simple and consistent tensor (the Einstein tensor $G_{\mu\nu}$) representing the geometry of space-time, satisfying the required properties in order to be related with the energy-momentum tensor. These are the famous Einstein field equations:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -k T_{\mu\nu}$$

Where

$$R_{\mu\nu} = \partial_{\mu} \Gamma_{\rho\delta}^{\nu} - \partial_{\nu} \Gamma_{\rho\delta}^{\mu} + \Gamma_{\nu\beta}^{\mu} \Gamma_{\beta\delta}^{\rho} - \Gamma_{\beta\delta}^{\mu} \Gamma_{\nu\beta}^{\rho}$$

are respectively the Riemann (curvature) tensor, Ricci tensor and the 4-dimensional curvature invariant. The constant $k$ is $\frac{8\pi G}{c^4}$ and the minus sign on the right hand side is due to the metric signature (+ - - -). Both $G_{\mu\nu}$ and $T_{\mu\nu}$ are symmetric and their covariant derivative is zero! The Riemann tensor has the following properties:

**Symmetry properties**

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$$

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\gamma\beta\delta}$$

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$$

**Cyclic identity**

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0$$

**Bianchi identity**

$$\nabla_{\xi} R_{\alpha\beta\gamma\delta} + \nabla_{\gamma} R_{\alpha\beta\delta\xi} + \nabla_{\delta} R_{\alpha\beta\xi\gamma} = 0$$

(Using the symetries it can be expressed as $\nabla_{\xi} R_{\alpha\beta\gamma\delta} = 0$)
The fundamental space-time interval determining space-time distances is given by (the generalized Phythagorian theorem)

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$$

Free (test) particles follow geodesics $x^\alpha(\tau)$ obeying the equations

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0$$

and the connection $\Gamma^\alpha_{\beta\gamma}$ is the so called metric-connection ($\nabla _\mu g^{\mu\nu} = 0$) or Levi-Civita connection already mentioned in chapter 1 and is completely determined by the metric $g_{\alpha\beta}$:

$$\Gamma^\delta_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left( \partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\gamma\alpha} - \partial_\alpha g_{\beta\gamma} \right)$$

This is a symmetric connection compatible with the metricity condition $\nabla _\mu g^{\mu\nu} = 0$, where

$$\nabla_G = \nabla_a g_{b\gamma} e^a \otimes e^\beta \otimes e^\gamma = 0$$

$$\nabla_a g_{b\gamma} = \partial_a g_{b\gamma} - g_{\delta a} \Gamma^\delta_{\beta\gamma} - g_{a\delta} \Gamma^\delta_{\beta\gamma}$$

The field equations are non-linear second-order differential equations for the metric components and difficult to solve in situations where there isn’t any defined symmetry. Nevertheless, when applied to specific cases of spherical spatial symmetry or axial symmetry or even to maximally symmetric spaces, these equations have revolutionized Astrophysics and Cosmology leading to the study of Relativistic stars (such as neutron stars or pulsars), blackholes, active galaxies and the Friedmann-Robertson-Walker-Lemaître cosmological models. All the research and debate around Big-Bang models, the cosmic singularity, blackhole singularities, space-time wormholes and many other “hot” topics in Physics were fundamentally driven or impelled by these famous equations.

Raising one indice in Einstein’s equations (applying the metric tensor) and contracting the indices we see that

$$R^a_{\beta} - \frac{1}{2} \delta^a_{\beta} R_{\beta} = -kT^a_{\beta} \implies R = kT$$

Where $T \equiv T^a_{\beta}$ is the trace of the energy-momentum tensor. Substituting this relation in the initial Einstein equations we arrive at a different form:

$$R_{\mu\nu} = -k \left( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right)$$

So we see that in the absence of any sort of matter-energy, the field equations in “vacuum” are given by:

$$R_{\mu\nu} = 0$$

Notice that to a first look we might think that the curvature tensor has $n^4$ components ($n$ is the dimension of the manifold) but the cyclic and symmetry properties actually allow us to conclude that there are $n^2(n^2 - 1)/12$ independent components. The total number of independent field equations equals the independent components in the metric which are $n(n + 1)/2$. The following table shows the number of field equations and independent components in $R_{a\beta\gamma\delta}$ as a function of space-time dimension:

<table>
<thead>
<tr>
<th>Space-time dim n</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>N° of field equations</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>15</td>
</tr>
<tr>
<td>N° of independent components in $R_{a\beta\gamma\delta}$</td>
<td>1</td>
<td>6</td>
<td>20</td>
<td>50</td>
</tr>
</tbody>
</table>

So we see that for n<4 the Einstein equations in vacuum guarantee that the full curvature tensor must vanish (in vacuum). Only in four or more dimensions gravitational fields can exist in “empty space”.

**Weyl curvature tensor**

The **Weyl curvature tensor**, is a measure of the curvature of a pseudo-Riemannian manifold. Like the Riemann tensor, it expresses the tidal force that a body feels when moving along a geodesic. It differs from the Riemann tensor in that it does not convey information on how the volume of the body changes, but only how the shape of the body is distorted by the tidal force. The Ricci curvature (trace component of the Riemann tensor) contains precisely the information about how volumes change in the presence of tidal forces, so the **Weyl tensor** is the traceless component of the Riemann tensor. It has the same symmetries as the Riemann tensor with the extra condition that it be trace-free (metric contraction on any pair of indices yields zero). The Weyl curvature is the only part of the curvature that exists in *free space* (absence of matter-energy fields). It governs the propagation...
The cosmological constant

As is well known Einstein introduced in his equations a term proportional to the metric – “the cosmological constant term”, motivated by the hypothesis that the universe should be static (with no contraction nor expansion). This term re-emerged later as providing a representation of the contribution of a different kind of energy with negative pressure - dark energy – in order to explain the apparent acceleration of the cosmological expansion. Since \( \nabla_a T^{a\beta} = 0 \) and \( \nabla_a g^{a\beta} = 0 \), we can always add in Einstein’s tensor a term proportional to the metric and get a consistent result. The Einstein equations with a cosmological term are:

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + g_{\mu\nu} \Lambda = -\frac{8\pi G}{c^4} T_{\mu\nu}
\]

Interpreting this term as representing some “source” we see that the corresponding energy-momentum is given by

\[
T_{\mu\nu}^{(\Lambda)} = \frac{c^4}{8\pi G} g_{\mu\nu} \Lambda
\]

Now, the definition of the energy-momentum tensor for a fluid \( T^{\mu\nu} \equiv \rho(m) u^\mu u^\nu - \frac{\rho}{c^2} u^\mu u^\nu \) (where \( \rho(m) \) is the mass density, \( \rho \) is the energy density and \( u^\mu \) are the fluid 4-velocity components in some reference system), allows us to derive the case of a perfect fluid with no viscosity and isotropic pressure:

\[
T_{\mu\nu}^{(\text{perfect fluid})} = \left( \rho(m) + \frac{p}{c^2} \right) u_\mu u_\nu - p g_{\mu\nu} = (\rho + p) \frac{u_\mu u_\nu}{c^2} - p g_{\mu\nu}
\]

So, if we consider the application of Einstein equations to a Universe spatially isotropic and homogeneous (Friedmann models) with a cosmological constant interpreted as source of “dark energy”, the contribution of this mysterious fluid is given by \( T_{\mu\nu}^{(\Lambda)} \) which should be a perfect fluid. Therefore we conclude that:

\[
\begin{align*}
\rho_\Lambda &= -\frac{c^4}{8\pi G} \Lambda \\
\rho_\Lambda &= -p_\Lambda
\end{align*}
\]

We see in fact that for \( \Lambda > 0 \) this corresponds to a negative pressure fluid! In fact, it is well known from the Friedman equations with a cosmological term that the condition for accelerated expansion requires that \( p < -\frac{1}{3} \rho \).

Cosmological constant from a scalar field

The Euler-Lagrange equations for a scalar field with Lagrangian \( \mathcal{L} = \frac{1}{2} g^{\mu\nu} (\nabla_\mu \phi)(\nabla_\nu \phi) - U(\phi) \) determines the evolution equation given by

\[
g^{\mu\nu} \nabla_\mu \nabla_\nu \phi + \frac{dU}{d\phi} = 0
\]

Which reduces to the familiar Klein-Gordon equation \( (g^{\mu\nu} \nabla_\mu \nabla_\nu \phi + \mu^2 \phi = 0) \) if we take the “exterior potential energy density” to be \( U(\phi) = \frac{1}{2} \mu \phi^2 \). The respective energy-momentum tensor is

\[
T_{ij} = (\partial_i \phi)(\partial_j \phi) - g_{ij} \left( \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - U(\phi) \right)
\]
If we interpret this field as representing some “dark energy” and allowing $T_{ij}$ to represent a fluid consistent with spatial homogeneity and isotropy (a perfect fluid), one can show that the pressure and energy density of this scalar field are

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + U(\phi) + \frac{1}{2} (\square \phi)^2 \quad p_\phi = \frac{1}{2} \dot{\phi}^2 - U(\phi) - \frac{1}{6} (\square \phi)^2$$

We see that if the field is constant in space and time we get an equation of state $p(\rho)$ that is consistent with that of a cosmological “constant”

$$\rho_\phi = U(\phi) = -p_\phi \quad \Rightarrow \quad \Lambda = \frac{8 \pi G}{c^4} \rho_\phi$$

The effect of cosmic acceleration can in fact be reproduced by the dynamics of a scalar field governed by some potential energy, whose functional form might be tuned in order to rest at the present “epoch” within some value compatible with the cosmological observations that constrain $\Lambda$. What sort of matter might constitute this scalar field? No one knows exactly! It is also interesting to note that for a massive scalar field, the term $U(\phi)$ can be interpreted as an interaction between the scalar field and another scalar field like the Higgs field which results in the inertia of the particle in consideration. In this sense, it is interesting to see that the cosmological observations on $\Lambda$ over time and space may be seen as indirect measurements of the coupling between the “repulsive scalar field” and the Higgs field.

### Curiosity

The cosmological term may also be interpreted as something related to the “geometrical side of the equations”, instead of being associated with the introduction of new kinds of completely hypothetical forms of energy. There are in fact some examples from alternative theories of gravity, such as f(R) theories or torsion theories, which reproduce the cosmic accelerated expansion without introducing dark energy (see C.7). There are also some works that relax the spatial homogeneity and isotropy assumption, in order to solve some of the cosmological problems without additional “dark energy” [11].

### Some notes on the cosmological term and the “vacuum crisis”

By contracting the indices in the Einstein equations with cosmological term one can easily arrive at ($n>2$ is the dimension of the space-time manifold):

$$R = \frac{2}{n-2} (kT + n\Lambda) \quad \Rightarrow \quad R_{\mu \nu} = -k T_{\mu \nu} + g_{\mu \nu} k T \frac{1}{n-2} + g_{\mu \nu} \Lambda \frac{2}{n-2}$$

This equation (and also $T_{\mu \nu}^{(A)} = \frac{c^4}{8 \pi G} (g_{\mu \nu} \Lambda)$) shows very clearly that the curvature associated to the cosmological term is:

$$R(A) = \frac{2n}{n-2} \Lambda \quad n > 2, \quad \Lambda > 0 \Rightarrow R > 0$$

### Is this a “vacuum curvature”??

This might be interpreted as the curvature of “vacuum” ($R(A) = R_{(\text{vacuum})}$) if our definition of “vacuum” corresponds to space devoted of ordinary particles. Notice that interpreting this as the curvature arising from the “vacuum” equations (where $T_{\mu \nu} = 0$) corresponds to assume that $\Lambda$ plays a role in the geometrical side of Einstein equations instead of being associated to any kind of energy-momentum source. Otherwise, the vacuum equations would be the usual $R_{\mu \nu} = 0$ and the ricci curvature would be zero (but not the Weyl curvature as already mentioned).

We see that in principle one could test the $\Lambda$ term by somehow inspecting experimentally the “intrinsic” curvature of vacuum. Suppose we create “vacuum” - space devoted from ordinary matter – and develop a ultra precise system to check for curvature: if this “vacuum curvature” is of Ricci type ($R(A) \neq 0$) then, the validity of the presence of $\Lambda$ in Einstein equations is reinforced; if is of Weyl type (with zero Ricci curvature) then, either there is some dark energy fluid and it was removed together with ordinary matter (suggesting some sort of coupling) or there is no dark energy fluid and $\Lambda$ should not appear in Einstein’s equations! In the first alternative there is a degeneracy of interpretations since the result is compatible with the existence of some exotic fluid (decoupled from matter) and with the idea that is some geometrical property appearing from the form of Einstein equations without assuming any exotic fluid (in the first case one has to introduce an unknown fluid and in the other case there is an effect with no apparent cause other than the quantum fluctuations). Nevertheless $R(A) = \frac{2n}{n-2} \Lambda$ is the curvature associated to the cosmological term even if we interpret it as being representative of some sort of unknown matter-energy fluid with $T_{\mu \nu}^{(A)} = \frac{c^4}{8 \pi G} (g_{\mu \nu} \Lambda)$ (since $R = \frac{2}{n-2} k T$).
We saw that for $\Lambda > 0$, $R$ is positive. Nevertheless, this term (interpreted as some fluid or not) has an anti-gravity effect. One can illustrate this also in the weak-field (Newtonian) limit of the Einstein equations with $\tilde{\Lambda}$ term. This limit turns out to be [12]:

$$\nabla^2 \Phi_g = 4\pi G \rho_{(m)} - \tilde{\Lambda} c^2$$

Applying to a spherical mass $M$ one obtains a gravitational field given by

$$\tilde{g} = -\nabla \Phi_g = -G \frac{M}{r^2} \vec{e}_r + \frac{\tilde{\Lambda} c^2}{3} r \vec{e}_r$$

So in this case a positive cosmological constant corresponds to a gravitational repulsion whose strength increases linearly with $r$.

### Note

In the case of Earth there would be equilibrium at a distance:

$$r_{eq} = \left(\frac{3GM_{Earth}}{\tilde{\Lambda} c^2}\right)^{1/3}$$

Note that if $\tilde{\Lambda}$ is very small, then this distance is very far indeed. Notice also that the Einstein equations give us local information, so this so-called “cosmological term” does not necessarily correspond to the “usual” cosmological constant that is inferred from cosmological observations and related to the cosmic expansion. In fact, it should not be the same, since if we assume that it is the same, then when we recalculate the mass of the Earth, compatible with the following constrain:

$$9.8 \text{ m/s}^2 = \left| -\frac{G M_{Earth}}{R_{Earth}^2} + \frac{\tilde{\Lambda} c^2}{3} R_{Earth} \right| = G \frac{M_{Earth}}{R_{Earth}^2} - \frac{\tilde{\Lambda} c^2}{3} R_{Earth} = M_{Earth} = \frac{|\tilde{g}| + \frac{\tilde{\Lambda} c^2}{3} R_{Earth}}{G} (R_{Earth})^2$$

Using $R_{Earth} \approx 6.371 \times 10^6$ m [14], $\tilde{\Lambda} = \Lambda_{cosmo} \approx 10^{-52}$ m$^{-2}$ [15], we obtain $M_{Earth} \approx 10^{-7}$ Kg which is ridiculous. Therefore, if there is a $\tilde{\Lambda}$ associated to the Earth field, it can’t be the same that is assumed for FRWL cosmological applications. In fact for $M_{Earth} \approx 10^{24}$ Kg [16] we need $\tilde{\Lambda} \approx 10^{-21}$ m$^{-2}$. In that case the equilibrium would be at a distance $r_{eq} \approx 10^{20}$ m $= 10^{17}$ Km. I enhance the fact that anomalies in the acceleration of a spacecraft leaving the Earth would most probably start to be detected at distances far inferior than the $r_{eq}$.

Now returning to the idea of associating the cosmological term to some energy source ($T_{\mu\nu}^{(\Lambda)} = \frac{\epsilon^4}{8\pi G} g_{\mu\nu} \Lambda$), the so-called vacuum quantum energy is a natural candidate for this “unknown energy” although there are enormous discrepancies between the predictions coming from quantum field theory and the indirect cosmological measurements. Above we present some estimation from different quantum field theories, comparing with constraints from the cosmological observations for the dark cosmological energy [12]

| $\rho^{\mu\nu}_{\text{vacuum}}$ | $\sim 3 \times 10^{47}$ erg cm$^{-3}$ |
| $\rho^{QCD}_{\text{vacuum}}$ | $\sim 1.6 \times 10^{36}$ erg cm$^{-3}$ |
| $|\rho^{(obs)}_{\Lambda}|$ | $\sim 2 \times 10^{10}$ erg cm$^{-3}$ |

**Table 4** – Some vacuum energy density values, predicted from quantum field theories, compared to the energy density associated to $\Lambda$ and the estimated from cosmological observations [12]
Energy-momentum tensor and obtaining the geodesic motion from Einstein equations

The conservation of energy and momentum for a system of particles or fluid in special relativity can be expressed by:

\[ \partial_\alpha T^{\alpha\beta} = 0 \]

Where the energy-momentum tensor has the following components

\[
T = \begin{pmatrix}
\rho & j_1/c & j_2/c & j_3/c \\
j_1/c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\
j_2/c & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\
j_3/c & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz}
\end{pmatrix}
\]

\( \rho \) – Energy density

\( J \) – Energy flux density

\( \sigma \) – Momentum flux density

For a matter fluid we have:

**Rest frame**

\[ T^{00} = \rho = \rho_0c^2 \]  
energy density

\[ T^{0i} = \frac{\rho}{c} u^i \]  
energy flux \( \times \ c^{-1} \) in the i – direction

\[ T^{ij} = \frac{\rho}{c^2} u^i u^j \]  
momentum flux density (internal stress)

**Arbitrary inertial frame**

\[ T^{00} = \gamma_u^2 \rho \quad T^{0i} = \gamma_u^2 \frac{\rho}{c} u^i \quad T^{ij} = \gamma_u^2 \frac{\rho}{c^2} u^i u^j \quad [u^a] = \gamma_u(c, \vec{u}) \]

Still working in Minkowski space-time, in an arbitrary coordinate system the conservation equation is written with the covariant derivative,

\[ \nabla_\alpha T^{\alpha\beta} = 0 \]

Now, if we consider the presence of a gravitational field (and hence curved space-time) the appropriate interpretation for this important equation is that it represents the equation of motion of the matter (or “energy fluid”) under the influence of the gravitational field \( \square \). To see this I will be using two results not proven here. One is simply an expression to compute the covariant derivative of a second-rank tensor and the other expression is derivable from the relation between the metric and the (Levi-Civita) connection. This last expression is:

\[ \frac{\sqrt{-\gamma}}{\sqrt{\epsilon}} \]

Consider the case of a single test particle of rest mass \( m \). We may write

\[ T^{\alpha\beta}(x) = \rho m u^\alpha u^\beta = \frac{m}{\sqrt{-g}} \int \frac{dy^\alpha}{d\tau} \frac{dy^\beta}{d\tau} \delta^4(x - y(\tau)) d\tau \]

Where \( y(\tau) \) is the particle world line. Notice that \( dV^4 = \sqrt{-g} dx^4 \) is the invariant 4-dimensional volume element so that the invariant “delta-function” \( \delta^4 \) is given by \( \delta^4(x - y(\tau)) = \delta^4(x - y(\tau))/\sqrt{-g} \) so that we can have, for any function \( \phi \):

\[ \int \phi(x) \delta^4(x - y)dV^4 = \phi(y) \]

After substitution in the covariant derivation of \( T^{\alpha\beta} (\nabla_\alpha T^{\alpha\beta} = 0) \) and after some algebra we arrive at the following result:

\[ \int \left( \frac{d^2 y^\beta}{d\tau^2} + \Gamma^\beta_\gamma \frac{dy^\gamma}{d\tau} \right) \delta^4(x - y(\tau)) d\tau = 0 \]
Clearly, for every point $x$ that belongs to the particle world line, we see that the expression in parenthesis must vanish, and therefore the correct worldline is a solution of the geodesic equation! The same result would have been obtained if we had considered a pressless perfect fluid or “dust”. This is very interesting indeed: the position of the particle is where the field equations become singular, but the solution of the field equations in the empty space surrounding the “singularity” determines how it should move. It obeys the same equations as that of a test particle moving in the space-time geometry that is influenced by the particle itself [12].

To the extent that the expression $\nabla_\alpha T^{\alpha\beta} = 0$ is implicitly contained in Einstein equations we can say that the field equations predict the equation of motion and this is in contrast to the situation in Maxwell’s electrodynamics. In this latter case the electromagnetic field equations do not contain the corresponding equation of motion for a charged particle (Lorentz force), it has to be postulated separately. Mathematically, the origin of this distinction between gravity and electromagnetism lies in the non-linearity of Einstein equations. Physically, the reason for this non-linear nature comes from the fact that the gravitational field itself carries energy-momentum and can therefore act as its own source, whereas electromagnetic field carries no charge and so cannot act as its own source. In this sense one can say strictly speaking that according to general relativity there are no such thing as an ideal test particle.

<table>
<thead>
<tr>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>In spite of what was said, note however that the electromagnetic excitations $\mathbf{D}, \mathbf{H}$, are defined in all space (not only in the regions where there are sources) and have dimensions of charge/L$^2$ and current/L respectively. In fact, Maxwell’s equations may give some reasons to re-think the nature of electromagnetic fields from this point of view. For instance, the product of a changing electric flux with the permittivity of “vacuum” (a property existing throughout space) has dimensions of current (Maxwell displacement current). In this sense, although the Maxwell fields do not have charge, the excitations do have.</td>
</tr>
</tbody>
</table>

Taking this procedure to re-think the equivalence principle

Another intriguing aspect of this procedure suggests itself when dealing with inertial forces and the issue of finding a coherent explanation for their origin. When we considered the expression $\partial_\alpha T^{\alpha\beta} = 0$ we were dealing with the flat space-time of Minkowsky. We know that in arbitrary coordinate systems (such as non inertial ones) we should replace this by $\nabla_\alpha T^{\alpha\beta} = 0$. Even in that case, all the above reasoning still applies and we conclude that from the point of view of a non-inertial observer the equations of motion for a “free” particle are:

$$\frac{d^2 y^\beta}{d\tau^2} + \Gamma^\beta_{\alpha\gamma} \frac{dy^\gamma}{d\tau} \frac{dy^\delta}{d\tau} = 0 \iff \frac{d^2 y^\beta}{d\tau^2} = -\Gamma^\beta_{\alpha\gamma} \frac{dy^\gamma}{d\tau} \frac{dy^\delta}{d\tau}$$

which reflect the fact that the so called “fictitious” forces are given by $-\Gamma^\beta_{\alpha\gamma} \frac{dy^\gamma}{d\tau} \frac{dy^\delta}{d\tau}$ as mentioned in the appendix. Now, the usual interpretation is that it is always possible to make a coordinate transformation such that the connection coefficients vanish proving that the geometry is flat and that the “inertial” forces are nothing but a “coordinate effect”. This transformation always correspond to a transition into an inertial frame! It seems to me that, although coherent, this isn’t a satisfying explanation since it fails to explain how this effect appears in a way that is consistent with the equivalence principle. Notice that according to this principle the inertial effects should be locally equivalent to gravitational fields. Now, gravitational fields are a manifestation of space-time curvature and possibly torsion, therefore, does this mean that “inertial” forces should have something to do with the curvature (or torsion) of space-time? Normally this equivalence is only taken to be a formal equivalence, valid locally, but it may actually be a physical equivalence. Two “natural” objections may arise:

- One may object this idea in the first place since tidal forces are the “signature of real gravitational fields” and therefore of curvature. Actually, it should be noticed that it is always possible to imagine non-inertial reference systems with such characteristics that one would have to define a field of accelerations (which may have the same intensity but different directions) such that the effects due to “inertial forces” would be similar to the existence of “tidal forces” - with the respective convergence of the free-particle paths. Tidal effects strictly speaking doesn’t have a local character, so we should not use them to argue against a “curvature explanation of inertial forces” based on comparisons between non-uniform gravitational fields (such as the Earth) and effects due to uniform acceleration fields (like a uniformly and linearly accelerating spaceship);
- Secondly, the point of view that the inertial forces could be explained by geometrical perturbations of space-time such as curvature (and eventually torsion) seems incoherent from the point of view that if we change to the inertial reference system there is no evidence of curvature and the free particle moves in straight lines \( \frac{d^2y^\mu}{dt^2} = 0 \).

This last objection is very strong indeed since it is related to the important idea that geometrical quantities such as the full curvature of a manifold representing space-time should be invariant and not observer dependent. What if we wish to pursue the hypothesis that from the point of view of the accelerated observer in the non-inertial frame, the free particle is moving through a non-Euclidean manifold (and possibly non-Riemann)? Notice that within the specific non-inertial frame under consideration there is no coordinate transformation that makes the connection components null - that happens only when we change into any inertial frame. In this case, there are only to possibilities: either the inertial and non-inertial observers experience two different space-time manifolds, or they experience the same space-time manifold and, in the case of curvature, in order to have a situation where curvature is observer dependent, the curvature in question can’t be the full curvature but in fact just some part of it.

Consider the last possibility (since it is slightly less disturbing). If we raise the dimension of the manifold we can in principle construct a geometrical explanation for the inertial forces consistent with the idea that whoever is feeling these forces is in fact experiencing gravitational fields. Measurements within this system of reference may conclude the existence of curvature. The simplest solution is to make: \( 4d \rightarrow 5d \). Alternatively, one maintains 4 dimensions and postulate that it is the 3d curvature (and not the full 4d curvature) that is responsible for the effect.

Let’s return to special relativity. Notice that all observers and particles are moving along the 4-dimensional space-time. There is always motion through the 4th physical direction (which gives the experience of time). Consider a given particle; an observer in the proper frame of the particle is one for which the motion through the 4th direction is exactly the same as for the particle, making precisely the same angle with the instantaneous 3-dimensional hypersurface. Their 3d-surface of simultaneity (imagining also an observer where the particle is, to make sense of the word simultaneity) is exactly the same! Different inertial observers are moving along the 4th dimension through different directions (relative to each other). This makes simultaneity, time distances and spatial distances relative, although 4-dimensional distances are absolute, invariant.

The idea here presented (an observer dependent curvature to explain inertial forces) would be somehow similar. Considering that an accelerated observer is moving along the (instantaneous) 4th direction which is continuously changing, he/she gradually “sees” different 3d-hypersurfaces of simultaneity that aren’t parallel to each other. This effect would cause them to detect/feel curvature. In the 5d extension we would say that 3d or even 4d curvature are relative but 5d curvature is absolute. As it is known the idea of five dimensions is extremely interesting when considering the issue of unifying the gravitational and electromagnetic fields.

**Comment on the expressions** \( \nabla_\alpha T^{\alpha\beta} = 0 \) and \( R = kT \)

- **Fermions**

Consider a free fermion obeying the Dirac equation on a curved space-time with pseudo-Riemann geometry. The equation \( \nabla_\alpha T^{\alpha\beta} = 0 \) allowed us to conclude that the particle moves along the geodesics of the space-time manifold that is solution of the Einstein equations in the space “surrounding” the particle. This curvature is created by the particle itself. Now, according to the quantum theory the probability density of localization in space-time is given by \( \tilde{\psi}y^\mu \psi \) so, given some initial probability distribution, we see also that the regions of high curvature correspond to those of high position probability according to \( R = kT \). In fact we have a energy density probability distribution and therefore, via the Einstein equations (\( R = kT \)) a curvature probability distribution

\[
\rho = mc^2 \tilde{\psi} \gamma^0 \psi \Rightarrow T = mc^2 \tilde{\psi} \gamma^0 \psi + T_k^k \Rightarrow R = \frac{8\pi G}{c^2} m \tilde{\psi} \gamma^0 \psi + \frac{8\pi G}{c^4} T_k^k \\
\Rightarrow curvature \ fluctuations
\]

Reconciling this idea with the notion of the particle following geodesics and the quantum state reduction is a challenge!

The Einstein equations may be rewritten
Electromagnetic field

In this case we see that electric fields, magnetic fields and light are also gravitational fields (or produce gravitational fields). We therefore conclude that accompanying any electromagnetic wave there is always a curvature (gravitational) wave:

\[ T = -\frac{1}{\mu_0} \left( F^\alpha_{\beta} F^{\beta}_{\alpha} - \frac{1}{4} \delta^\alpha_{\beta} F_{\gamma\epsilon} F^{\gamma\epsilon} \right) \]

\[ \sigma^k = \frac{1}{4} \left( \varepsilon_0 E^2 + \mu_0 H^2 - \frac{3}{2} (\varepsilon_0 E^2 + \mu_0 H^2) \right) = \frac{1}{4} \left( \varepsilon_0 E^2 + \frac{B^2}{\mu_0} - \frac{3}{2} \left( \varepsilon_0 E^2 + \frac{B^2}{\mu_0} \right) \right) = -\frac{1}{8} \left( \varepsilon_0 E^2 + \frac{B^2}{\mu_0} \right) \]

\[ \Rightarrow T = \rho_{em} + \frac{1}{8} \left( \varepsilon_0 E^2 + \frac{B^2}{\mu_0} \right) = \frac{9}{8} \left( \varepsilon_0 E^2 + \frac{B^2}{\mu_0} \right) \]

Vacuum (Ricci) curvature

Let us now construct an action for gravitation from which the Einstein field equations can be derived, considering first gravitation in vacuum. We must define a Lagrangian \( L \) which is a scalar under general coordinate transformations, depending on the components of the metric tensor \( g_{\mu\nu} \) (the dynamical fields), and their first and possibly higher order derivatives. The simplest non-trivial scalar that can be constructed from the metric and its derivatives is the Ricci scalar \( R \), which depends on \( g_{\mu\nu} \) and its first and second order derivatives. In fact, \( R \) is the only scalar derivable from the metric that depends on derivatives no higher than second order. From our knowledge of gravitation as being a manifestation of space-time curvature it is actually:

\[ R \]
very plausible to expect the action to be constructed from the curvature tensor! As a result, searching the simplest plausible variational principle for gravitation, one is led to the Einstein–Hilbert action and respective Lagrangian density:

\[ S_{EH} = \int_{\mathcal{M}} R \sqrt{-g} \, d^4x \quad \mathcal{L}_{EH} = R \sqrt{-g} \]

The resulting Euler-Lagrange equations take the form:

\[
\frac{\partial \mathcal{L}}{\partial g_{\mu \nu}} - \frac{\partial}{\partial x^\alpha} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\alpha g_{\mu \nu})} \right) + \partial_\rho \partial_\sigma \left( \frac{\partial \mathcal{L}}{\partial (\partial_\rho \partial_\sigma g_{\mu \nu})} \right) = 0
\]

The task of evaluating each term in this equation involves a formidable amount of algebra. Alternatively we can derive the field equations by considering directly the variation in the action resulting from variation in the metric tensor given by:

\[ g_{\mu \nu} \to g_{\mu \nu} + \delta g_{\mu \nu} \]

Where \( \delta g_{\mu \nu} \) and its first derivatives vanish on the boundary of the space-time region \( \mathcal{M} \). To determine the corresponding variation in the inverse metric we first note that the constant unit tensor \( \delta g^{\alpha \beta} = g^{\alpha \mu} g_{\mu \beta} \) doesn’t change under a (virtual) variation. Therefore

\[ g_{\mu \beta} \delta g^{\alpha \mu} + g^{\alpha \mu} \delta g_{\mu \beta} = 0 \]

After some arrangement we have

\[ \delta g^{\mu \nu} = -g^{\mu \alpha} g^{\nu \beta} \delta g_{\alpha \beta} \]

Now we write the Ricci scalar as \( R = g^{\mu \nu} R_{\mu \nu} \) and the first order variation in the Einstein-Hilbert action may be written as

\[
\delta S_{EH} = \delta S_1 + \delta S_2 + \delta S_3 = \int_{\mathcal{M}} \delta g^{\mu \alpha} R_{\mu \nu} \sqrt{-g} \, d^4x + \int_{\mathcal{M}} g^{\mu \nu} \delta R_{\mu \nu} \sqrt{-g} \, d^4x + \int_{\mathcal{M}} g^{\mu \nu} \delta (\sqrt{-g}) \, d^4x
\]

We need to factor out the (inverse) metric variations \( \delta g^{\mu \nu} \) in the second and third terms. Let us focus on the second term. It is useful to consider the variation in the full curvature tensor \( \delta R^\alpha_{\beta \gamma \delta} \), from which the corresponding variation in the Ricci tensor. Remembering that \( R^\alpha_{\beta \gamma} = \partial_\gamma \Gamma^\alpha_{\beta \gamma} - \partial_\delta \Gamma^\alpha_{\beta \gamma} + \Gamma^\mu_{\beta \delta} \Gamma^\alpha_{\mu \gamma} - \Gamma^\mu_{\beta \gamma} \Gamma^\alpha_{\mu \delta} \), we must consider the variation in the connection components

\[ \Gamma^\alpha_{\beta \gamma} \to \Gamma^\alpha_{\beta \gamma} + \delta \Gamma^\alpha_{\beta \gamma} \]

Notice that \( \delta \Gamma^\alpha_{\beta \gamma} \) is a difference between two connections and therefore a tensor. In a local geodesic coordinate system at some arbitrary point \( P \) we have \( \Gamma^\alpha_{\beta \gamma}(P) \) and so

\[ \delta R^\alpha_{\beta \gamma \delta} = \partial_\gamma (\delta \Gamma^\alpha_{\beta \delta}) - \partial_\delta (\delta \Gamma^\alpha_{\beta \gamma}) \]

In this system at \( P \) partial and covariant derivatives coincide

\[ \delta R^\alpha_{\beta \gamma \delta} = \nabla_\gamma (\delta \Gamma^\alpha_{\beta \delta}) - \nabla_\delta (\delta \Gamma^\alpha_{\beta \gamma}) \]

The quantities on the right hand side are tensors and therefore this expression holds not only in geodesic coordinates at \( P \) but in any arbitrary coordinate system. Since \( P \) was arbitrary this result is generally valid and is know as the Palatini equation. We now form the corresponding variation in the Ricci tensor

\[ \delta R_{\beta \gamma} = \nabla_\gamma (\delta \Gamma^\alpha_{\beta \alpha}) - \nabla_\alpha (\delta \Gamma^\alpha_{\beta \gamma}) \]

Finally the second term in the action is written as

\[
\delta S_2 = \int_{\mathcal{M}} g^{\beta \gamma} \left[ \nabla_\gamma (\delta \Gamma^\alpha_{\beta \alpha}) - \nabla_\alpha (\delta \Gamma^\alpha_{\beta \gamma}) \right] \sqrt{-g} \, d^4x
\]
\[ \int_{\mathcal{R}} \nabla_{\gamma} \left[ g^{\beta\gamma} \left( \delta \Gamma^\sigma_{\beta\alpha} - g^{\beta\gamma} \left( \delta \Gamma^\gamma_{\beta\sigma} \right) \right) \right] \sqrt{-g} d^4x \]

Here we used the fact that for pseudo-Riemann manifolds we have \( \nabla_{\gamma} g^{\beta\gamma} = 0 \). Using the divergence theorem this integral can be written as a “surface” integral at the boundary of \( \mathcal{R} \) which vanishes, assuming that the variation in the connection vanishes on the boundary. Consider now the third term in the action \( \delta S_3 \). We must express \( \delta \sqrt{-g} \) in terms of the variation \( \delta g^{\mu\nu} \) and it can be shown that

\[ \delta g = g g^{\mu\nu} \delta g_{\mu\nu} = -g g_{\mu\nu} \delta g^{\mu\nu} \]

We used the previous result \( g_{\mu\rho} \delta g^{\alpha\mu} + g^{\alpha\mu} \delta g_{\mu\beta} = 0 \). Therefore we have

\[ \delta \sqrt{-g} = -\frac{1}{2} (-g)^{-1/2} \delta g = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \]

After substitution in \( \delta S_3 \), we finally conclude that the variation in the Einstein-Hilbert action can be written as

\[ \delta S_{EH} = \int_{\mathcal{R}} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu} \sqrt{-g} d^4x \]

Requiring \( \delta S_{EH} = 0 \) and given the fact that the variation \( \delta g^{\mu\nu} \) is arbitrary we recover the Einstein field equations in vacuum:

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \]

We obtained the field equations by varying an action which we constructed very naturally upon basic ideas of symmetry and simplicity. This illustrates the power of the variational approach. This formalism suggests how Einstein’s theory might be modified by adding to the Lagrangian terms proportional to \( R^2, R^3 \), etc. It provides also a mean of investigating alternative gravitational Lagrangians constructed by invariants obtained from the Riemann tensor (such as \( R_{\alpha\beta\gamma} R^{\alpha\beta\gamma} \delta \) or from the torsion tensor for example (see part C).

**Palatini approach**

The Palatini approach to general relativity provides a more elegant and illuminating method to obtain the Einstein field equations from an action depending only on dynamical fields and their first derivatives. In this formalism one treats the metric \( g_{\mu\nu} \) and the connection \( \Gamma_{\mu\gamma}^{\nu} \) as independent fields — one does not assume any explicit relationship between the metric and the connection. The following derivation is essentially based on [13]

Let us start from the Einstein-Hilbert Lagrangian density

\[ \mathcal{L}_{EH} = \sqrt{-g} R_{\mu\nu} g^{\mu\nu} = \sqrt{-g} g^{\mu\nu} (\nabla_{\nu} \Gamma_{\mu\alpha}^{\alpha} - \nabla_{\alpha} \Gamma_{\mu\nu}^{\alpha} + \Gamma_{\mu\nu}^{\alpha} \Gamma^{\gamma}_{\alpha \nu} - \Gamma_{\mu\nu}^{\alpha} \Gamma_{\gamma \nu}^{\alpha}) \]

Which we now consider as a functional of the metric, the connection and the first derivatives of the connection \( \mathcal{L}_{EH} = \mathcal{L}_{EH}(g_{\mu\nu}, \Gamma_{\mu\gamma}^{\alpha}, \partial_{\lambda} \Gamma_{\mu\nu}^{\alpha}). \) Consider a variation in the action due to variation in the metric alone:

\[ \delta S_{EH} = \int_{\mathcal{R}} \delta(\sqrt{-g} g^{\mu\nu}) R_{\mu\nu} d^4x \]

By demanding that \( \delta S_{EH} = 0 \) for an arbitrary variation in the metric, we immediately find the Einstein field equations in vacuum:

\[ R_{\mu\nu} = 0 \]

Consider now varying the action with respect to the connection, which yields

\[ \delta S_{EH} = \int_{\mathcal{R}} \delta R_{\mu\nu} \sqrt{-g} g^{\mu\nu} d^4x = \int_{\mathcal{R}} \sqrt{-g} g^{\mu\nu} \left( \nabla_{\nu} (\delta \Gamma_{\mu\alpha}^{\alpha}) - \nabla_{\alpha} (\delta \Gamma_{\mu\nu}^{\alpha}) \right) d^4x \]

where we have used a contracted version of the already mentioned Palatini equation. Relabeling some indices and using Leibniz rule for differentiation, we get
At this stage the procedure is not assuming a pseudo-Riemann manifold with the Levi-Civitta connection so we do not impose that $\nabla_\gamma g^{\mu \nu} = 0$. The first integral may be written as an integral on the boundary $\partial \mathcal{R}$ which vanishes if we assume that the variation in the connection vanishes on the boundary. We thus find that (after relabeling indices again)

$$\delta S_{EH} = - \int_{\partial \mathcal{R}} \left( \delta_{\alpha}^\mu \nabla_\beta g^{\mu \beta} - \nabla_\alpha g^{\mu \nu} \right) \delta \Gamma_{\mu \nu} \sqrt{-g} d^4 x$$

Now we should specify the characteristics of the manifold appropriate to general Relativity. We should therefore assume that it is a torsionless manifold (see chapter 7 and part B) which means that our connection coefficients should be symmetric in the lower indices (this implication is valid in coordinate basis). Therefore the variation $\delta \Gamma_{\mu \nu}^{\alpha}$, although arbitrary, should have this symmetry. As a result, demanding $\delta S_{EH} = 0$ only requires the symmetric part of the factor in parenthesis to vanish (when contracted with $\delta \Gamma_{\mu \nu}^{\alpha}$, the antisymmetric part vanishes automatically). Thus stationarity of the action requires that

$$\frac{1}{2} \delta_{\alpha}^\nu \nabla_\beta g^{\mu \beta} - \frac{1}{2} \nabla_\alpha g^{\mu \nu} + \frac{1}{2} \delta_{\alpha}^\mu \nabla_\beta g^{\nu \beta} - \frac{1}{2} \nabla_\alpha g^{\nu \mu} = \frac{1}{2} \delta_{\alpha}^\nu \nabla_\beta g^{\mu \beta} + \frac{1}{2} \delta_{\alpha}^\mu \nabla_\beta g^{\nu \beta} - \nabla_\alpha g^{\nu \mu} = 0$$

$$\Rightarrow \nabla_\mu g^{\nu \mu} = 0$$

Thus demanding the Einstein-Hilbert action to be stationary with respect to variations in the (symmetric) connection, we derive that the covariant derivative of the metric must vanish, which can be written as

$$\partial_\rho g_{\mu \nu} = \Gamma_{\mu \rho}^{\alpha} g_{\alpha \nu} + \Gamma_{\nu \rho}^{\alpha} g_{\mu \alpha}$$

Cyclically permuting the indices $\rho \mu \nu$ and combining the results we then arrive at the result that $\Gamma_{\mu \rho}^{\alpha}$ should be the Levi-Civitta (metric) connection $\Gamma_{\rho \nu}^{\alpha} = \frac{1}{2} g^{\alpha \sigma} \left( \partial_\rho g_{\sigma \nu} + \partial_\nu g_{\rho \sigma} - \partial_\sigma g_{\rho \nu} \right)$.

### Variational principle for gravity with matter-energy source. The dynamical energy-momentum tensor

The full Einstein equations, in the presence of other (non gravitational) fields, can also be obtained from the variational principle. One simply as to add an extra term to te action:

$$S = \frac{1}{2k} S_{EH} + S_M = \int_{\mathcal{R}} \left( \frac{1}{2k} \mathcal{L}_{EH} + \mathcal{L}_M \right) d^4 x \quad \mathcal{L}_{EH} = R \sqrt{-g} \quad k = \frac{8\pi G}{c^4}$$

The factor $\frac{1}{2k}$ is chosen for convenience. If we vary the action with respect to the (inverse) metric we get

$$\frac{1}{2k} \delta \mathcal{L}_{EH} + \frac{\delta \mathcal{L}_M}{\delta g^{\mu \nu}} = 0$$

Remember that when deriving the Einstein equations from the Einstein-Hilbert action we arrived at

$$\delta S_{EH} = \int_{\mathcal{R}} \left( R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} \right) \delta g^{\mu \nu} \sqrt{-g} d^4 x$$

therefore

$$\frac{\delta \mathcal{L}_{EH}}{\delta g^{\mu \nu}} = g_{\mu \nu} \frac{\delta}{\delta g^{\mu \nu}}$$

We conclude that we recover the full Einstein equations if we take the energy-momentum tensor to be

$$T_{\mu \nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_M}{\delta g^{\mu \nu}}$$

This is in fact a symmetric tensor and it can be shown to obey $\nabla_\alpha T^{\alpha \beta} = 0$. This result follows from the general covariance of the matter action just as the gauge invariance of the action for electromagnetism implies charge conservation!
B.2 Gravitoelectromagnetism and GP-B experiment

B.2.1 Gravitoelectromagnetism. Linear form of the field equations and gauge invariance.

There are two alternative ways to expose the so called gravitoelectromagnetism formalism. One is through explicit geometric reasoning (the curvature approach) and the other consists in a linear perturbation procedure. I will deal with the linear or perturbative method. Eventually one ends up by defining gravitoelectric scalar and gravitomagnetic vector potentials and associated gravitoelectric and gravitomagnetic fields. These fields obey to analogous (Einstein-Maxwell) equations. The linear approach, often called the weak field limit of gravity, is not metric independent in the sense that it assumes a background metric. It starts by a small perturbation \( h_{\alpha\beta} \) of this background geometry and one can apply it to the solar system or to weak gravitational waves propagating through interstellar space for example. For Solar System situations we have: \( |h_{\alpha\beta}| \ll \frac{M_{\text{Sun}}}{R_{\text{Sun}}} \sim 10^{-6} \) [16]. The small perturbation of the background geometry is represented in the expression:

\[ g_{\alpha\beta} = g_{\alpha\beta}^{\text{backgr}} + h_{\alpha\beta} \]

Usually one chooses small perturbations over Minkowsky (flat) background space-time geometry.

\[ g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \quad |h_{\alpha\beta}| \ll 1 \]

As we will see in this weak field situation one can expand the field equations in powers of \( h_{\alpha\beta} \) and, without much loss of accuracy, keep only the linear terms. It is precisely this theory that we get when we search for the classical field corresponding to a gravitational boson (in a semi classical approach) satisfying the following properties: i) the stress-energy content of the linearized gravitational field; ii) the influence of this acting as source for corrections to the field; iii) the stress-energy content of the corrections \( h_{\alpha\beta}^{(1)} \) and the influence of this acting as source for corrections \( h_{\alpha\beta}^{(2)} \). and so on. This alternative way to derive general relativity was developed and explored by Gupta (1954, 1957, 1962), Kraichnan (1955), Thirring (1961), Feynman (1963a), Weinberg (1965), and Deser (1970), see [16]. The following table (taken from), [17]) presents a comparison of this derivation with the geometric approach:

<table>
<thead>
<tr>
<th>Derivation of General Relativity from geometric viewpoint and from spin-two viewpoint, compared and contrasted</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Nature of primordial space-time geometry</strong></td>
</tr>
<tr>
<td><strong>Topology of space-time</strong></td>
</tr>
<tr>
<td><strong>Vision of physics</strong></td>
</tr>
<tr>
<td><strong>Starting points</strong></td>
</tr>
<tr>
<td><strong>Resulting dynamical equations</strong></td>
</tr>
<tr>
<td><strong>Resulting assessment of the space-time geometry from which derivation started</strong></td>
</tr>
<tr>
<td><strong>View about complete gravitational collapse</strong></td>
</tr>
</tbody>
</table>

Table 5 - Comparison between 2 ways of deriving the field equations: the (geometric) Einstein approach; and the Spin-2 derivation. Taken from [17]
It is widely stated in the literature that through this bootstrapping process, one arrives back at the original non-linear field equations of general relativity, although this claim was also brought into question [18]:

“The two classical fields—electromagnetism and gravity—are described by a vector field and second rank symmetric tensor field, respectively. Considerations based on Lorentz group suggest interpreting them (when suitable restrictions are imposed) as corresponding to massless spin-1 and spin-2 fields. The vector field A, couples to a conserved current J, but does not contribute to this current (that is, the photon does not carry charge). In contrast, the tensor field is believed to be coupled to the energy momentum tensor; since the field itself carries energy, it has to couple to itself in a nonlinear fashion (the situation is similar to Yang-Mills fields which carry isotopic charge and hence are non-linear). It may, therefore, be possible to obtain a correct theory for gravity by starting with a massless spin-2 field h_{ab} coupled to the energy momentum tensor T_{ab} of other matter sources to the lowest order, introducing self-coupling of h_{ab} to its own energy momentum tensor at the next order and iterating the process. This will lead to a field theoretic description of gravity in a Minkowski background and is conceptually quite attractive. This attempt has a long history. The field equation for a free massless spin-2 field was originally obtained by Fierz and Pauli. The first attempt to study the consequences of coupling this field to its own energy momentum tensor seems to have been by Kraichnan in unpublished work done in 1946-47. The first published attempt to derive the non-linear coupling is by Gupta and Kraichnan published some of his results soon after. Feynman has provided a derivation in his Caltech lectures on gravitation during 1962-63. The problem was re-addressed by a clever technique by Deser. Virtually all these approaches claim to obtain not only Einstein’s field equations but also the Einstein-Hilbert action (this result is widely quoted in literature) and, at first sight, seems eminently reasonable. However, deeper examination raises several disturbing questions (…)”

( T. Padmanabhan, in [18])

Returning to the linearized theory, adopting the perturbed form for the metric, the resulting connection coefficients when linearized in the metric perturbation are given by:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} \eta^{\mu\nu} (\partial_\beta h_{\alpha\nu} + \partial_\alpha h_{\beta\nu} - \partial_\nu h_{\alpha\beta}) = \frac{1}{2} (\partial_\beta h^\mu_\alpha + \partial_\alpha h^\mu_\beta - \partial^\mu h_{\alpha\beta})$$

A similar linearization of the Ricci tensor yields

$$R_{\alpha\beta} = \partial_\gamma \Gamma^\gamma_{\alpha\beta} - \partial_\beta \Gamma^\gamma_{\alpha\gamma} = \frac{1}{2} \left( \partial_\gamma \partial_\beta h^\gamma_\alpha + \partial_\alpha \partial_\beta h^\gamma_\beta - \partial_\beta \partial_\gamma h_{\alpha\beta} - \partial_\gamma \partial_\alpha h \right)$$

where

$$h \equiv \eta^{\mu\nu} h_{\mu\nu} = h^\alpha_\alpha$$

After a further contraction to form the curvature scalar

$$R = g^{\alpha\beta} R_{\alpha\beta} \approx h^{\alpha\beta} R_{\alpha\beta}$$

and defining the field $\tilde{h}_{\mu\nu}$ as

$$\tilde{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h_{\mu\nu} h$$

one gets the

Einstein equations in its linearized form

$$\partial_\alpha \partial^\alpha \tilde{h}_{\mu\nu} + h_{\mu\nu} \partial_\rho \partial_\sigma \tilde{h}^{\rho\sigma} - \partial_\rho \partial_\nu \tilde{h}^\rho_\mu - \partial_\rho \partial_\mu \tilde{h}^\rho_\nu = -2 k T_{\mu\nu}$$

These equations are general for any (small) perturbation field of Minkowsky geometry.

In B.1 it was shown that the equation $\nabla_\mu T^{\mu\beta} = 0$ in the full non-linear theory leads directly to the geodesic equation of motion for the world line of a test particle (which moves in the space-time geometry that it perturbs). Performing a similar calculation using $\partial_\mu T^{\mu\beta} = 0$ for the linear theory in Minkowsky space-time (and using the appropriate coordinate system) one is led to the conclusion that $x^\mu = 0$ which doesn’t make sense. This means that the linearized theory is a useful approximation, provided we are interested only in the far field of sources whose motion we know a priori, and that we are willing to neglect the “gravity of gravity”. The effect of gravitational fields on test particles can be computed by inserting the (linearized) connection coefficients into the geodesic equations but, to calculate how some body moves under the action of its own gravity, one need the non-linear terms in the equations that the linear theory discards [12].
Symmetries and gauge invariance

- Consider a global Lorentz transformation

\[ x^\mu = \Lambda^\mu_\nu x^\nu \quad \eta_{\mu\nu} = \Lambda^\rho_\mu \Lambda^\sigma_\nu \eta_{\rho\sigma} \]

where \( \Lambda^\mu_\nu \) are 6 independent constant parameters. It is easy to show that the new metric can also be expressed as a small perturbation of a flat Minkowsky space-time background geometry.

\[ g_{\mu\nu} = \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x^\nu}{\partial x'^\nu} g_{\mu\nu}' = \Lambda^\mu_\mu A^\nu_\nu (\eta_{\mu\nu} + h_{\mu\nu}) = \eta_{\mu\nu} + A^\mu_\mu A^\nu_\nu h_{\mu\nu} \]

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad h_{\mu'\nu'} = \Lambda^\mu_\mu A^\nu_\nu h_{\mu\nu} \]

In this framework a weak gravitational field means slightly curved space-time geometry but on the other hand, since under a Lorentz transformation the perturbation transforms just like a tensor field in special relativity, it is equivalently valid to consider \( h_{\mu\nu} \) as a gravitational field in Minkowsky space-time, just like \( A_\mu \) describes the electromagnetic field in this space.

As a consequence in the linear theory any index manipulation on \( h_{\mu\nu} \) may be performed by the Minkowsky metric instead of using the full metric. For instance by demanding that \( g_{\mu\nu} g^{\mu\rho} = \delta^\rho_\mu \) one get:

\[ g^{\mu\rho} = \eta^{\mu\rho} - h^{\mu\rho} \]

Consider now a general and infinitesimal coordinate transformation and let us see how the metric transforms

\[ x^\mu = x^\mu + \xi^\mu (x) \]

\[ g_{\mu\nu} = \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x^\nu}{\partial x'^\nu} g_{\mu\nu}' = \delta^\mu_\mu - \partial_\mu \xi^\mu (\delta^\nu_\nu - \partial_\nu \xi^\nu) (\eta_{\mu\nu} + h_{\mu\nu}) \]

We see that the transformed metric may once again be written in the familiar perturbed way:

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} - \partial_\mu \xi^\nu - \partial_\nu \xi^\mu \]

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]

\[ h_{\mu\nu} = - \partial_\mu \xi^\nu - \partial_\nu \xi^\mu \]

In this way the new field also obeys the same (linear) Einstein equations. In other words, these equations are invariant with respect to transformations of \( h_{\mu\nu} \) completely analogous to the Gauge transformations of Electromagnetic 4-potential

\[ A_\mu^{(new)} = A_\mu + \partial_\mu \psi \quad h_{\mu\nu}^{(new)} = h_{\mu\nu} - \partial_\mu \xi^\nu - \partial_\nu \xi^\mu \]

\[ \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h_{\mu\nu} h \quad \Rightarrow \quad \partial_\rho \bar{h}^{\mu\rho} = \partial_\rho \bar{h}^{\mu\rho} - \partial_\sigma \partial^\sigma \xi^\mu \]

According to this procedure a choice of a gauge translates in a given choice for the arbitrary function \( \xi^\mu (x) \). In particular let us define the Lorentz gauge:

\[
\text{Lorentz gauge} \quad \partial_\mu \bar{h}^{\mu\nu} = 0
\]

Using the linear expression for the connection coefficients it’s easy to show that this condition can also be expressed by the expression:

\[ g^{\alpha\beta} T^\mu_{\alpha\beta} = 0 \]

In this gauge the field equations \( G_{\alpha\beta} = kT_{\alpha\beta} \) read

\[
\partial^\gamma \partial_\gamma \bar{h}_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\alpha\beta} \\
\frac{1}{c^2} \frac{\partial^2 \bar{h}_{\alpha\beta}}{\partial t^2} - \nabla^2 \bar{h}_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\alpha\beta}
\]

A new gauge transformation that preserves the Lorentz gauge is obtained by choosing \( \xi^\mu (x) \) such that:

\[ \partial^\gamma \partial_\gamma \xi^\mu (x) = 0 \]
There are some common gauges that are worth to mention and we can think of them as different choices for defining the constant time 3-d hypersurfaces. Some common examples are [19]:

- **Synchronous gauge**

  It confines any perturbations to Minkowsky space-time to the spatial part of the metric

  \[ g^{0\mu} = (1,0,0,0) \]

  and is commonly used in the study of cosmological perturbations

- **Newtonian gauge**

  The deviations to Minkowsky metric are expressed in terms of a function that looks like the Newtonian potential, \( \Phi \). The metric is diagonal and given by

  \[
  ds^2 = \left(1 + \frac{\Phi}{c^2}\right)c^2 dt^2 - \left(1 - \frac{\Phi}{c^2}\right)(dx^2 + dy^2 + dz^2)
  \]

  This gauge in fact doesn’t allow gravitational waves but is the appropriate choice when dealing with weak gravitational fields generated by static mass distributions.

- **Transverse traceless gauge**

  This is the gravitational analogue of the Coulomb gauge (\( \nabla \cdot \vec{A} = 0 \)), which allows electromagnetic radiation in free space to be described in terms of the vector potential only. In gravity the corresponding gauge definition is given by

  \[
  h^{0\mu} = 0 \quad h^{\mu}_{\mu} = 0
  \]

**Solutions in the Lorentz gauge**

Let us consider the vacuum solutions first. The equations admit (free) plane harmonic wave solutions

\[
\partial_{\mu}\partial^\alpha \bar{h}^{\mu\alpha} = 0
\]

\[
\bar{h}^{\mu\nu} = A^{\mu\nu}e^{ikx^\alpha} \quad k^\alpha = \left(\frac{w}{c}, \frac{k}{c}\right)
\]

After substituting this expression into the wave equation and using the Lorentz condition we can conclude that these are transverse waves propagating at the speed of light \( c \)

\[
\text{wave equation} \Rightarrow k_\mu k^\mu = 0 \quad \text{=} \quad \frac{w}{k} = c
\]

\[
\text{Lorentz condition} \Rightarrow A^{\mu\nu}k_\nu = 0 \quad \text{=} \quad \text{transverse waves}
\]

Without boundary constraints, the most general solution (in unlimited “vacuum”) is a linear and continuous superposition of harmonic plane waves:

\[
\bar{h}^{\alpha\beta}(x) = Re \left[ \int \frac{d^3 \tilde{k}}{(2\pi)^{3/2}} A^{\alpha\beta}(\tilde{k})e^{ikx_\mu} \right]
\]

The solutions for waves created by a given matter-energy source in a region containing such sources is easily shown to be (using a Green’s function method):

\[
\bar{h}^{\mu\nu}(ct, \vec{x}) = -\frac{4G}{c^4} \int \frac{T^{\mu\nu}(ct - |\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|} d^3 \vec{y}
\]
\[
t' = t - \frac{\|\vec{x} - \vec{x}'\|}{c} < t \quad \text{retarded solutions (past – future causality)}
\]
\[
t' = t - \frac{\|\vec{x} - \vec{x}'\|}{c} > t \quad \text{advanced solutions (future – past causality)}
\]

Observer’s perspective:

![Diagram of retarded and advanced solutions](image)

Point source perspective:

![Diagram of retarded and advanced solutions](image)

The physical interpretation of the various components \( T^{\mu\nu} \) is as follows:

\[
\int T^{00}(\vec{x}', ct')d^3\vec{x}' \quad \text{total energy } e \text{ of the source (including rest mass } M \text{c}^2); \\
\int T^{0i}(\vec{x}', ct')d^3\vec{x}' \quad \text{total momentum of the source } (\times c) \text{ in the } x^i \text{ direction } = P^i \times c; \\
\int T^{ij}(\vec{x}', ct')d^3\vec{x}' \quad \text{integrated internal stresses in the source}
\]

In the linear theory for an isolated source the quantities \( M \) and \( P^i \) are constants which can easily be proved from the conservation equation \( \partial_{\alpha}T^{a\beta} = 0 \) (It is also possible to choose our spatial coordinates to correspond to the centre of momentum frame for the case of a system of particles, in which case \( P^i = 0 \). In fact more realistically, the source loses energy via emission of gravitational radiation, but the energy-momentum carried away by the gravitational field is quadratic in \( h_{\mu\nu} \) [12].

When the particles are non-relativistic \( u << c \), the following expressions are valid

\[
T^{00} = \rho_m c^2 \quad T^{0i} = \rho_m c u^i \quad T^{ij} = \rho_m u^i u^j \quad \frac{|T^{ij}|}{|T^{00}|} \sim u^2/c^2 \Rightarrow T^{ij} \approx 0
\]

Generally, the gravitational source is dynamic and may have a spatial extent that isn’t small comparing with the distance to the point at which one wishes to calculate the field. In such cases, it is often very hard (or impossible) to obtain an analytical
expression for the solution. In analogy with what is usually done with electromagnetism it is convenient to make a *multipole expansion* which allows successive approximations to the solution. One begins with the following Taylor expansion:

$$\frac{1}{||\vec{x}-\vec{x'}||} = \frac{1}{||\vec{x}||} - x'^k \partial_k \left( \frac{1}{||\vec{x}||} \right) + \cdots$$

After some algebra one obtains the so called multipole expansion of the field \([12]\):

$$\bar{h}^{\mu\nu}(\vec{x},ct) = -\frac{4G}{c^4} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} M^{\mu\nu k_2...k_l}(ct') \partial_{k_2}...\partial_{k_l} \left( \frac{1}{||\vec{x}||} \right)$$

$$M^{\mu\nu k_2...k_l}(ct') = \int T^{\mu\nu}(\vec{x}',ct') x'^{k_1} x'^{k_2}...x'^{k_l} d^3\vec{x}'$$

$$k_1 k_2...k_l = 1,2,3$$

<table>
<thead>
<tr>
<th>$l$</th>
<th>$\bar{h}^{\mu\nu}(\vec{x},ct)$</th>
<th>$M^{\mu\nu}(ct')$</th>
<th>$M^{\mu\nu k_1}(ct')$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-\frac{4G}{c^4} M^{\mu\nu}(ct')$</td>
<td>$\int T^{\mu\nu}(\vec{x}',ct') d^3\vec{x}'$</td>
<td>$\int T^{\mu\nu}(\vec{x}',ct') x'^{k_1} d^3\vec{x}'$</td>
</tr>
<tr>
<td>1</td>
<td>$-\frac{4G}{c^4} M^{\mu\nu k_1}(ct') \partial_{k_1} \left( \frac{1}{</td>
<td></td>
<td>\vec{x}</td>
</tr>
<tr>
<td>2</td>
<td>$-\frac{4G}{c^4} M^{\mu\nu k_2 k_3}(ct') \partial_{k_1} \partial_{k_2} \left( \frac{1}{</td>
<td></td>
<td>\vec{x}</td>
</tr>
</tbody>
</table>

**Compact source approximation**

In the compact source approximation the extent of the source is small comparing with the distance to the point at which one wishes to calculate the field. In this case the instant $t'$ of the source event does not contribute to the integral. The resultant expression is similar to the zero\(^{th}\) order of the multipole expansion and is given by:

$$\bar{h}^{\mu\nu}(\vec{x},ct) = -\frac{4G}{c^4 ||\vec{x}||} \int T^{\mu\nu}(\vec{x}',ct') d^3\vec{x}'$$

$$t' = t - \frac{||\vec{x}||}{c}$$

We thus consider only the far-field solution to the linearized gravitational equations, which varies as $1/r$.

Using the compact source approximation, we have

**Solutions in the compact source approximation**

$$\bar{h}^{\mu\nu}(\vec{x},ct) = -\frac{4G}{c^4 ||\vec{x}||} \int T^{\mu\nu}(\vec{x}',ct') d^3\vec{x}'$$

$$t' = t - \frac{||\vec{x}||}{c}$$

$$\bar{h}^{00} = -\frac{4Ge}{||\vec{x}|| c^4}, \quad \bar{h}^{10} = \bar{h}^{01} = -\frac{4G}{c^4 ||\vec{x}||} \int T^{0i}(\vec{x}',ct') d^3\vec{x}'$$

(in the center of momentum coordinate system $\bar{h}^{00} = \bar{h}^{0i} = 0$)

$$\bar{h}^{ij}(ct,\vec{x}) = -\frac{4G}{c^4 ||\vec{x}||} \int T^{ij}(\vec{x}',ct') d^3\vec{x}'$$

**Integrated stress within the source**

$$\bar{h}^{ij}(ct,\vec{x}) = -\frac{2G}{c^6 r} \left[ \frac{d^2 I^{ij}(ct')}{dt'^2} \right]$$

$I^{ij}$ is the quadrupole-moment tensor of the energy density of the source:

$$I^{ij}(ct') = \int T^{00}(ct',\vec{x}) x'^i x'^j d^3\vec{x}'$$

51
As we will see, this formula can be used to determine the far-field gravitational radiation generated by a time varying matter source.

Stationary sources
For these kinds of sources we have $\partial_\alpha T^{\mu\nu} = 0$ and the “gravitational tensor field” is given by:

$$\bar{h}^{\mu\nu}(\vec{x}) = -\frac{4G}{c^4} \int \frac{T^{\mu\nu}(\vec{y})}{|\vec{x} - \vec{y}|} d^3\vec{y}$$

The condition of stationarity does not necessarily imply staticity, which would additionally require the form of metric and of $T^{\mu\nu}$ to be invariant under a time reflection $t \rightarrow -t$. One typical example of a stationary but non-static situation is a uniform rigid axisymmetric body rotating with constant angular velocity.

Gravitoelectric and gravitomagnetic potentials
We can introduce the gravitomagnetic potentials $\Phi_g, A_g$ through the following definitions:

$$\bar{h}^{00}(ct, \vec{x}) \equiv \frac{4\Phi_g}{c^2}(ct, \vec{x}) \quad \bar{h}^{0i}(ct, \vec{x}) \equiv \frac{A_g}{c}(ct, \vec{x})$$

$$\bar{h}^{\mu\nu}(ct, \vec{x}) = -\frac{4G}{c^4} \int \frac{T^{\mu\nu}(ct - |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|} d^3\vec{x}'$$

In general for non-relativistic sources we have

$$\Phi_g(x) \equiv \frac{c^2}{4} \bar{h}^{00}(x) = -\frac{G}{c^2} \int \frac{\rho(\vec{x}', t')}{\|\vec{x} - \vec{x}'\|} d^3\vec{x}'$$

$$A_g(x) \equiv c\bar{h}^{0i}(x) = -c \frac{4G}{c^4} \int \frac{j^i(\vec{x}', t')}{\|\vec{x} - \vec{x}'\|} d^3\vec{x}' = -\frac{4G}{c^4} \int \frac{\rho u^i}{\|\vec{x} - \vec{x}'\|} d^3\vec{x}'$$

Where $j^i(\vec{x}') \equiv \rho(\vec{x}')u^i(\vec{x}')$ is the energy current density. For non-relativistic stationary sources we have:

$$\Phi(\vec{x}) = -G \int \frac{\rho(\vec{y})}{|\vec{x} - \vec{y}|} d^3\vec{y} \quad A^i(\vec{x}) = -\frac{4G}{c^2} \int \frac{j^i(\vec{y})}{|\vec{x} - \vec{y}|} d^3\vec{y}$$

Taking the point of view that our fields describe a perturbation in Minkowsky geometry ($g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$), what is the line element associated to non-relativistic sources? For these sources we have

$$\bar{h}^{00}(ct, \vec{x}) \equiv \frac{4\Phi_g}{c^2}(ct, \vec{x}) \quad \bar{h}^{0i}(ct, \vec{x}) \equiv \frac{A_g}{c}(ct, \vec{x}) \quad \bar{h}^{ij} = 0$$

We see that $\bar{h} = \bar{h}^{00}$. Now considering that $h^{\mu\nu} = \bar{h}^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} \bar{h}$ we arrive at $h_{00} = h_{11} = h_{22} = h_{33} = \frac{2\Phi}{c^2}, h_{0i} = \frac{A_i}{c}, h_{ij} = 0$, concluding the following:

**Line element for non-relativistic stationary sources**

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right)c^2dt^2 + 2A_i dx^i dt - \left(1 - \frac{2\Phi}{c^2}\right)((dx^1)^2 + (dx^2)^2 + (dx^3)^2)$$

Note that $A_i dx^i = \eta_{ij} A_j dx^j = -\vec{A} \cdot d\vec{x}$. In the Newtonian limit, for static sources we get the following
We see that for a static, spherical object of mass M, the following familiar expressions arise:

\[
ds^2 = \left(1 + \frac{2\Phi}{c^2}\right)c^2dt^2 - \left(1 - \frac{2\Phi}{c^2}\right)((dx^1)^2 + (dx^2)^2 + (dx^3)^2)
\]

For stationary, non-relativistic sources, the following quantities,

\[
\Phi = -\frac{GM}{r} \quad \vec{A}_g = \nabla \Phi_g, \quad \vec{B}_g = \nabla \times \vec{A}_g
\]

obey the linear Einstein equations:

\[
\begin{align*}
\nabla^2 \Phi_g &= 4\pi G \rho \\
\nabla^2 \vec{A}_g &= \frac{16\pi G}{c^2} j \\
\n.\vec{E}_g &= -4\pi G \rho \\
.\vec{B}_g &= 0 \\
\n\nabla \times \vec{E}_g &= 0 \\
\n\nabla \times \vec{B}_g &= -\frac{16\pi G}{c^2} j
\end{align*}
\]

These are analogous to Maxwell equations according to the following the correspondence:

\[
\epsilon_0 \leftrightarrow -\frac{1}{4\pi G} \quad \mu_0 \leftrightarrow -\frac{16\pi G}{c^2}
\]

The (unbounded) stationary gravitoelectromagnetic potentials are:

\[
\Phi(\vec{x}) = -G \int \frac{\rho(\vec{y})}{|\vec{x} - \vec{y}|} d^3\vec{y} \quad A^i(\vec{x}) = -\frac{4G}{c^2} \int \frac{j^i(\vec{y})}{|\vec{x} - \vec{y}|} d^3\vec{y}
\]

For non-stationary sources the Einstein-Maxwell equations are:

\[
\begin{align*}
\nabla \vec{E}_g &= -4\pi G \rho \\
\n\nabla \vec{B}_g &= 0 \\
\n\nabla \times \vec{E}_g &= -\frac{\partial \vec{B}_g}{\partial t} \\
\n\nabla \times \vec{B}_g &= -\frac{16\pi G}{c^2} j + 4 \frac{\partial \vec{E}_g}{\partial t}
\end{align*}
\]

The gravitoelectromagnetic potentials are given by the expressions:

\[
\begin{align*}
\tilde{h}^{\mu\nu}(ct,\vec{x}) &= -\frac{4G}{c^4} \int \frac{T^{\mu\nu}(ct - |\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|} d^3\vec{y} \\
\tilde{R}^{00} &= \frac{4\Phi}{c^2} \\
\tilde{R}^{0i} &= \frac{A^i}{c} \\
\partial_t \rho + \nabla \cdot j &= 0
\end{align*}
\]
Something that is interestingly different from Maxwell’s theory, is that a Lorentz-like force equation (expressing the acceleration on a moving charged particle in terms of the gravitoelectric and gravitomagnetic fields) is derivable from a linearization of the geodesic equation. The geodesic equation, as previously mentioned, follows from $\nabla_\alpha T^{\alpha\beta} = 0$ which is implicitly contained in GR field equations.

There are many interesting astrophysical applications of gravitoelectromagnetism (see next section), but there is a great potential for technological applications also [20-21]. For example, just as rotating superconductors give rise to strong magnetic fields, the so called anomalous London moment, one should also expect unusually strong gravitomagnetic and gravitoelectric fields produced by rotating superfluids. The following picture illustrates torus-shaped system with a superconducting and superfluid in helicoidal movement around the torus, giving rise to gravitomagnetic field oriented along the torus. When this field is time-varying, it produces a gravitoelectric field which could be used as a propulsion system creating an adequate velocity field on the surrounding atmosphere. This structure might be a component of a more complex gravitodynamical system that I call an “Einstein-DaVinci system”

![Fig 3– Gravitomagnetic and gravitoelectric fields around a rotating superconducting an superfluid, on a torus structure – an “Einstein-DaVinci device”](image)

### B.2.2 Gravitoelectromagnetism and the GP-B experiment

**About the origins of GP-B**

Around 1917 Willem de Sitter (1872-1934) used GR to study the precession of the Earth-Moon system due to the Sun’s gravitational field, an effect known as the *de Sitter solar geodetic effect*. After this kind of analysis was applied for rotating bodies like the Earth, by the mathematician Jan Schouten (1883-1971) and the physicist and musician Adriaan Fokker (1887-1972). The calculation made by Sitter, Schouten and Fokker were popularized with an article by Arthur Eddington, entitled *The Mathematical Theory of Relativity* (1923). Later in 1930, P.M.S. Blackett (1897-1974) considered the idea of studying the general relativistic effects of the Earth’s gravitational field on a gyroscope. At that time the technology was simply unable to obtain the required precision to measure the predicted effects. Simple calculations show that to measure general relativistic precession rates of gyros of about 10 miliarcsec with a 1% precision, requires a suppression of Newtonian dynamical
perturbations with a precision of about 0.1 miliarcsec/yr ($\sim 10^{-18}$ rad/s), or less. To understand the engineering challenge, note that this kind of precision implies, for the case of spherical gyros, an extremely high degree of homogeneity, with deviations from the perfect sphere characterized by $\frac{\delta r}{r} < 10^{-16}$, and assuming an extremely low gravitational field $a \sim 10^{-11}g$ (space environment).

The ideas of Blackett were taken in consideration by George E. Pugh and Leonard I. Schiff (1915-1971) independently. Apparently, Pugh was influenced by a talk of Huseyin Yilmaz proposing a test that required the use of a satellite to test an alternative theory of gravity. On the other hand, Schiff was partially inspired by the *Physics Today* magazine, where he found interesting information about the use of cryogenics technology to suppress undesired accelerations. In November 1959, Pugh published an article where he introduced the concept of “drag-free motion”. Finally in 1960, L. I. Schiff described 2 effects of the gravitational field on an ideal gyroscope put in orbit around the Earth:

- The “precession” (angle deviation) on the plane of the orbit: **The Geodetic effect**;
- A precession on a plane perpendicular to the earth’s axis, due to the Earth’s rotation: **The Lens-Thirring/frame-dragging effect**

The Schiff calculations [22] show that the conjugation of both effects result in the following (total) precession angular velocity

$$\tilde{\Omega}_{tot} = \frac{3GM}{2c^2r^3} (\vec{r} \times \vec{v}) + \frac{3Gl}{c^2r^3} \left( \frac{3\vec{r}}{r^2} (\vec{w}.\vec{r}) - \vec{w} \right)$$

One has to take into account correction due to Earth’s oblaticity which were computed by Barker and O’Connell in 1970 and by Breakwell in 1988. The basic ideas for the development of the GP-B experiment were already established

**GP-B Experiment**

The GP-B experiment results from an important collaboration between the Stanford University and NASA [22]. Its purpose was to detect the extremely small geodetic and Lens-Thirring effects on gyroscopes orbiting the Earth, with a precision less than 0.5 miliarcsec/yr. Although the first funding started on March 1964, the launching of the spacecraft carrying the gyros was only on the 20th of April, 2004, it collected data for approximately 17 months and the final official results were presented on December 2010. The reason why it took 40 years to start the experiment is simply the fact that the smallness of the effects to measure and the precision required were so challenging that many inexistent sophisticated technologies had to be designed and elaborated.

The Lens-Thirring and geodetic effects, for the case of the GP-B orbit, predicted from Einstein’s GR were:

- **Geodetic effect**: 6606 miliarcsec/yr
- **Frame-dragging**: 39 miliarcsec/yr

(Required precision: < 0.5miliarcsec/yr)

This experiment, designed to test GR, consists in putting gyroscopes, with axis initially aligned with a guide star (IM Pegasi HR 8703), orbiting the Earth in a polar orbit with an altitude of 642Km above the Pole.
Brief description of the experiment

Because this is an historical experiment, I will briefly describe the way it was accomplished. The experiment was the successful result of many complex technologies. The major aspects of these consisted in the following [22]:

-A spacecraft in a dragg-free orbit;
-Cryogenic technology;
-4 almost perfect spherical gyros;
-SQUID –Superconducting Quantum Interferometer Devices;
-Orbiting telescope;
- VLB (Very Long Baseline) interferometry.

Inside the spacecraft was a Dewar containing cryogenic technology in order to highly reduce thermal disturbances. Inside the Dewar lies the probe containing four gyroscopes and the measurement electronic systems such as the SQUID’s. Inside the spacecraft, attached to the probe, was a telescope which was always pointing to a guide star (IM Pegasi – HR 8703). The axes of the gyro were all initially aligned with the guide star and, due to the distorted space-time geometry, it was expected to detect the geodetic and frame dragging effects.

One of the gyros was used as a reference gyro to guarantee the dragg-free motion. In order to achieve an almost perfect dragg-free motion, that is, following the geometry of space-time without the disturbances of forces, the spacecraft had thrusters and sensors. Inside the spacecraft the gyro were protected but, the spacecraft itself was subjected to two major sources of disturbance. These came from the atoms and molecules of Earth’s atmosphere, which even at the height of 642 Km has its presence, and from the Solar wind and solar radiation pressure. The sensors were always detecting any deviations from a solidary motion with the reference gyro and would share this info with the thrusters. These were activated in a determined way (expelling gas in the required directions) in order to always align the spacecraft with the dragg-free motion of the reference gyro.

The gyros were constructed with extremely precise technology such that the Guinness record was attributed, being the most perfect spheres that humans ever made - if one of these gyros had the size of the Earth, its highest mountains would be only 2,4 m high! These gyros consisted in rotating quartz spheres, electrically suspended in a vacuum compartment. The gyro were covered with a thin layer of a superconducting material, a fact that was crucial for the measurement process. How can we measure an axis deviation of a rotating perfect homogeneous sphere? Superconductivity solves the problem. A rotating superconductor produces an anomalous magnetic field and magnetic moment – the London moment. This London moment has the property that it is perfectly aligned with the rotating axis of the gyro. The GP-B experiment used a measurement system based on a SQUID’s (Superconducting Quantum Interferometer Devices), which measured with very high precision the deviation of the magnetic moment direction from the initial reference direction (IM Pegasi).

Finally, the GP-B experiment also used VLB interferometry in order to correct the intrinsic motions of the reference star, including orbital motions since the guide star is part of a binary system, but also star pulsations. The telescope was aligned with high precision with the star and in order to compute the geodetic and frame-dragging effects relatively to the initial orientation, the star’s motion is subtracted. The VLB method allowed an extremely precise tracking of the motion of IM Pegasi.
General relativistic effects on gyroscopes

a) Schwarzschild solution and the geodetic effect

The geodetic effect follows naturally from the Schwarzschild solution [12]. As is well known, Schwarzschild space-time, corresponds to the static gravitational field with spherical symmetry around a spherical energy-mass distribution. It is a static and spatially isotropic space-time. The staticity condition means that the metric components are independent of the time coordinate and the line element is invariant for a time inversion, whereas the spatial isotropy implies that the line element depends only on the rotational invariants of the coordinates and their differentials. From these symmetry considerations, the Einstein equations in vacuum give the Schwarzschild line element

$$ds^2 = c^2 \left(1 - \frac{2GM}{c^2r}\right)dt^2 - \left(1 - \frac{2GM}{c^2r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Suppose a gyroscope moving along a circular geodesic on a Schwarzschild space-time. The important geometrical objects describing the kinematics are the tangent 4-velocity $u$, which obeys the geodesic equation $\frac{du}{dt} + \Gamma^\beta_{\gamma\delta}u^\gamma u^\delta = 0$, and the “spin” 4-vector, which is defined such that it is a spatial 4-vector. In the instantaneous inertial reference frame of the gyro these 4-vectors are $u = (c, 0, 0, 0)$ and $s = (0, \vec{s})$. Since Riemann space-time obeys the metricity condition $\nabla_a g^{\gamma\beta} = 0$, the inner product is preserved and therefore $s \cdot u = g_{a\beta}s^\alpha u^\beta = 0$ is always obeyed. We conclude that the spin 4-vector is parallely transported along the geodesics:

$$\frac{ds^\beta}{dt} + \Gamma^\beta_{\gamma\mu}u^\gamma s^\mu = 0$$

To obtain the evolution $s(\tau)$ of this 4-vector, one uses the Schwarzschild metric to compute for $u(\tau)$ and then solves the above equation for $s(\tau)$.

$$g_{a\beta} \rightarrow \Gamma^a_{\beta\gamma} \rightarrow u(\tau) \rightarrow s(\tau)$$

It is easy to show ([12]) that after a complete revolution around the circular geodesic, the total (geodetic) deviation angle from the original axis orientation is given by

$$\Delta \phi_{\text{geodetic}} = 2\pi \left(1 - \left(1 - \frac{3GM}{rc^2}\right)^{1/2}\right)$$

When $\frac{3GM}{c^2r} < 1$ we can use the Taylor expansion of $(1 - x)^{1/2}$ to obtain

$$\Delta \phi_{\text{geodetic}} \approx \frac{3\pi GM}{c^2r}$$

Which for the case of Earth, it can also be expressed as

$$\Delta \phi_{\text{geodetic}} \approx 6.5 \times 10^{-9} \left(\frac{R_{\text{Earth}}}{r}\right) \text{ rad} => \frac{\Delta \phi_{\text{geodetic}}}{\text{year}} \approx 8.4 \times 10^{5/2} \left(\frac{R_{\text{Earth}}}{r}\right)^{5/2}$$

The geodetic effect can be partially (1/3) interpreted within the framework of gravitoelectromagnetism. In this framework, this effect is the gravitational analogue of a spin-orbit coupling giving rise to an analogous Thomas precession – on the instantaneous inertial frame of the gyro, the Earth is orbiting around it and the gravitomagnetic field thus produced interacts with the gyro’s gravitomagnetic moment, producing an induced gravitomagnetic torque. The other part of the geodetic effect (2/3) cannot be interpreted gravitoelectromagnetically and it is a signature of the Schwarzschild space-time curvature. One can visualize this by the “missing inch” diagram. The geometry around Earth causes a shortening of circumferences and for the case of the GP-B orbit this corresponds to ~1.1 inch. The figure below illustrates a pedagogical way of showing how this effect can be responsible for a deviation of the angular momentum vector orientation.
b) Kerr solution and the Lens-Thirring effect

The geodetic effect follows naturally from the Kerr solution \([12,13]\). As is well known, Kerr space-time, corresponds to the stationary axially symmetric gravitational field around a axially symmetric energy-mass distribution in stationary rotation. The metric components are independent of the time coordinate and of the rotational angle coordinate. Besides it is invariant with respect to both time inversions \(t \to -t\) and rotational inversions \(\phi \to -\phi\), which guarantees that the source’s movement is purely rotational around the symmetry axis. From these symmetry considerations, after solving the Einstein’s equations in vacuum, we get the Kerr line element:

\[
\begin{align*}
\frac{\Delta}{\Sigma^2} c^2 dt^2 &- \frac{\Sigma^2 \sin^2 \theta}{\rho^2} (d\phi - w dt)^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 \\
\rho^2 &\equiv r^2 + a^2 \cos^2 \theta, \quad \Delta \equiv r^2 - 2\mu r + a^2, \quad \Sigma^2 \equiv (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \quad w = 2\mu c a / \Sigma^2.
\end{align*}
\]

Here \(\mu\) and \(a\) are constants. The Kerr solution goes to Schwarzschild solution when \(a \to 0\) showing that \(\mu \equiv GM/c^2\). The constant \(a\) is connected to the source’s angular momentum \(\vec{j}\) and the study of the slow rotation and weak-field limit allows the conclusion that \(\|\vec{j}\| = J = Mac\). Expanding \(ds^2_{Kerr}\) to first order in \(a\) (slow rotation), gives

\[
ds^2_{Kerr} = ds^2_{Schwarzschild} + \frac{4GJ}{c^2 r} \sin^2 \theta d\phi dt
\]

On the other hand, the weak-field limit gives, in spherical and Cartesian coordinates, respectively:

\[
ds^2 = c^2 \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 + \frac{2GM}{c^2 r}\right) \left(dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)\right) + \frac{4GJ}{c^2 r} \sin^2 \theta d\phi dt
\]

\[
ds^2 = c^2 \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 + \frac{2GM}{c^2 r}\right) (dx^2 + dy^2 + dz^2) + \frac{4GJ}{c^2 r^3} (xyd - ydx) dt
\]

The first term corresponds exactly to the (first order) weak-field limit approximation of the Schwarzschild solution.

Considering the motion of a gyroscope in a circular orbit in a Kerr space-time, and using the same method as before,

\[
\begin{align*}
g_{\alpha\beta} &\to \Gamma^\gamma_{\beta\gamma} \to u(\tau) \to s(\tau)
\end{align*}
\]

we get the Lens-Thirring precession rate:

\[
\vec{\Omega}_{LT} = \frac{2GJ}{c^2 r^3} (3(j, \vec{e}_r)\vec{e}_r - \vec{j})
\]

The plane of the circular orbit rotates (is “dragged”) around the Earth’s rotation axis because the space-time structure is dragged around the axis. Most astrophysical systems such as galaxies, stars and planets have space-time vortexes around them!

Within the gravitomagnetic analogy, the Lens-Thirring effect is analogous to a spin-spin interaction between a magnetic moment of a magnetic dipole and the magnetic field of a charged rotating object. In this case the gyro and the central
rotating mass interact via gravitomagnetism. In fact, using the slow rotation and weak-field limit of the Kerr metric, the gravitoelectric and gravitomagnetic fields of the rotating (approximately spherical) mass [12]:

$$\vec{E}_g(\vec{x}) = -\frac{GM}{|\vec{x}|^2} \vec{e}_r \quad \vec{B}_g(\vec{x}) = \frac{2G}{c^2|\vec{x}|^3} (\vec{J} - 3(\vec{J} \cdot \vec{e}_r) \vec{e}_r)$$

Now, just like a magnetic dipole (with magnetic moment $\vec{m}$) “immersed” in a magnetic field $\vec{B}$, experiences a magnetic force $\vec{F} = (\vec{m}, \vec{\nabla}) \vec{B}$ and torque $\vec{T} = \vec{m} \times \vec{B}$, here we also have gravitomagnetic force and torque experienced by the gyro, and these give rise to a gravitomagnetic precession

$$\vec{F}_g = \frac{1}{2} (\vec{s}, \vec{\nabla}) \vec{B}_g \quad \vec{T}_g = \frac{1}{2} (\vec{s} \times \vec{B}_g) \Rightarrow \frac{d\vec{s}}{dt} = \frac{1}{2} (\vec{s} \times \vec{B}_g)$$

As a result one obtains the expression for the gravitomagnetic Lens-Thirring effect

$$\bar{\Omega}_{LT} = \frac{G}{c^2 |\vec{x}|^3} (3(\vec{J} \cdot \vec{e}_r) \vec{e}_r - \vec{J})$$

The gravitomagnetic Lens-Thirring or frame-dragging effect can be considered to be somehow linked to Mach’s principle, according to which the inertia of a body is determined by the mass distribution of the Universe. GR is not fully Machian in the sense that accelerations might be defined with respect to the space-time structure and therefore, contrarily to Mach’s ideas, Newton’s rotating bucket full of water would still exhibit a concave water surface even on an empty universe. Nevertheless, according to GR the inertial frames are influenced by the local geometry which is influenced by the local mass-energy distributions. In this respect the theory maintains some of the Mach’s ideas that served as philosophical inspirations for its development. The frame-dragging effect can be viewed as a Machian effect in the sense that the inertial frame is affected by the (local) mass distribution. In fact it is possible that the inertial frames are completely determined by the mass, when this tends to the entire mass of the universe, in which case one speaks of “total” or “perfect” dragging.

"Although Einstein's theory of gravity does not, despite its name 'general relativity,' yet fulfill Mach's postulate of a description of nature with only relative concepts, it is quite successful in providing an intimate connection between inertial properties and matter, (...) Perhaps against majority expectation, this connection is instantaneous in nature. Furthermore, general relativity has brought us nearer to an understanding of the observational fact that the local inertial compass is fixed relative to the most distant cosmic objects, but there is surely desire for still deeper understanding.”

(Pfister in Mach's Principle: From Newton's Bucket to Quantum Gravity (1995))

**GP-B experiment, PPN formalism and alternative theories of gravity**

The PPN (post-Newtonian parameterized) formalism allows one to parameterize geometrical gravitational theories within the Solar System. Many parameters are defined describing deviations from GR. For example, two of the most important ones can arise from an expansion in powers of $1/c$ of the most general static spherically symmetric space-time [13]:

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) = \text{Newtonian limit} + \text{post} - \text{Newtonian corrections}$$

One gets

$$A(r) = 1 - \frac{2GM}{c^2r} + 2(\beta - \gamma) \left(\frac{GM}{c^2r}\right)^2 + \cdots \quad B(r) = 1 + 2\gamma \left(\frac{GM}{c^2r}\right) + \cdots$$

For GR we have $\gamma = 1$ and $\beta = 1$. Many gravitational “classical” tests can be expressed in terms of these parameters. For example the deflection of light due the Sun’s gravitational field $\delta \phi_{\text{defl}} \approx \frac{4GM}{c^2b} (b$ is the smallest distance between the light ray and the Sun), the precession of a planet’s perihelion $\delta \phi_{\text{prec}} \approx \frac{\pi}{2} (2 + 2\gamma - \beta)$ and the Shapiro time delay of light signals due to the Sun’s field $\Delta \tau_{\text{excess}} \approx \frac{\pi}{2} (2 + 2\gamma - \beta)$. For the case we are interested in GP-B, we have:

$$\bar{\Omega}_{LT} = \left(\frac{7}{8} \Delta_1 + \frac{1}{8} \Delta_2\right) \frac{2G}{c^2|\vec{x}|^3} (3(\vec{J} \cdot \vec{e}_r) \vec{e}_r - \vec{J}) \quad \Delta \phi_{\text{geodetic}} \approx \left(\gamma + \frac{1}{2}\right) \frac{2\pi GM}{c^2r}$$

Here, $\Delta_1$ and $\Delta_2$ are other PPN parameters coming from the off-diagonal elements characteristic of a Kerr-like metric. In spite of this the GP-B Lens-Thirring measurement doesn’t compete with other Solar System tests.
Although GP-B was constructed to test GR, some works showed that it was also potentially useful to constrain alternative/extended theories of gravity. For example, the geodetic effect can be useful to constrain theories of gravity such as:

- Scalar-tensor theories (Kamal Nandi et al, 2001);
- Torsion theories (see part C) (Kenji Hayashi e Takeshi Shirafuji (1979), Leopold Halpern (1984), Yi Mao et al. (2006));
- Gravity with extra dimensions (Hongya Liu, James Overduin (2000)).

The GP-B Lens-Thirring can also be used to constrain torsion theories [24,25]

**Astrophysical relevance of gravitoelectromagnetism and GP-B**

Gravitomagnetic fields are important in astrophysical systems such as rotating black holes and Quasars or AGN’s [26-28]. Gravitomagnetism seem to provide a link between the central rotating black hole and the jets emitted. Sources like NGC 6251 have huge jets that are aligned for millions of years. Calculations show that the gravitomagnetic field seems to be crucial in the production and alignment of the jets. The interaction between the gravitomagnetic field and the magnetic field of these astrophysical objects promotes the creation of the jets [26]. The so called Blandford-Znajek effect converts rotational gravitomagnetic energy into the energy of relativistic particles. On the other hand the Bardeen-Petterson effect describes how the gravitomagnetic field causes a (Lens-Thirring) precession of the accretion disk, which in conjunction with the fluid’s viscosity, causes the inner regions to merge into the equatorial plane resulting in two preferred directions for jets. Therefore:

*GP-B is a crucial test of the physical mechanism that promotes the most violent explosions known in the Universe*

**GP-B Results**

The GP-B official results were the following [22]

<table>
<thead>
<tr>
<th>Source</th>
<th>$r_{NS}$ (mas/yr)</th>
<th>$r_{WE}$ (mas/yr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gyroscope 1</td>
<td>$-6,588.6 \pm 31.7$</td>
<td>$-41.3 \pm 24.6$</td>
</tr>
<tr>
<td>Gyroscope 2</td>
<td>$-6,707.0 \pm 64.1$</td>
<td>$-16.1 \pm 29.7$</td>
</tr>
<tr>
<td>Gyroscope 3</td>
<td>$-6,610.5 \pm 43.2$</td>
<td>$-25.0 \pm 12.1$</td>
</tr>
<tr>
<td>Gyroscope 4</td>
<td>$-6,588.7 \pm 33.2$</td>
<td>$-49.3 \pm 11.4$</td>
</tr>
<tr>
<td>Joint (see text)</td>
<td>$-6,601.8 \pm 18.3$</td>
<td>$-37.2 \pm 7.2$</td>
</tr>
<tr>
<td>GR prediction</td>
<td>$-6,606.1$</td>
<td>$-39.2$</td>
</tr>
</tbody>
</table>

**Table – GP-B official results [22].**

<table>
<thead>
<tr>
<th>Contribution</th>
<th>NS (mas/yr)</th>
<th>WE (mas/yr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Statistical (modeled)</td>
<td>16.8</td>
<td>5.9</td>
</tr>
<tr>
<td>Systematic (unmodeled)</td>
<td>7.1</td>
<td>4.0</td>
</tr>
<tr>
<td>Parameter sensitivity</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Guide star motion</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Solar geodetic effect</td>
<td>0.3</td>
<td>0.6</td>
</tr>
<tr>
<td>Telescope readout</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Other readout uncertainties</td>
<td>$&lt; 1$</td>
<td>$&lt; 1$</td>
</tr>
<tr>
<td>Other classical torques</td>
<td>$&lt; 0.3$</td>
<td>$&lt; 0.4$</td>
</tr>
<tr>
<td>Total Stat. + Sys.</td>
<td>18.3</td>
<td>7.2</td>
</tr>
</tbody>
</table>

**Table – Contributions to experimental uncertainty [22].**

The geodetic and Lens-Thirring effects were successfully detected with the following experimental errors [22].

Geodetic: 0.25%  
Lens-Thirring: 19%

**Graphic 1 – Combined results of the measurements on the 4 gyroscopes [22].**
B.3.1 Introduction

The question of coupling gravity and electromagnetism is quite old. In GR this issue follows naturally from the recently, this topic has gained an increasing interest mainly because there are new experimental capabilities and also because it is relevant for testing alternative/extended theories of gravity. In fact, the metric, the non-metricity tensor and torsion (see part C) can play a role in the coupling between gravity and electromagnetism in some of these theories. The bending of light can be understood in terms of an effective space-time refraction index ([29] and references therein) due to the coupling between the electromagnetic field and the gravitational field. The Einstein-Maxwell coupling correctly describes some experimental facts such as the bending of light rays by the gravitational field, the gravitation red shift, the Shapiro time delay and also effects due to gravitational lenses. In fact, in all such experiments, we usually study the propagation of light along null geodesics in a given gravitational field which is a solution of the vacuum Einstein’s equations. This might be viewed as an inconsistency since one should use the Einstein-Maxwell system of equations with $\tau_{\alpha\beta}^{(\text{Maxwell})} \neq 0$. This kind of “vacuum” could be called “electrovacuum”. A completely new effect that could come out of the Einstein-Maxwell coupling would be the production of the electromagnetic waves by gravitational waves, but this effect hasn’t been observed yet. One should keep trying to search for other consequences of this coupling and compare with observation.

B.3.2 Electromagnetic waves as source of gravitational waves

Electromagnetic waves as source of gravity

I am interested in studying the solutions to the Einstein linear equations in the case where the source is the energy-momentum $T_{\alpha\beta}^{(\text{Maxwell})}$ associated with electromagnetic waves! The dynamical (symmetric) energy-momentum tensor for electromagnetic field is:

$$T_{\alpha\beta}^{(\text{Maxwell})} = \frac{2}{\sqrt{-g}} \frac{\delta L_{\text{Maxwell}}}{\delta g^{\alpha\beta}} = -\frac{1}{\mu_0} \left( F_{\gamma\beta} F_{\alpha} - \frac{1}{4} g_{\alpha\beta} F_{\gamma\epsilon} F_{\epsilon} \right)$$

The Maxwell stress-tensor is

$$\sigma_{\alpha\beta} = \frac{1}{4} \left( \varepsilon_0 F_{\alpha} F_{\beta} + H_{\alpha} H_{\beta} - \frac{1}{2} \delta_{\alpha\beta} (\varepsilon_0 E^2 + H^2) \right) = \frac{1}{4} \left( \varepsilon_0 F_{\alpha} F_{\beta} + \frac{B_{\alpha} B_{\beta}}{(\mu_0)^2} - \frac{1}{2} \delta_{\alpha\beta} \left( \varepsilon_0 E^2 + \frac{B^2}{\mu_0} \right) \right)$$

Now, in this linear regime it is useful to make use of gravitoelectromagnetism. In the case I am interested in, the gravitoelectric scalar and the gravitomagnetic vector potentials will be linked to the energy density and energy flow (poynting vector) of the electromagnetic field:

$$\text{gravitational perturbation: } \tilde{h}_{\alpha\beta}(\tilde{x},ct) = -\frac{4G}{c^4} \int \frac{T_{\alpha\beta}(\tilde{x}',ct')}{||\tilde{x} - \tilde{x}'||} d^3\tilde{x}'$$
$$\text{gravitoelectric: } \phi_g \equiv \frac{c^2}{4} \tilde{h}^{00} = -\frac{G}{c^2} \int \frac{\rho(\tilde{x}',t')}{||\tilde{x} - \tilde{x}'||} d^3\tilde{x}'$$
$$\text{gravitomagnetic: } A^i \equiv \tilde{h}^{0i} = -c \frac{4G}{c^4} \int \frac{J^i(\tilde{x}',t')/c}{||\tilde{x} - \tilde{x}'||} d^3\tilde{x}' = -\frac{4G}{c^4} \int \frac{\rho u^i}{||\tilde{x} - \tilde{x}'||} d^3\tilde{x}'$$
The electromagnetic energy density is quadratic in the fields, \( \rho = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2 \mu_0} B^2 \) and the energy flux (pointing vector) is given by \( \vec{f} = \rho \vec{c} \vec{e}_k = \rho \vec{c} \vec{S} \). Remember that using Maxwell’s equations for plane harmonic waves,
\[
\vec{e}_k \times \vec{E} = c \vec{B} \quad \vec{e}_k \times \vec{B} = -\frac{1}{c} \vec{E} \quad \vec{k} \cdot \vec{E} = 0 \quad \vec{k} \cdot \vec{B} = 0
\]
we see that \( E = cB \) and therefore the total energy density is \( \rho = \varepsilon_0 E^2 = \frac{1}{\mu_0} B^2 \). This said, the gravitoelectric and gravitomagnetic perturbations are given by
\[
\phi_g = -\frac{G \varepsilon_0}{c^2} \int \frac{E^2(\vec{x}', t')}{\| \vec{x} - \vec{x}' \|} d^3 \vec{x}' \quad \vec{A}_g = -\frac{4G}{c^3} \varepsilon_0 \left( \int \frac{E^2(\vec{x}', t')}{\| \vec{x} - \vec{x}' \|} d^3 \vec{x}' \right) \vec{e}_k = \frac{4}{c} \phi_g \vec{e}_k
\]

### Searching solutions

a) Consider a 3d volume filled with plane, linearly polarized electromagnetic waves:

\[
V = L^3 \quad \vec{E} = \vec{E}_{max} \cos(k \mu x) = \vec{E}_{max} \cos(w t - \vec{k} \cdot \vec{x})
\]

The gravitoelectric and gravitomagnetic perturbations are given by:
\[
\phi_g(\vec{x}, x, y, z) = -\frac{G \varepsilon_0 E_m^2}{c^2} \int \cos^2\left( wt' - k_x x' + k_y y' + k_z z' \right) d^3 \vec{x}'
\]
\[
\vec{A}_g(\vec{x}, x, y, z) = \frac{4}{c} \phi_g \vec{e}_k
\]

Instead of calculating these integrals we shall proceed with a different approach! One can use the compact source approximation and the expressions for \( \vec{h}^{ij}(\vec{x}, t) \) and \( l^{ij}(\vec{x}, t) \). So the quantities we are interested in are: \( \phi_g, \vec{A}_g \) and \( \vec{h}^{ij}(\vec{x}, t) \propto \frac{d^2 l^{ij}(\vec{x}, t)'}{dt'^2} \)
\[
\phi_g \equiv \frac{c^2}{4} \vec{h}_{00} = -\frac{G}{c^2 \| \vec{x} \|} \int \rho(\vec{x}', t') d^3 \vec{x}' = -\frac{G}{c^2 \| \vec{x} \|} \frac{e(t')}{\| \vec{x} \|}
\]
\[
A^i \equiv \vec{h}^{0i} = -\frac{4G}{c^4 \| \vec{x} \|} \int \frac{l^i(\vec{x}', t')}{c} d^3 \vec{x}' = -\frac{4G}{c^4 \| \vec{x} \|} \int \rho(\vec{x}', t') \left( \vec{e}_i \cdot \frac{\vec{k}}{c} \right) d^3 \vec{x}'
\]
\[
\vec{A}_g = -\frac{4G}{c^3 \| \vec{x} \|} \frac{\rho(\vec{x}', t')}{\vec{e}_k} d^3 \vec{x}'
\]

For plane electromagnetic waves as source

The calculations obtained are as follows:

a.1) Source: Plane, linearly polarized EM waves
\[
\epsilon = \int T^{00} d^3 \vec{x}' = \varepsilon_0 E_m^2 \int \cos^2\left( wt' - (k_x x' + k_y y' + k_z z') \right) dx' dy' dz'
\]
\[
\epsilon = \varepsilon_0 E_m^2 \frac{1}{2} \left( L^3 + \frac{\cos((k_x + k_y + k_z)L - 2wt') \sin(k_x L) \sin(k_y L) \sin(k_z L)}{k_x k_y k_z} \right)
\]
\[ \phi_g(t, \vec{x}) = -G \frac{e(t')}{c^2 \| \vec{x} \|} = -G e_0 E_m^2 \frac{1}{2} \left( \frac{L^2}{c^2} \right) \left( \frac{\cos((k_x + k_y + k_z)L - 2w t') \sin(k_x L) \sin(k_y L) \sin(k_z L)}{k_x k_y k_z} \right) \]

\[ \vec{A}_g(t, \vec{x}) = - \frac{4G e(t')}{c^3 \| \vec{x} \|} \vec{e}_k \]

\[ I^{ij}(ct') = e_0 E_m^2 \int_0^L \iiint x'^i x'^j \cos^2(w t' - k_x x'^1 + k_y x'^2 + k_z x'^3) dx'^1 dx'^2 dx'^3 \]

\[ I^{11}(ct') = e_0 E_m^2 \int_0^L \iiint (x'^1)^2 \cos^2(w t' - k_x x'^1 + k_y x'^2 + k_z x'^3) dx'^1 dx'^2 dx'^3 \]

\[ I^{12}(ct') = I^{12}(ct') = e_0 E_m^2 \int_0^L \iiint x'^1 x'^2 \cos^2(w t' - k_x x'^1 + k_y x'^2 + k_z x'^3) dx'^1 dx'^2 dx'^3 \]

\[ I^{13}(ct') = e_0 E_m^2 \int_0^L \iiint x'^1 x'^3 \cos^2(w t' - k_x x'^1 + k_y x'^2 + k_z x'^3) dx'^1 dx'^2 dx'^3 \]

\[ I^{22}(ct') = e_0 E_m^2 \int_0^L \iiint (x'^2)^2 \cos^2(w t' - k_x x'^1 + k_y x'^2 + k_z x'^3) dx'^1 dx'^2 dx'^3 \]

\[ I^{23}(ct') = e_0 E_m^2 \int_0^L \iiint x'^2 x'^3 \cos^2(w t' - k_x x'^1 + k_y x'^2 + k_z x'^3) dx'^1 dx'^2 dx'^3 \]

\[ I^{33}(ct') = e_0 E_m^2 \int_0^L \iiint (x'^3)^2 \cos^2(w t' - k_x x'^1 + k_y x'^2 + k_z x'^3) dx'^1 dx'^2 dx'^3 \]

\[ I^{11}, I^{22}, I^{33} \]

\[ I^{11} = e_0 E_m^2 \frac{1}{24 k_x^3 k_y k_z L^5} \left( 4 k_x^3 k_y k_z L^5 \right) \]

\[ + 3 \sin(k_y L) \sin(k_z L) \left( 2 k_x L \cos \left( (2k_x + k_y + k_z)L - 2w t' \right) + \sin((k_y + k_z)L - 2w t') \right) + (2 k_x^2 L^2 \right) \]

\[ - 1) \sin \left( (2k_x + k_y + k_z)L - 2w t' \right) ) \]

\[ I^{22} = e_0 E_m^2 \frac{1}{24 k_y^3 k_x k_z L^5} \left( 4 k_y^3 k_x k_z L^5 \right) \]

\[ + 3 \sin(k_x L) \sin(k_z L) \left( 2 k_y L \cos \left( (2k_y + k_x + k_z)L - 2w t' \right) + \sin((k_x + k_z)L - 2w t') \right) + (2 k_y^2 L^2 \right) \]

\[ - 1) \sin \left( (2k_y + k_x + k_z)L - 2w t' \right) ) \]

\[ I^{33} = e_0 E_m^2 \frac{1}{24 k_z^3 k_y k_z L^5} \left( 4 k_z^3 k_y k_z L^5 \right) \]

\[ + 3 \sin(k_y L) \sin(k_z L) \left( 2 k_z L \cos \left( (2k_z + k_y + k_x)L - 2w t' \right) + \sin((k_y + k_z)L - 2w t') \right) + (2 k_z^2 L^2 \right) \]

\[ - 1) \sin \left( (2k_z + k_y + k_x)L - 2w t' \right) ) \]
\( I^{12} = I^{21} \)

With similar expressions for \( I^{13} = I^{31} \) and \( I^{23} = I^{32} \)

Let us assume a wave traveling in the x direction:

\[
\vec{E} = E_{\max} \cos(k \cdot x) = E_{\max} \cos(wt - kx)
\]

In this case,

\[
\phi_g(t, \vec{x}) = -\frac{G \varepsilon_0 E_{\max}^2}{c^2 \| \vec{x} \|} \frac{L^2}{4k_x} [2k_x L + \sin(2wt') + \sin(2k_x L - 2wt')]
\]

\( \| \vec{x} \| = \sqrt{x^2 + y^2 + z^2} \equiv \text{distance between the observer and the reference sistem located on the source} \)

\[ t' = t - \frac{\| \vec{x} \|}{c} \quad (\text{we are using the compact source approximation}) \]

Now, we calculate the gravitomagnetic vector potential \( \vec{A}_g(t, \vec{x}) = -\frac{4G \varepsilon(t')}{c^3 \| \vec{x} \|} \vec{e}_x \)

\[
\varepsilon(t') = \varepsilon_0 E_{\max}^2 \frac{L^2}{4k_x} [2k_x L + \sin(2wt') + \sin(2k_x L - 2wt')]
\]

The gravitoelectric and gravitomagnetic fields created by the plane electromagnetic wave are:

\[
\vec{B}_g = \nabla \times \vec{A}_g = -\frac{G \varepsilon_0 E_{\max}^2}{c^3} \frac{L^2}{k_x} [2k_x L + \sin(2wt') + \sin(2k_x L - 2wt')] \nabla \left( \frac{1}{\| \vec{x} \|} \right) \times \vec{e}_x
\]

\[
\vec{E}_g = -\nabla \phi_g - \frac{\partial \vec{A}_g}{\partial t} = \frac{G \varepsilon_0 E_{\max}^2}{c^3 \| \vec{x} \|^3} \frac{L^2}{k_x} [2k_x L + \sin(2wt') + \sin(2k_x L - 2wt')] (z \vec{e}_y - y \vec{e}_z)
\]

And for the second term we need to evaluate the expression:

\[
\nabla(\sin(2wt') + \sin(2k_x L - 2wt')) = \nabla(\sin(2wt')) + \nabla(\sin(2k_x L - 2wt')) =
\]

\[
= -\cos(2wt') \frac{2w}{c} \frac{\vec{x}}{\| \vec{x} \|} - \cos(2k_x L - 2wt') \frac{2w}{c} \frac{\vec{x}}{\| \vec{x} \|} = -2 \frac{w}{c} \frac{1}{\| \vec{x} \|} (\cos(2wt') + \cos(2k_x L - 2wt')) \vec{x}
\]

We arrive at the following expression for \( \vec{B}_g \)

\[
\vec{B}_g = \frac{G \varepsilon_0 E_{\max}^2}{c^3 \| \vec{x} \|^3} \frac{L^2}{k_x} [2k_x L + \sin(2wt') + \sin(2k_x L - 2wt')] (z \vec{e}_y - y \vec{e}_z)
\]

\[
+ \frac{G \varepsilon_0 E_{\max}^2}{c^3} \frac{L^2}{k_x} \frac{2w}{c} \frac{1}{\| \vec{x} \|} (\cos(2wt') + \cos(2k_x L - 2wt')) (z \vec{e}_y - y \vec{e}_z)
\]

\[
\vec{B}_g = \frac{G \varepsilon_0 E_{\max}^2}{c^3 \| \vec{x} \|^3} \frac{L^2}{k_x} \left[ \frac{1}{\| \vec{x} \|^2} (2k_x L + \sin(2wt') + \sin(2k_x L - 2wt')) + \frac{2w}{c} (\cos(2wt') + \cos(2k_x L - 2wt')) \right] (z \vec{e}_y - y \vec{e}_z)
\]
To compute the components we can use the expressions

\[ -\nabla \phi_g = \frac{G\varepsilon_0 E_m^2}{c^2 \| \vec{x} \|} \frac{L^2}{4k_x} \left( \sin(2w t') + \sin(2k_x L - 2w t') \right) \]

\[ -\nabla \phi_g = \frac{G\varepsilon_0 E_m^2}{c^2 \| \vec{x} \|} \frac{L^2}{4k_x} \frac{2}{c} \frac{w}{\| \vec{x} \|} \left( \cos(2w t') + \cos(2k_x L - 2w t') \right) \vec{x} \]

\[ -\nabla \phi_g = \frac{G\varepsilon_0 E_m^2}{c^2 \| \vec{x} \|^2} \frac{L^2}{4k_x} \left( \frac{2 w}{c} \cos(2w t') + \cos(2k_x L - 2w t') \right) + \frac{1}{\| \vec{x} \|} \left( 2k_x L + \sin(2w t') + \sin(2k_x L - 2w t') \right) \vec{x} \]

\[ \frac{\partial \vec{A}_g}{\partial t} = \frac{G\varepsilon_0 E_m^2}{c^2 \| \vec{x} \|} \frac{L^2}{k_x} \frac{\partial}{\partial t} \left[ \sin(2w t') + \sin(2k_x L - 2w t') \right] \vec{e}_x \]

\[ \frac{\partial \vec{A}_g}{\partial t} = \frac{G\varepsilon_0 E_m^2}{c^2 \| \vec{x} \|} \frac{L^2}{k_x} \left( \cos(2w t') 2w + \cos(2k_x L - 2w t') 2w \right) \vec{x} \]

\[ \vec{E}_g = -\frac{G\varepsilon_0 E_m^2}{c^2 \| \vec{x} \|^2} \frac{L^2}{4k_x} \left[ \frac{2 w}{c} \left( \cos(2w t') + \cos(2k_x L - 2w t') \right) \right] + \frac{1}{c^2 \| \vec{x} \|^2} \left( 2k_x L + \sin(2w t') + \sin(2k_x L - 2w t') \right) \vec{x} \]

\[ \vec{E}_g = -\frac{G\varepsilon_0 E_m^2}{c^2 \| \vec{x} \|^2} \frac{L^2}{4k_x} \left[ \frac{2 w}{c} \left( \cos(2w t') + \cos(2k_x L - 2w t') \right) \right] + \frac{1}{\| \vec{x} \|^2} \left( 2k_x L + \sin(2w t') + \sin(2k_x L - 2w t') \right) \vec{x} \]

\[ \vec{E}_g = -\frac{G\varepsilon_0 E_m^2}{c^2 \| \vec{x} \|^2} \frac{L^2}{4k_x} \left[ \frac{2 w}{c} \left( \cos(2w t') + \cos(2k_x L - 2w t') \right) \right] + \frac{1}{\| \vec{x} \|^2} \left( 2k_x L + \sin(2w t') + \sin(2k_x L - 2w t') \right) \vec{x} \]

\[ \vec{B}_g = \frac{G\varepsilon_0 E_m^2}{c^2 \| \vec{x} \|} \frac{L^2}{k_x} \frac{1}{\| \vec{x} \|^2} \left( 2k_x L + \sin(2w t') + \sin(2k_x L - 2w t') \right) \vec{x} \]

We have seen that in the compact source approximation (where \( t' = \frac{\| \vec{x} \|}{c} \)) we have the following gravitational wave quantities created by the plane, linearly polarized electromagnetic waves traveling in the x direction:

\[ \phi_g(t, \vec{x}) = \frac{-G\varepsilon_0 E_m^2}{c^2 \| \vec{x} \|} \frac{L^2}{4k_x} \left( 2k_x L + \sin(2w t') + \sin(2k_x L - 2w t') \right) \]

\[ \vec{A}_g(t, \vec{x}) = \frac{-G\varepsilon_0 E_m^2}{c^2 \| \vec{x} \|} \frac{L^2}{k_x} \left( 2k_x L + \sin(2w t') + \sin(2k_x L - 2w t') \right) \vec{e}_x \]

\[ \vec{E}_g = \frac{-G\varepsilon_0 E_m^2}{c^2 \| \vec{x} \|} \frac{L^2}{4k_x} \left[ \frac{2 w}{c} \left( \cos(2w t') + \cos(2k_x L - 2w t') \right) \right] + \frac{1}{\| \vec{x} \|^2} \left( 2k_x L + \sin(2w t') + \sin(2k_x L - 2w t') \right) \vec{x} \]

\[ \vec{B}_g = \frac{G\varepsilon_0 E_m^2}{c^2 \| \vec{x} \|} \frac{L^2}{k_x} \frac{1}{\| \vec{x} \|^2} \left( 2k_x L + \sin(2w t') + \sin(2k_x L - 2w t') \right) \vec{x} \]

To compute the \( \vec{h}^{ij}(ct, \vec{x}) \) components we can use the expressions

\[ \vec{h}^{ij}(ct, \vec{x}) = -\frac{2 G}{c^6 r} \left[ \frac{d^{ij}(ct)}{dt^2} \right] t^{ij}(ct) = \varepsilon_0 E^2 \int_0^t \int x' x' \cos^2 \left( w t' - k_x x' \right) dx' dy' dz \]

Alternatively we can use the information we have about the energy momentum tensor to compute

\[ \vec{h}^{ij}(ct, \vec{x}) = \frac{-4 G}{c^4} \int \frac{T^{ij}(\vec{x}', ct')}{\| \vec{x} - \vec{x}' \|^2} d^3 \vec{x}' \Rightarrow \vec{h}^{ij}(\vec{x}, ct) = \frac{G}{c^4} \int \frac{1}{\| \vec{x} - \vec{x}' \|^2} \left( \varepsilon_0 E_i E_j + \frac{B_i B_j}{\mu_0} - \frac{1}{2} \delta_{ij} \left( \varepsilon_0 E^2 + \frac{B^2}{\mu_0} \right) \right) d^3 \vec{x}' \]
Which in the compact source approximation becomes:

\[ \tilde{h}^{ij}(ct, \vec{x}) = \frac{G}{c^4 \| \vec{x} \|} \int \left( E_i E_j + \frac{B_i B_j}{(\mu_0)^2} - \frac{1}{2} \delta_{ij} \left( \frac{E^2}{\mu_0} + \frac{B^2}{\mu_0} \right) \right) d^3 \vec{x}' \]

\[ \tilde{h}^{11}(ct, \vec{x}) = \frac{G}{c^4 \| \vec{x} \|} \int \left( -\frac{1}{2} \left( \frac{\epsilon_0 E^2}{\mu_0} + \frac{B^2}{\mu_0} \right) \right) d^3 \vec{x}' \]

\[ \tilde{h}^{22}(ct, \vec{x}) = \frac{G}{c^4 \| \vec{x} \|} \int \left( \epsilon_0 (E_y)^2 + \frac{B_y^2}{(\mu_0)^2} - \frac{1}{2} \left( \epsilon_0 E^2 + \frac{B^2}{\mu_0} \right) \right) d^3 \vec{x}' \]

\[ \tilde{h}^{33}(ct, \vec{x}) = \frac{G}{c^4 \| \vec{x} \|} \int \left( \epsilon_0 (E_z)^2 + \frac{B_z^2}{(\mu_0)^2} - \frac{1}{2} \left( \epsilon_0 E^2 + \frac{B^2}{\mu_0} \right) \right) d^3 \vec{x}' \]

\[ \tilde{h}^{12}(ct, \vec{x}) = 0 = \tilde{h}^{13}(ct, \vec{x}) = \tilde{h}^{23}(ct, \vec{x}) = \frac{G}{c^4 \| \vec{x} \|} \int \left( \epsilon_0 E_y E_x + \frac{B_y B_x}{(\mu_0)^2} \right) d^3 \vec{x}' \]

The energy in these gravitational waves can be calculated in the gravitoelectromagnetism formalism using a similar expression as in the case of electromagnetic waves. Remember that in this linear approach the \( \tilde{h}^{ij}(ct, \vec{x}) \) field behaves like an ordinary tensor field propagating in the Minkowsky background, carrying energy \( \epsilon \), linear momentum \( \vec{p} \) and angular momentum \( \vec{j} \).

\[ \epsilon = \int \rho_g d^3 x \left( \frac{1}{2} |\epsilon_{0g}| \right) \int E^2 d^3 x + \frac{1}{2} |\mu_{0g}| \int B^2 d^3 x \]

\[ |\epsilon_{0g}| = \frac{1}{4\pi G}, \quad |\mu_{0g}| = \frac{16\pi G}{c^2} \]

\[ \vec{p} = \int \vec{k} d^3 x \quad \vec{j} = \rho_g c \vec{e}_k = \rho c \vec{e}_k \equiv \vec{S} \]
C – Gravity and geometry: space-time torsion

C.1 Introduction to extended theories of gravity

Several issues emerged in the last thirty years leading to the idea that we should search for other theories of the gravitational interaction (beyond Einstein’s General Relativity) that may have GR as a limit (see [30] and references therein).

From Newton to Einstein

Newton’s contribution to gravity is not restricted to the expression of the inverse square law. The conceptual basis of his gravitational theory incorporated two key ideas:

1. The idea of absolute space; a fixed arena where physical phenomena took place.
2. The (later called) weak equivalence principle which, in the language of Newtonian theory, states an equivalence between the inertial and the gravitational mass.

It doesn’t make sense to say that Newton’s theory, or any other physical theory, is right or wrong, since any (consistent) theory is apparently "right" within some limited domain of experimental validity. An appropriate question would be to ask about the extent of the set of physical phenomena that is sufficiently described by such a theory. One could also enquire on the uniqueness of the specific theory for the description of the relevant phenomena. In the first 20 years after the introduction of Newtonian gravity it was obvious that it did manage to explain all of the aspects of gravity known at that time. However, things would gradually change with time:

- In 1855, Urbain Le Verrier observed a 35 arc-second excess precession of Mercury’s orbit and later on, in 1882, Simon Newcomb measured this precession more accurately to be 43 arc-seconds (see [30] and references therein). This experimental fact was not predicted by Newton’s theory.

- In 1893, Ernst Mach stated what was later called by Albert Einstein "Mach’s principle". This was the first constructive opposition to Newton’s idea of absolute space after the 18th century debate between Gottfried Leibnitz and Samuel Clarke (defending Newton) on the same subject, known as the Leibnitz–Clarke Correspondence (see [30] and references therein). Mach’s idea was essentially brought into the mainstream of physics later on by Einstein along the following lines: “(...) inertia originates in a kind of interaction between bodies (...)”. According to Newton, inertia was always relative to the absolute frame of space. There exists also a later, different interpretation of Mach’s Principle, given byDicke ([30]): "The gravitational constant should be a function of the mass distribution in the Universe". This is different from Newton’s idea of the gravitational constant as being universal and unchanging. Mach’s principle will be revisited in the last part of this work.

Newton’s basic axioms were being questioned but it was not until 1905, when Albert Einstein completed special relativity that Newtonian gravity would have to face a serious challenge. Einstein’s new theory appeared to be incompatible with Newtonian gravity because according to Newton the gravitational interaction acts instantaneously. With special relativity a new conceptual arena to think about space and time emerged and relative motion and all the linked concepts had gone well beyond Galileo and Newton’s ideas. Even more, it seemed that special relativity should somehow be generalized to include non-inertial frames. In fact, in 1907, Einstein introduced the equivalence between gravitation and inertial phenomena in accelerated frames and successfully used it to predict the gravitational redshift. Finally, in 1915, he completed the theory of general relativity (GR), a generalizing special relativity by including gravity and any accelerated frame.

Remarkably, the theory matched perfectly the experimental result for the precession of Mercury’s orbit and was gradually validated by several experiments such as the “classical” tests (see [30] and references therein): bending of starlight (1919), precession of planetary orbits, Shapiro effect (light signal delay), gravitational redshift (Gravity Probe-A, 1956); and modern tests including indirect measurements of gravitational waves in binary pulsars, the geodetic and Lens-Thirring effects (Gravity Probe-B), cosmological tests, gravitational lensing, gravitomagnetic tests such as LLR (Laser Lunar Ranging) and LAGEOS (Laser Geodynamic Satellites). GR overthrew Newtonian gravity and continues to be up to now an extremely successful and well-accepted theory for gravitational phenomena.

Newtonian gravity is of limited validity compared to GR, but it is still sufficient for most applications related to gravity. What is more, in weak field limit of gravitational field strength and velocities, GR inevitably reduces to Newtonian gravity. Newton’s equations for gravity have been generalized and some of the axioms of his theory may have been abandoned, like the notion of an absolute frame, but some of the cornerstones of his theory still exist in the foundations of GR, the most prominent example being the Equivalence Principle, in a more suitable formulation.

Basic requirements for gravitational theories

There are some minimal requirements of phenomenological nature that any relativistic theory of gravity has to match. Firstly, it has to explain the astrophysical observations (such as self-gravitating structures and the orbits of planets). This means that it has to reproduce the Newtonian dynamics in the weak-energy limit and has to pass the classical Solar System tests which are all experimentally well founded. It should reproduce galactic dynamics considering the observed baryonic constituents (luminous components as stars, sub-luminous components as planets, dust and gas), radiation and a Newtonian potential which
is, by assumption, extrapolated to galactic scales. Lastly it should be able to largely describe the evolution of large scale structure (clustering of galaxies) and the cosmological dynamics, which means to reproduce, in a self-consistent way, the cosmological parameters such as the Hubble constant and the density parameters.

The simplest theory which tries to satisfy the above requirements is GR. Based on the assumption of a single space-time structure which in the limit of no gravitational forces has to reproduce the Minkowski space-time, the theory was profoundly influenced by ideas earlier put forward by Riemann, who stated that the Universe should be a curved manifold and that its curvature should be established on the basis of astronomical observations (see [30] and references therein). The theory, eventually formulated by Einstein in 1915, predicts that the distribution of matter influences point by point the local curvature of the space-time structure and was strongly based on three assumptions that the physics of gravitation has to satisfy:

- The "Principle of Relativity", that requires all frames to be good frames for Physics, so that no preferred inertial frame should be chosen a priori (if any exist);
- The "Principle of Equivalence", that amounts to require inertial effects to be locally indistinguishable from gravitational effects (in a sense, the equivalence between the inertial and the gravitational mass);
- The "Principle of General Covariance", that requires field equations to be "generally covariant" (to be invariant under the action of the group of all space-time diffeomorphisms).

- The requirement that causality has to be preserved: the "Principle of Causality". Each point of space-time should admit a universally valid notion of past, present and future. This means that the equations should be invariant under local conformal transformations, under which the light cone is (locally) an invariant set.

Hilbert and Einstein proved that the field equations for a metric tensor, related to a given distribution of matter-energy, can be achieved by starting from the invariant Ricci curvature scalar starting from the variation of the Einstein-Hilbert action. The choice of Hilbert and Einstein was completely arbitrary (as it became clear a few years later), but it was certainly the simplest, both from the mathematical and the physical point of view.

Levi-Civita clarified in 1919 that curvature is not a "purely metric notion" but, rather, a notion related to the "linear connection" fundamental to the notions of "parallel transport" and "covariant derivation" (see [30] and references therein). In some sense, this is the precursor idea of what would be called a "gauge theoretical framework" [30], after the pioneering work by Cartan in 1925 (see [30] and references therein). But at the time of Einstein, only metric concepts were at hands, but it was later clarified that the three principles of relativity, equivalence and covariance, together with causality, just require that the space-time structure has to be determined by either one or both of two fields, a metric and a linear connection, assumed at the beginning to be torsionless for the sake of simplicity. The metric fixes the causal structure of space-time (the light cones) as well as its metric relations (temporal and spatial distances); the connection fixes the free-fall - the locally inertial observers - and therefore the geodesics. They have, of course, to satisfy a number of compatibility relations which amount to require that photons follow null geodesics, so that the connection and the metric can be independent à priori but constrained à posteriori by some physical restrictions. These, however, do not impose that the connection has necessarily to be the Levi-Civita connection of GR.

Therefore at least on a purely theoretical basis, it is somehow natural to consider the so-called "alternative theories of gravitation" or "extended Theories of Gravitation" (ETGs). This last designation is in most cases more appropriate since their starting points are exactly those considered by Einstein and Hilbert: theories in which gravitation is described by either a metric (the so-called "purely metric theories"), or by a linear connection (the so-called "purely affine theories") or by both fields (the so-called "metric-affine theories", also known as "first order formalism theories"). In these theories, the Lagrangian is a scalar density of the curvature invariants constructed out of both the metric and the connection. The choice of Hilbert-Einstein Lagrangian is not at all unique and in fact, it turns out that the Hilbert-Einstein Lagrangian is the only choice that produces an invariant that is linear in the second derivatives of the metric (or first derivatives of the connection). A Lagrangian that is rather singular from the Hamiltonian point of view, in much the same way as Lagrangians, linear in canonical momenta, are rather singular in Classical Mechanics (see [30] and references therein).

A number of attempts to generalize GR (and unify it to Electromagnetism) along these lines were followed by Einstein himself and many others (Eddington, Weyl, Schrodinger, just to quote some, [30]) but they were eventually given up in the fifties, mainly because of a number of difficulties related to the more complicated structure of a non-linear theory ("non-linear" in the sense that is based on non-linear invariants of the curvature tensor), and also because of the new understanding of physics, based on four fundamental forces, that requires the more general "gauge framework" to be adopted (see [30]). Still a number of more or less rare investigations about "alternative theories" continued even after 1960 ([30] ).

The search for a coherent quantum theory of gravitation or the idea that gravity can be considered as a sort of low-energy limit of string theories [30], which won’t be consider here, has more or less recently revitalized the idea that there is no reason to follow the simple prescription of Einstein and Hilbert and to assume that gravity should be classically governed by a Lagrangian linear in the curvature. Further curvature invariants or non-linear functions of them should be also considered, especially in view of the fact that they have to be included in both the semi-classical expansion of a quantum Lagrangian or in the low-energy limit of a string Lagrangian (see [30] and references therein).
On the other hand, it is clear from the recent astrophysical observations and from the current cosmological hypotheses that Einstein equations are a good test for gravitation at Galactic, extra-galactic and cosmic scales, only if one assumes that the matter side of the field equations contains some kind of exotic matter-energy such as the "dark matter" and "dark energy".

Other alternative suggests itself within homogeneous and isotropic models: instead of changing the matter side of the field equations in order to fit the "missing matter-energy" content of the currently observed Universe by adding any sort of unknown and strangely behaving matter and energy, one can change the "geometrical" side of the equations, admitting corrections coming from non-linear terms in the effective Lagrangian. However, this is nothing else but a matter of theoretical choice and it is enriching to explore such an alternative approach. Naturally, the Lagrangian can be conveniently tuned up (chosen from a huge family of allowed Lagrangians) on the basis of its best fit with all possible observational tests, at all scales (solar, galactic, extragalactic and cosmic). Something that can and should be done before rejecting à priori a theory of gravitation based on a non-singular Lagrangian and insisting that the Universe has to be necessarily described by a rather singular gravitational Lagrangian (from a good Hamiltonian point view) accompanied by matter that does not follow the behavior that standard baryonic matter, usually satisfies. Nevertheless all possibilities are still open and it may be the case that Nature surprises us with post-Riemann geometries and different previously unknown matter-energy fields.

**General Relativity and its shortcomings – motivations for extended/alternative theories of gravity**

During the last thirty years people began to investigate whether GR is the only fundamental theory capable of explaining the gravitational interaction. Some discussed issues, coming essentially from cosmology and quantum field theory, can be seen as evidences of shortcomings of GR. These are related to many theoretical aspects and also to observational results. Most physicists agree that modern physics is based on two main pillars: GR and quantum field theory. Each of these two theories have been very successful in its own arena of physical phenomena: GR in describing gravitating systems and non-inertial frames from a classical point of view on large enough scales and quantum field theory at high energy or small scale regimes where a classical description breaks down. However, quantum field theory assumes that space-time is flat and its extensions such as quantum field theory in curved space time consider space-time as a rigid (non-dynamical) arena inhabited by quantum fields. GR, on the other hand, does not take into account the quantum nature of matter. Therefore, it comes naturally to ask what happens if a strong gravitational field is present at quantum scales. How do quantum fields behave in the presence of gravity and how is their feedback on the structure of space-time? Does a continuous description of space-time remain valid? To what extent are these theories compatible?

Rigorously, it should be mentioned that there is no final proof that gravity should have some quantum representation at high energies or small scales, or even that it will retain its nature as an interaction. On the other hand, it may be the case that a coherent theory of quantum gravity will not only require a new and more general gravitational theory but also some changes on the structure of quantum theory itself. According to Penrose a successful theory for the quantum gravity issue should also be able to solve the measurement problems of quantum theory.

The gravitational interaction is so weak compared with other interactions that the characteristic scale under which one would expect to experience non-classical effects relevant to gravity, the Planck scale, is $10^{-33}$ cm and such a scale is not accessible by any current experiment (although imprints of quantum gravity phenomenology might be found in lower energy experiments). However, there are a number of reasons for which one would prefer to fit together GR and Quantum Field Theory. Curiosity is one of the motivations leading scientific research and from this perspective it is natural to accept the theoretical and experimental challenges of studying the regime where both quantum and gravitational effects are important. The fact that the Planck scale seems currently experimentally inaccessible does not, in any way, imply that it is physically irrelevant. On the contrary, one can easily name some very important open issues of contemporary physics that are related to the Planck scale. A particular example is the Big Bang scenario in which the Universe inevitably goes through an era in which its dimensions are smaller than the Planck scale (the Planck era). Another typical example lies at the core of black hole structures where, according to the present theoretical constructs, one expects to reach the Planck scale and a singularity. On the other hand, space-time in GR is a continuum and so in principle all scales are relevant. From this perspective, in order to derive conclusions about the nature of space-time one has to answer the question of what happens on very small and very large scales.

**What can generate gravity?**

One can ask if there are invariance principles leading to the gravitation [30]. Utiyama (see [30] and references therein) was the first to propose that GR can be seen as a gauge theory based on the local Lorentz group SO(3, 1) in much the same way that the Yang-Mills gauge theory was developed on the basis of the internal isospin gauge group SU(2). In this formulation the Riemannian connection is the gravitational counterpart of the Yang-Mills gauge fields. While in the Yang-Mills theory SU(2) is an internal symmetry group, the Lorentz symmetry represents the local nature of space-time rather than internal degrees of freedom.

The Einstein Equivalence Principle, asserted for GR, requires that the local space-time structure can be identified with the Minkowski space-time possessing Lorentz symmetry. In order to relate local Lorentz symmetry to the external space-time, we need to solder the local space to the external space. The soldering tools are the tetrad fields that I will introduce latter. Utiyama regarded the tetrads as objects given à priori. Soon after, Sciama (see [30] and references therein) recognized that space-time should necessarily be endowed with torsion in order to accommodate spinor fields. In other words, the gravitational interaction
of spinning particles requires the modification of the Riemann space-time of GR to be a (non-Riemannian) curved space-time with torsion. Although Sciama used the tetrad formalism for his gauge-like handling of gravitation, his theory had some shortcomings in treating tetrad fields as gauge fields. Kibble (see [30] and references therein) made a comprehensive extension of the Utiyama gauge theory of gravitation by showing that the local Poincaré symmetry containing SO(3, 1) and T(3, 1) (T (3, 1) is the translation group) can generate a space-time with torsion as well as curvature.

The gauge fields introduced by Kibble include the tetrads as well as the local affine connection. There have been a variety of gauge theories of gravitation based on different local symmetries which have given rise to several interesting applications in theoretical physics [ref]. Following the Kibble approach, it can be demonstrated how gravitation can be formulated starting from a pure gauge point of view [30]. In particular, a theory of gravitation is a gauge theory which can be obtained starting from the local Poincaré symmetry and this feature works not only for GR but also for ETGs. A gauge theory of gravity based on a nonlinear realization of the local conformal-affine group of symmetry transformations can be formulated in any case (see [30] and references therein). This means that the coframe fields (tetrads) and gauge connections of the theory can be always obtained.

The tetrads and Lorentz group metric have been used to induce a space-time metric [30]. The connection coefficients (which transform in a non-homogeneous way under the Lorentz group) give rise to gravitational gauge potentials used to define covariant derivatives accommodating the minimal coupling of matter and gauge fields. On the other hand, the tensor valued connection forms can be used as auxiliary dynamical fields associated with the dilation, to special conformal and deformation (shear) degrees of freedom inherent to the bundle manifold. This allows to define the bundle curvature of the theory. Then boundary topological invariants can be constructed [30]. In [30] one starts from the Invariance Principle, considering first the Global Poincaré Invariance and then the Local Poincaré Invariance. The gravitational Lagrangian is constructed and the Bianchi identities, covariant field equations and gauge currents are obtained. This approach leads to a given theory of gravity as a gauge theory and could be useful to formulate self-consistent schemes for any theory of gravity.

In summary, a relativistic theory of gravitation can be obtained from a gauge theory of local Poincaré symmetry. In fact gauge fields can be obtained by requiring the invariance of the Lagrangian density under local Poincaré invariance. The resulting gravity theory describes a space endowed with non-vanishing curvature and torsion. The simplest prototype of this approach is the Einstein-Cartan theory which is the lowest order gravitational action linear in the curvature scalar and quadratic in torsion. As will become clear, the Dirac spinors can be introduced as matter sources which couple with gravity via the torsion stress-tensor, which is a component of the total energy-momentum. In view of the structure of the generalized energy-momentum-tensor, the gravitational field equations obtained are of Yang-Mills type and it is worth noticing that the torsion tensor plays the role of the Yang-Mills field strength. Besides the gauge approach, the gravitational field theories can also be recovered by space-time deformations, an approach connected to the topic of conformal transformations [30].

C.2 Fundamentals of differential geometry and torsion

Preliminaries in differential geometry and torsion

There are different ways to introduce torsion in differential geometry. In order to this in the ordinary language of tensor analysis on differentiable manifolds and given the importance of this geometrical concept, I will start by briefly review some basic concepts for the sake of completeness and clarity in this exposure.

We start with a differentiable manifold $M$. In order to define at each point some set of coordinates, we establish a local homeomorphism $\Phi : U \subset M \rightarrow V \subset \mathbb{R}^N$ linking $M$ to $\mathbb{R}^N$ in the following way:

The coordinates of a point $p$ are defined as $x^a(p) \equiv \pi^a \circ \phi(p)$ and on each point $p$ we define the tangent $T_p(M)$ and cotangent $T^*_p(M)$ manifolds. Vectors are elements of the first and co-vectors belong to the second. Analogously any tensor $F$ of type $(m,n)$, with $m$ contravariant indices and $n$ covariant indices, is an element of a new manifold that is constructed using $T^*_p(M)$ $m$ times and $T_p(M)$ $n$ times, $(F \in T^*_p(M) \times ... \times T^*_p(M) \times ... \times T_p(M) \times ...)$.

Defining a basis $\{e_a\}$ for $T_p(M)$ and the corresponding dual basis $\{e^a\}$ for $T^*_p(M)$, that is $e^a(e_b) = \delta^a_b$, a general tensor can always be expanded using these basis ($\otimes$ represents the tensor product):
The tensor components in any other system of coordinates obey the following transformation rule:

$$F_{r's'...w'} = \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} \ldots \frac{\partial x^h}{\partial x^{h'}} \frac{\partial x^r}{\partial x^{r'}} \ldots \frac{\partial x^w}{\partial x^{w'}} \hat{F}^{ab..h}_{r's'...w'}$$

Before introducing any (affine) connection on $M$, there is no notion of any relation between tensors defined on manifolds at different points $p$ and $q$ (like $T_p(M)$ and $T_q(M)$). Similarly, before introducing a metric there is no notion of distance between different points on $M$. The metric $g$ and its inverse $g^{-1}$ introduce a correspondence between the tangent and cotangent spaces allowing the operations of lowering and raising indices (respectively) of tensor components by contracting with these objects.

$$g: T_p(M) \rightarrow T^*_p(M) \quad g^{-1}: T^*_p(M) \rightarrow T_p(M)$$

As a consequence, one defines the usual scalar product of vectors $g(v, u) = g_{ab}v^a u^b = v^a \omega_a^b$, where $\omega \in T^*_p(M)$ is a covector. It is well known that the metric is a bilinear, symmetric, application with which one constructs the line element

$$ds^2 = g_{ab} dx^a dx^b.$$ 

A curve $\Gamma$ on $M$ may be defined as the result of the following application $\Gamma: \lambda \in \mathbb{R} \rightarrow \Gamma(\lambda) \in M$, originating a continuous parametric description of a line on $M$ with coordinates $x^a(\lambda) = x^a(\Gamma(\lambda)) = \pi^a \circ \Phi(\Gamma(\lambda))$.

Geodesics are very specific paths whose length connecting two points $\int_A^B ds$ is stationary and can be obtained as solutions of the geodesic equations which correspond to Euler-Lagrange equations

$$\frac{d}{d\lambda} \left( \frac{\partial L}{\partial \dot{x}^a} \right) = \frac{\partial L}{\partial x^a},$$

with a Lagrangian given by $L = (g_{ab} \dot{x}^a \dot{x}^b)^{1/2}$ (here $\dot{x}^a \equiv dx^a/d\tau$ and $\tau$ is some affine parameter).

Since very ancient times, science have been carefully engaged in observing and representing the motion of physical entities, such as pendula, projectiles, planets, stars, galaxies, dust particles, atoms and subatomic particles, fluids, living organisms and people, electromagnetic fields, etc., in short, mass and energy transferences in space-time. A mathematical representation of motion is therefore crucial for mathematical sciences and this is strictly connected to geometry (and time), namely to the geometry of paths. A parametric description of a path may be thought of as representing some evolution between related points (in some abstract space) associated to some motion obeying some dynamics or verified causal pattern (“a physical law”).

At each point on a curve or path $\{x^a(\lambda)\}$ we may consider the following (operational) tangent vector $\frac{d}{d\lambda} = \frac{\partial x^a}{\partial \lambda} \frac{\partial}{\partial x^a}$ where we have introduced a coordinate basis $\{\partial_a\}$. The objects we may be interested to study may have as suitable representation a field, such as a vector field $J: p \in M \rightarrow J(p) \in T_p(M)$. For a given (“regular”) vector field we can find its integral curves, that is, the set of curves for which at each point the tangent vector and the vector field itself at that point coincide. For vector fields that cover the whole manifold $M$, the set of its integral curves constitute a congruence for $M$. Now, given any two curves $\{x^a(\lambda)\}, \{x^a(\mu)\}$, the vectors $\frac{d}{d\lambda}$ and $\frac{d}{d\mu}$ may not be elements of a coordinate basis, that is $\left[ \frac{d}{d\lambda}, \frac{d}{d\mu} \right] \neq 0$. For coordinate basis the commutator of any pair of elements is zero

$$\left[ \frac{d}{d\lambda}, \frac{d}{d\mu} \right] = 0.$$ As mentioned in the appendix, defining the vector fields $V = \frac{d}{d\lambda}$ and $W = \frac{d}{d\mu}$, the commutator $\left[ \frac{d}{d\lambda}, \frac{d}{d\mu} \right]$ is called the Lie derivative of $W$ with respect to $V$

$$\left[ \frac{d}{d\lambda}, \frac{d}{d\mu} \right] = L_v W = \frac{d}{d\lambda} W - V \frac{d}{d\mu}$$

The general expression for the commutator between two arbitrary vector fields expanded using a coordinate basis can be easily obtained:
These notions are crucial for the (operational) definition of the curvature and torsion tensors.

In a general manifold there is à priori no global notion of parallelism for vectors in different points. One needs a rule in order to define a parallel transport of a vector between two points. The figure below shows two alternative ways of transporting a vector at A to C which give two different orientations.

A connection provides a rule for parallel transport. To see this, let us recall the notion of covariant differentiation. The ordinary partial derivative of the components of a general (m,n) tensor $\partial F^{a_1a_2...a_m}_{b_1b_2...b_n}$ doesn’t transform like a tensor! We need a definition of a covariant derivative $\nabla f$ such that $\nabla f^{a_1a_2...a_m}_{b_1b_2...b_n}$ does transform like the components of a (m, n+1) tensor. We therefore introduce the connection whose components $\Gamma^p_{qr}$ do not transform like a tensor but instead transform as

$$\Gamma^p_{qr} = \Gamma^d \frac{\partial x^f}{\partial x^q} \frac{\partial x^g}{\partial x^r} + \frac{\partial x^d}{\partial x^q} \frac{\partial^2 x^d}{\partial x^r \partial x^p}$$

The covariant derivative defined as

$$\nabla f^{a_1a_2...a_m}_{b_1b_2...b_n} = \partial f^{a_1a_2...a_m}_{b_1b_2...b_n} + \Gamma^d f^{a_1a_2...a_m}_{b_1b_2...b_n} + \cdots + \Gamma^d b_1 b_2 ... b_n f^{a_1a_2...a_m}_{b_1b_2...b_n} - \Gamma^k b_1 b_2 ... b_n f^{a_1a_2...a_m}_{b_1b_2...b_n} - \cdots - \Gamma^k b_1 b_2 ... b_n f^{a_1a_2...a_m}_{b_1b_2...b_n}$$

can be shown to transform as a tensor. From these definitions we see that the covariant derivation of the components of vectors and conectors are $\nabla_i v^c = \partial_i v^c + v^d \Gamma^c_{id}$ and $\nabla_i w_c = \partial_i w_c - w_d \Gamma^d_{ic}$ respectively. In fact while

$$\frac{\partial v^a}{\partial x^b} = \frac{\partial}{\partial x^b} \left( \frac{\partial x^a}{\partial x^d} v^d \right) = \frac{\partial x^b}{\partial x^b} \frac{\partial x^a}{\partial x^d} v^d + \frac{\partial x^b}{\partial x^d} \frac{\partial x^a}{\partial x^d} v^d$$

on the other hand

$$\nabla_i v^a = \frac{\partial x^a}{\partial x^b} \frac{\partial x^b}{\partial x^b} v^a$$

The covariant derivation of a vector field $Y$ along some curve (with tangent vector $u$) can be symbolically represented by the following limit

$$\nabla_u Y = \lim_{\varepsilon \to 0} \left( \frac{Y(\lambda + \varepsilon) - Y(\lambda)}{\varepsilon} \right)$$

Where, $Y(\lambda + \varepsilon)$ represents the vector field at $(\lambda + \varepsilon)$ parallel transported to the point where $Y(\lambda)$ is defined.

A Tensor $F$ is therefore parallel transported along some curve whose tangent vector $u$ has components $\frac{\partial x^a}{\partial x}$, if its covariant derivative along that curve vanishes.
\[ u/\nabla_j F_{_b_1 a_2 \ldots a_m} = 0 \quad \nabla_u F = 0 \]

We then see that geodesics are curves for which the tangent vector is parallel transported along itself

\[ u/\nabla_j u^a = 0 \quad \nabla_u u = 0 \]

Having these definitions in mind we can introduce the torsion tensor \( \tau \). It is a tensor that when applied (contracted) to two vector fields gives the following vector field

\[ \tau(X, Y) \equiv \nabla_X Y - \nabla_Y X - [X, Y] \]

The property of having a symmetric connection, rigorously speaking, means that

\[ \nabla_X Y - \nabla_Y X = [X, Y] \iff \"\text{symmetric connection}\" \]

We have zero torsion! If our rule for parallel transport has this property, then if we apply the torsion tensor to two elements of a coordinate basis \( ([e_a, e_b] = 0) \), we see that

\[ \nabla_{e_a} e_b - \nabla_{e_b} e_a = [e_a, e_b] = 0 \implies (\Gamma^c_{ab} - \Gamma^c_{ba}) e_c = 0 \]

Therefore if we have zero torsion this is equivalent of saying that the connection components are symmetric in their lower indices only if we are working with a coordinate basis. Using such basis, the components of the torsion tensor may be defined as:

\[ \tau^c_{ab} \equiv \Gamma^c_{ab} - \Gamma^c_{ba} \]

In fact, for non-coordinate basis we see that

\[ [e_a, e_b] = C^c_{ab} e_c \implies \tau^c_{ab} = \Gamma^c_{ab} - \Gamma^c_{ba} + C^c_{ab} \]

We see also that using a coordinate basis, the symmetric part of a connection \( \Gamma^c_{(ab)} \equiv \frac{1}{2} (\Gamma^c_{ab} + \Gamma^c_{ba}) \) can be expressed as

\[ \Gamma^c_{(ab)} = \Gamma^c_{ab} - \frac{1}{2} \tau^c_{ab} \]

One thing that is very interesting is the fact that an affine manifold does not have a well-defined (affine) connection because under the transformation:

\[ \Gamma^c_{ab} \rightarrow \Gamma^c_{ab} + \Psi^c_{ab} \]

where \( \Psi^c_{ab} \) is some tensor possibly representing some new field, the quantities \( \nabla J F_{_b_1 a_2 \ldots a_m} \) constructed out of the new connection, still behave as a tensor components under arbitrary (local) diffeomorphisms (coordinate transformations)! A special choice for the connection is the Einstein or Christoffel connection of GR. It obeys the requirements

\[ \begin{cases} \text{symmetry} & \Gamma^c_{ab} = \Gamma^c_{ba} \\ \text{metricity of the covariant derivative} & \nabla J g_{ab} = 0 \end{cases} \]

The only solution is \( \{ \delta \}_{\beta Y} = \frac{1}{2} g^{a\delta} (\partial_y g_{\alpha \beta} + \partial_\rho g_{\alpha \beta} - \partial_\rho g_{\gamma \beta}) \) and it is a very particular case of the affine connection. It is a very important object depending on the metric only; it is the simplest one among all possible affine connections and it’s useful to consider it as some “reference point” for all possible connections. Other connections may be considered by adding an appropriate tensor and it is easy to prove that the difference between two connections is always a tensor. When the space-time is the flat Minkowsky manifold of special relativity, the metric \( g_{ab} \) and the Christoffel symbols \( \{ \Gamma^c_{ab} \} \) depend just on the choice of the coordinates and one can choose them in such a way that \( \{ \Gamma^c_{ab} \} = 0 \) everywhere. On the contrary, if we consider a connection of the kind

\[ \Gamma^c_{ab} = \{ \Gamma^c_{ab} \} + \Psi^c_{ab} \]

Then, the tensor \( \Psi^c_{ab} \) and consequently the whole connection cannot be eliminated by a choice of coordinates. Even if one takes the flat metric, the covariant derivative based on this connection is different than the partial derivative. This means that in general geometry is not completely described by the metric, but has another absolutely independent characteristic, in this case, the tensor \( \Psi^c_{ab} \). We see that if this tensor has a non-symmetrical part, we will have a manifold with torsion \( (\Gamma^c_{ab} - \Gamma^c_{ba} \neq 0) \) independently of having curvature or not.
The ambiguity in the definition of the connection is very important, for it enables one to extend gravitation theories and/or to introduce gauge fields different from gravity, and thus describe various interactions.

There are several different classes of physical theories of space-time corresponding to the various combinations of the curvature, the torsion and the non-metricity conditions. The following diagram taken from [24] illustrates these manifolds (in this diagram torsion is represented by the letter S):

In each of these manifolds there will be different gravitational theories according to the way a specific Lagrangian is constructed. In any case, one natural procedure for the free gravitational Lagrangian is to consider one that is constructed from the curvature and torsion invariants: 

\[ S_{\text{free}} = \int \mathcal{L}_{\text{free}}(\text{inv Curvature}; \text{inv Torsion}), \sqrt{|g|} dx^4. \]

One should take into account that the above ambiguity in the definition of the connection means that any dynamical equation, in some physical theory, containing covariant derivatives will not lose its covariance if we make such transformations on the space of all possible affine connections, but the geometrical properties of the space-time manifold will change with possible physically observable consequences. In this sense, it is not completely analogous to gauge transformations, because the equations change (although their invariance under local diffeomorphisms does not).

For a connection with torsion we obtain new Riemann and Ricci tensors and therefore new gravitational equations and different applications such as astrophysical or cosmological ones. Let us consider the Riemann-Cartan space-time and derive the expressions for the (full) Riemann. We start with a connection of the form (for reasons that will become clearer I will be using Greek letters)

\[ \Gamma_{\beta \gamma}^\alpha = \left\{ \begin{array}{c} \alpha \\ \beta \gamma \end{array} \right\} + \Psi_{\beta \gamma}^\alpha \]

We assume that it is not symmetric,

\[ \tau_{\beta \gamma}^\alpha = \left( \Gamma_{\beta \gamma}^\alpha - \Gamma_{\gamma \beta}^\alpha \right) \neq 0 \]

and the metricity condition allows one to express the connection through the metric and torsion in a unique way as

\[ \Gamma_{\beta \gamma}^\alpha = \left\{ \begin{array}{c} \alpha \\ \beta \gamma \end{array} \right\} + K_{\beta \gamma}^\alpha \]

Where \( K_{\beta \gamma}^\alpha \) is the contorsion tensor depending on the torsion tensor

\[ K_{\beta \gamma}^\alpha \equiv \frac{1}{2} \left( \tau_{\beta \gamma}^\alpha - \tau_{\gamma \beta}^\alpha - \tau_{\beta \gamma}^\alpha \right) \]

The indices are raised and lowered by means of the metric. It is worthwhile noticing that the contortion is antisymmetric in the first two indices, \( K_{\beta \gamma}^\alpha = -K_{\gamma \beta}^\alpha \), while torsion is antisymmetric in the last two indices \( \tau_{\alpha \beta}^\gamma = 2 \Gamma_{[\alpha \beta]}^\gamma \). We now proceed to obtain some important quantities in Riemann-Cartan space-time. The following table reminds us of the definitions for the (full) Riemann and torsion tensors.
Let us first see the dependence of the commutator of covariant derivatives (in Riemann-Cartan space-time) with torsion and curvature. The commutator acting on a scalar field yields:

\[
\left[ \nabla_\alpha, \nabla_\beta \right] \phi = -\left( \partial_\mu \phi \right) (K^\mu_{\alpha \beta} - K^\mu_{\beta \alpha}) = -\left( \partial_\mu \phi \right) 2\Gamma^\mu_{[\alpha \beta]} = -\left( \partial_\mu \phi \right) \tau^\mu_{\alpha \beta}
\]

For the commutator of a vector field we have

\[
\left[ \nabla_\alpha, \nabla_\beta \right] V^\mu = \partial_\alpha \left( \nabla_\beta V^\mu + \nabla_\beta \Gamma^\mu_{\beta \lambda} \right) - \partial_\beta \left( \nabla_\alpha V^\mu + \nabla_\alpha \Gamma^\mu_{\alpha \lambda} \right) + \Gamma^\mu_{\alpha \lambda} \nabla_\beta V^\lambda - \Gamma^\mu_{\beta \lambda} \nabla_\alpha V^\lambda = R^\mu_{\alpha \beta \lambda} \nabla_\lambda V^\nu + \nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu
\]

Using these results and the fact that $V^\mu B_\mu$ is a scalar, one can easily derive the commutator of covariant derivatives acting on a 1-form $B_\mu$, and on any tensor. In all cases, the commutator is a linear combination of the Riemann tensor and torsion.

The (full) Riemann tensor can be expressed in terms of the curvature tensor depending only on the metric $\tilde{\gamma}_{\beta \gamma}$, the torsionless covariant derivative $\tilde{\nabla}$ and the contorsion tensor $\tilde{\kappa}_{\beta \gamma}$, and similar formulas [31] can be found for the Ricci tensor and scalar curvature with torsion.

It is useful to divide torsion into three irreducible components [31]:

1. The (trace) vector
   \[ \tau_\mu \equiv \tau^\mu_{\beta \mu} \]

2. The (sometimes called pseudo-trace) axial vector
   \[ \alpha^\delta \equiv \epsilon_{\beta \gamma} \epsilon_{\delta \epsilon} \tau_{\beta \gamma} \]

3. The tensor
   \[ q^\alpha_{\beta \gamma} \quad (q^\mu_{\beta \mu} = 0, \quad \epsilon_{\beta \gamma} \epsilon_{\delta \epsilon} q_{\alpha \beta \gamma} = 0) \]

Using these fields, torsion can be expressed as [31]

\[ \tau_{\alpha \beta \gamma} = \frac{1}{3} (\tau_{\beta \gamma} \epsilon_{\alpha \mu} - \tau_{\gamma} \epsilon_{\beta \alpha \mu}) - \frac{1}{6} \epsilon_{\beta \gamma} \epsilon_{\delta \epsilon} \alpha^\delta + q_{\alpha \beta \gamma} \]

One can re-obtain the above formulas for the Riemann tensor, Ricci tensor and Ricci scalar expressed in terms of the irreducible components of torsion. For example, the Ricci scalar is written as follows [31]

\[ R = \tilde{R} - 2\nabla_\rho \tau^\rho - \frac{4}{3} \tau^\beta \tau^\beta + \frac{1}{2} q_{\epsilon \mu \tau} q^{\epsilon \mu} + \frac{1}{24} \alpha_{\beta \alpha} \]
Tetrads

In any theory of gravitation, the basic geometrical setting is the tangent bundle, a natural construction associated to the space-time manifold [29]. Space-time is the bundle base space and at each point there is always a tangent space – the fiber – on which a space-time transformation group acts. In the case of the Lorentz group, this tangent space provides the vector representation for the group. This bundle formalism provide a natural extention for other representations – tensorial or spinorial.

From this section and until the end (except when there is no ambiguity on the nature of indices), the Greek alphabet will denote indices related to space-time and Latin letters will denote indices related to the tangent space, endowed with the Minkowsky metric $\eta_{ab} = diag(+1, -1, -1, -1)$. The systems of space-time coordinates $\{x^\nu\}$ and tangent space coordinates $\{x^k\}$ define, on their domains, local (coordinate) basis for vector fields, formed by the gradients

$$\partial_\nu \equiv \{\partial/\partial x^\nu\} \quad \text{and} \quad \partial_k \equiv \{\partial/\partial x^k\}$$

As well as dual basis for covector fields or 1-forms

$$\{dx^\nu\} \quad \text{and} \quad \{dx^k\}$$

The coordinate basis, $\{\partial/\partial x^k\}$ generates the tangent space $T(M)$ at each point of the space-time manifold and the set of 1-forms $\{dx^k\}$ generates the co-tangent space $T^*(M)$. These bases are dual in the sense that

$$dx^\mu(\partial_\nu) = \delta^\mu_\nu \quad dx^k(\partial_l) = \delta^k_l$$

On the respective domains of definition, any vector or covector field can be expressed in terms of these bases, which can also be extended by direct product to constitute bases for general tensor fields. These coordinate or holonomic basis, related to coordinates (gradients and differentials), are very particular examples of linear basis. In fact any set of linearly independent fields $\{\hat{e}_a\}$ will form another base with its dual $\{\hat{e}^a\}$. These frame fields are the general linear basis of the space-time differentiable manifold. The set of all these fields constitute also a differentiable manifold - the bundle of linear frames. Naturally, on the common domains, the members of each base can be expressed in terms of the other

$$\hat{e}_a = e_a^\mu \partial_\mu \quad \partial_\mu = e^c_\mu \hat{e}_c \quad \hat{e}^b = e^b_\nu dx^\nu \quad dx^\nu = e^\mu_a \hat{e}^a$$

Here, the so called “Wierbein” are elements of the general linear transformation group $\{e_a^\mu\}, \{e^b_a\} \in \text{LG}(4, \mathbb{R})$, $\det[e^b_a] > 0$. So $\hat{e}_a$ form a new base obtained by a “rotation” $\in \text{LG}(4, \mathbb{R})$ of the coordinate basis $\{\partial/\partial x^\nu\}$ preserving orientation. We will be considering the tetrad fields $\{\hat{e}_a, \hat{e}^b\}$ as fields of linear frames connected with the presence of a gravitational field. Vectors and 1-forms may be expressed in the following ways

$$\mathbf{v} = v^\mu \partial_\mu = v^a \hat{e}_a \quad \mathbf{z} = z_\nu dx^\nu = z_b \hat{e}^b$$

We immediately obtain

$$v^\mu = e_a^\mu v^a \quad z_\nu = e_\nu^b z_b$$

And also

$$\mathbf{v} \cdot \mathbf{v} = v^a \mu \mu = e_a^\mu e_\mu^b v^a v^b = v^a v_a = v^a v_a \delta^b_a \quad \Rightarrow \quad e_a^\mu e_\mu^b = \delta^b_a$$

Therefore

$$v^a = e_\mu^a v^\mu \quad z_b = e_\nu^b z_{\nu}$$

Analogously, we have the relations

$$e_a^\mu e_\nu^a = \delta^\mu_\nu$$

On the other hand,

$$\mathbf{v} \cdot \mathbf{v} = g_{\mu \nu} v^\mu v^\nu = \eta_{ab} v^a v^b = \eta_{ab} e_a^\mu e_\mu^b v^\mu v^\nu$$

$$\Rightarrow g_{\mu \nu} = \eta_{ab} e_a^\mu e_\mu^b$$

and similarly,

$$\eta_{ab} = e_a^\mu e_b^\nu g_{\mu \nu} = g(\hat{e}_a, \hat{e}_b)$$
This means that a tetrad field is a linear frame whose members are pseudo-orthogonal by the metric $g_{\mu\nu}$. It is not difficult to show that: $\sqrt{-g} = \det(e_\mu^a) = \det((e_\nu^\mu)^{-1})$. The dual basis $\{\hat{e}_a\}$, considered as tetrads (associated to a gravitational field), constitute a non coordinate basis (or non holonómic) and are characterized by a non-null Lie bracket at each point $p$:

$$[\hat{e}_a, \hat{e}_b](p) = f^{c}_{ab}(p)\hat{e}_c(p)$$

**Anholonomy** is the property of a differential form which is not the differential of anything, or of a vector field which is not a gradient. It appears in Physics in different contexts (e.g. heat and work in thermodynamics, angular velocity of a generic rigid body in classical mechanics) and in the context of gravitation, anholonomy is related, through the equivalence principle, to the very existence of a gravitational field [32]. Given a Riemann metric given by $g_{\mu\nu} = \eta_{ab}e_\mu^b e_\nu^a$, the presence or absence of a gravitational field is fixed by the holonomic or holonomic character of the forms $\epsilon^b = e^b_\mu dx^\mu$. Consider the coordinate transformation $\{x^\gamma\} \leftrightarrow \{x^k\}$. We immediately have the following (local) relations between holonomic bases in the base manifold and tangent fiber:

$$dx^a = (\partial_a x^\mu) dx^\mu \quad \partial_a = (\partial_a x^\mu) \partial_\mu$$

$$(dx^a)(dx^\beta) = (\partial_a x^\mu)(\partial_\beta x^\nu) = (\partial_a x^\mu)(\partial_\beta x^\mu)(dx^\nu)$$

On the other hand an anholonomic form $h^a$, such that $dh^a \neq 0$, is not formed by differentials. When applied to $\partial_a$ the result $h^a(\partial_a) = h^a_\mu$ is just the component of $h^a = h_\mu^a dx^\mu$ along $dx^\mu$. Analogously $h_a^\mu = h^\mu_\alpha \partial_\alpha$ are not gradients. Closed forms are locally exact and a clear criterion for holonomy/anholonomy is simple: a form is holonomic if its exterior derivative vanishes! A holonomic base will always be of the form $h^a = dy^a$ for some coordinate set $\{y^\alpha\}$. For such a basis the tensor with components $g_{\mu\nu} = \eta_{ab}h_\mu^ah_\nu^b$ would simply be $g_{\mu\nu} = \eta_{ab}\partial_\alpha x^a \partial_\beta x^b$, that is, the Minkowsky Lorentz metric $\eta$ in the coordinate system $\{y^\mu\}$. We conclude in fact that

The assumption that in the presence of a gravitational field the metric of space-time $g$ should be different than the Minkowsky $\eta$, implies that the tetrad fields $(\hat{e}_a = e_\mu^a \partial_a; \hat{e}^b = e^b_\nu dx^\nu)$ associated to the gravitational field are anholonomic, obeying some Lie algebra:

$$[\hat{e}_a, \hat{e}_b] = f^{c}_{ab}(p)\hat{e}_c$$

This provides a representation for the Lie group characterizing the local space-time symmetries of the tangent fibers. This is one of the basic ideas behind the so called Poincaré Gauge theories of gravity.

From the above commutation relation (characterizing a Lie Algebra) one can also obtain after some algebra the following expression

$$d\hat{e}^c = -\frac{1}{2}f^{c}_{ab}\hat{e}^a \wedge \hat{e}^b = \frac{1}{2}(\partial_\mu e_\nu^c - \partial_\nu e_\mu^c)dx^\mu \wedge dx^\nu$$

We see that the structure coefficients represent the curls of the base members

$$f^{c}_{ab} = [\hat{e}_a, \hat{e}_b]\hat{e}^c = e_\mu^a e_\nu^b (\partial_\nu e_\mu^c - \partial_\mu e_\nu^c)$$

If $f^{c}_{ab} = 0$, then $d\hat{e}^c = 0$, implying the existence of some coordinates such that $\hat{e}^c = dy^c$.

**Connections**

Connections $\Gamma^\lambda_{\mu\nu}$ are essential in order to define derivatives with well-defined tensor behavior. They behave as vectors in the last index and the non-tensorial behavior in the first two indices compensates the non-tensoriality of the ordinary derivatives. Linear connections are intimately linked to the space-time manifold, being defined on the bundle of linear frames which is a constitutive part of the space-time geometric structure. This bundle has some properties not found in the bundles related to internal gauge theories, for example, linear connections, in particular Lorentz connections, always have torsion, while internal gauge potentials have not.

It is important to enhance the fact that formally, curvature and torsion are properties of a connection [32]. In rigor, there are no such things as curvature or torsion of space-time, but instead curvature or torsion of a connection. In fact, many different connections are allowed to exist in the very same space-time. Nevertheless when considering the specific case of GR, where the only connection involved is the Levi-Civita connection, universality of gravitation allows it to be interpreted as part of the space-time definition. Even so, in the presence of different connection with their curvatures and torsions, it seems convenient to
consider space-time as being represented by a manifold and interpret the connections (with their curvatures and torsions) as “additional” structures.

- **General linear connection and corresponding spin connection**

A *spin connection* is a connection of the form [32]

\[ A_\mu = \frac{1}{2} \Lambda^a_{b\mu} S_{ab} \]

Where \( S_{ab} = -S_{ba} \) are Lorentz generators in a given representation. Now, we know that the tetrad fields relate “internal” (associated to the fibers) with external tensors, i.e., it transforms indices defined on the fiber to indices related to space-time. For example:

\[ V^\lambda = e^a_\lambda V^a \quad \quad V^a = e^a_\nu V^\nu \]

Curiously, as we will see, the relation between a general linear connection \( \Gamma^a_{\mu\nu} \) and its corresponding spin connection \( A^a_{b\mu} \) involves an additional “vacuum” term [32]:

\[ \Gamma^a_{\mu\nu} = \Gamma^a_{\mu\nu}(A^a_{b\nu}) = e^a_\nu \partial_\nu e^a_\mu + e^a_\lambda A^a_{b\nu} e^b_\mu \]

\[ A^a_{b\nu} = A^a_{b\nu}(\Gamma^a_{\mu\nu}) = e^a_\mu \partial_\mu e^b_\nu + e^a_\lambda \Gamma^a_{\mu\nu} e^b_\mu \]

One can easily see that these equations are different ways of expressing the result that the total covariant derivative (acting on space-time *and* on Lorentz indices) of the tetrad vanishes identically:

\[ \partial_\mu e^a_\nu - \Gamma^a_{\mu\nu} e^a_\lambda + A^a_{b\nu} e^b_\mu = 0 \]

- **Metric compatibility**

A connection is said to be metric compatible if the covariant derivation of the metric vanishes

\[ \partial_\rho g_{\alpha\beta} - \Gamma^\alpha_{\rho\beta} g_{\alpha\beta} - \Gamma^\beta_{\rho\alpha} g_{\alpha\beta} = 0 \]

From the tetrad point of view, by using the relations \( \Gamma^a_{\mu\nu}(A^a_{b\nu}) \) and \( A^a_{b\nu}(\Gamma^a_{\mu\nu}) \), this condition can be rewritten in the form

\[ e_\mu(\eta_{ab}) - A^a_{a\mu} \eta_{ab} - A^a_{b\mu} \eta_{ab} = 0 \]

Here, \( e_\mu = e^c_\mu \partial_\nu \) and therefore \( e_\mu(\eta_{ab}) = 0 \). We obtain

\[ A_{ba\mu} = -A_{ab\mu} \]

Therefore, the underlying content of the metric preserving property of a connection is that the spin connection is “Lorentzian”.

- **Curvature and torsion of a connection**

Using this formalism, the curvature and torsion of a connection \( A^a_{b\nu} \) are defined by [32]

\[ R^a_{b\nu\mu} = \partial_\nu A^a_{b\mu} - \partial_\mu A^a_{b\nu} + A^a_{c\nu} A^c_{b\mu} - A^a_{c\mu} A^c_{b\nu} \]

\[ \tau^a_{\nu\mu} = \partial_\nu e^a_\mu - \partial_\mu e^a_\nu + A^a_{c\nu} e^c_\mu - A^a_{c\mu} e^c_\nu \]

Using the relation \( A^a_{b\nu}(\Gamma^a_{\mu\nu}) \), these can be expressed entirely in space-time terms [32]:

\[ R^a_{b\nu\mu} \equiv e^b_\beta e^a_\alpha R^\alpha_{b\nu\mu} = \partial_\nu \Gamma^a_{\beta\mu} - \partial_\mu \Gamma^a_{\beta\nu} + \Gamma^\lambda_{\beta\mu} \Gamma^a_{\lambda\nu} - \Gamma^\lambda_{\beta\nu} \Gamma^a_{\lambda\mu} \]

\[ \tau^\gamma_{\nu\mu} \equiv e^\gamma_\alpha \tau^a_{\nu\mu} = \Gamma^\gamma_{\nu\mu} - \Gamma^\gamma_{\mu\nu} \]
• Torsion, contortion, spin connection and anholonomy

We have seen that the connection coefficients can be decomposed, $\Gamma^a_{\beta\gamma} = \{a\} + K^a_{\beta\gamma}$, using the contortion tensor $K^a_{\beta\gamma} \equiv \frac{1}{2}(\tau^a_{\beta\gamma} - \tau^a_{\gamma\beta} - \tau^a_{\beta\gamma})$ and the Einstein (Lecic-Civita) connection $\delta_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}(\partial_{\gamma}g_{\alpha\beta} + \partial_{\beta}g_{\gamma\alpha} - \partial_{\alpha}g_{\beta\gamma})$. In terms of the spin connection, this decomposition assumes the form [32]

$$A^a_{\beta\mu} = \tilde{A}^a_{\beta\mu} + K^a_{\beta\mu}$$

Where $\tilde{A}^a_{\beta\mu}$ is the spin connection of general relativity, the so called Ricci coefficient of rotation. Since this is connection is a tensor in the last index, we can write

$$\tilde{A}^a_{\beta\mu} \equiv \tilde{A}^a_{\beta\mu} e^\mu_k$$

which can be expressed in terms of the anholonomy, that is, in terms of the structure coefficients ($f^c_{ab} = [\hat{e}_a, \hat{e}_b] \hat{e}^c = e^\mu_a e^\nu_b (\partial_{c} e^\mu_{k} - \partial_{k} e^\mu_{c})$):

$$\tilde{A}^a_{bc} = -\frac{1}{2}(f^a_{bc} + f^a_{cb})$$

Now, one can easily see that in the anholonomic basis $\hat{e}_a$, the curvature and torsion components are given by [32]

$$R^a_{bdc} = \hat{e}_e (A^a_{bd}) - \hat{e}_d (A^a_{be}) + A^a_{ec} A^e_{bd} - A^a_{ed} A^e_{bc} + f^e_{cd} A^a_{be}$$

$$\tau^a_{bc} = -f^a_{bc} + A^a_{cb} - A^a_{bc}$$

From this perspective, we see that torsion includes anholonomy. Using this last expression for $\tau^a_{v\mu}$, it is possible to obtain an expression for $A^a_{bc}$ in terms of the structure coefficients and torsion [32],

$$A^a_{bc} = -\frac{1}{2}(f^a_{bc} + f^a_{cb} + f^a_{cb} + \tau^a_{bc} + \tau^a_{cb} + \tau^a_{cb}),$$

which reduces to the general relativistic Ricci coefficient of rotation in the absence of torsion.

Geometrical interpretation of torsion I – Cartan’s circuit: A brief look on the RC-space, seen from Cartan’s perspective

In order to illustrate the geometrical significance of torsion and actually provide some sort of image containing this information, it is usual to invoke the operational definition of torsion. Let us recall that after contraction with two vector fields it originates a new vector field given by:

$$\tau(u, v) \equiv \nabla_u v - \nabla_v u - [u, v]$$

As a consequence, we get the familiar result that the local (infinitesimal) “parallelograms” do not actually close [33]:

![Figure 10 – Torsion and the non-closure of parallelograms [33].](image)

It is the so called closure failure – a parallelogram is only closed up to a small translation.
How can a local observer at a point P tell whether his or her space carries torsion and/or curvature? The local observer defines a small loop (or a circuit) originating from P and leading back to P. Then he/she rolls the local reference space without sliding—this is called Cartan displacement — along the loop and adds up successively the small relative translations and rotations, see [33]. As mentioned, the added up translation is a measure for the torsion and the rotation for the curvature. Since the loop encircles a small 2-dimensional area element, Cartan’s prescription attaches to an area element a small translation and a small rotation. Thus, torsion and curvature are both 2-forms in any dimensions n>1, the torsion is vector-valued, because of the translation vector, the curvature is bivector-valued, because of the rotations. In this way Cartan visualized a RC-space as consisting of a collection of small Euclidean granules that are translated and rotated with respect to each other.

Intuitively it is clear that this procedure of Cartan is similar to what one does in gauge field theory: A rigid (or global) symmetry, here the corresponding Euclidean motions of translation and rotations, is extended to a local symmetry. In four-dimensional space-time it is the Poincaré (or inhomogeneous Lorentz) group of Minkowski space that is gauged and that yields a RC-space-time. There are two degenerate cases: A RC-space with vanishing torsion is the conventional Riemannian space, a RC-space with vanishing RC-curvature is the Weitzenböck space, or a space with teleparallelism. We will come back to this notion later.

Geometrical interpretation of torsion II – Physical ideas that influenced Cartan: The Cosserat continuum and Crystallography

In the context of RC-geometry, Cartan was inspired by the brothers Cosserat [33] and their theory of a new type of continuum: The Cosserat continuum. The classical continuum of elasticity and fluid dynamics consists of unstructured points, and a displacement vector field is the only quantity necessary for specifying the deformation. The Cosserats conceived a specific medium with microstructure, consisting of structured points such that, in addition to the displacement field $u_i$, it is possible to measure the rotation of such a structured point by a bivector field $\omega_{ij} = -\omega_{ji}$. In the geometrical-physical interpretation of the structures of the RC-space, Cartan apparently made use of these results of the brothers Cosserat. The Riemannian space is the analogue of the body of classical continuum theory: points and their relative distances is all what is needed to describe it geometrically; for example, the analogue of the strain $\epsilon_{ij}$ of classical elasticity is the metric tensor $g_{ij}$ of the Riemannian space.

In the 1930s, the concept of a crystal dislocation was introduced in order to understand the plastic deformation of crystalline solids, as, for instance, of iron. Dislocations are one-dimensional lattice defects. If sufficiently many dislocations populate a crystal, then a continuum or field theory of dislocations is appropriate. Already in 1953, Nye [33] was able to derive a relation between a dislocation density and the contortion tensor, which describes the relative rotations between neighboring lattice planes. At the same time it becomes clear that, from a macroscopic, i.e., continuum theoretical view, the response of the crystal to its contortion induced by the dislocations are spin moment stresses $s_{ij}$. It is obvious that if one enriches the geometry with torsion, then in the dynamical side one should allow, besides stress (in 4D energy-momentum), spin moment stress (in 4D spin angular momentum).

In short, a general space-time can, in principle, present two different properties - curvature and torsion – and analogously, these are also known in Crystallography as mentioned, where these two properties are considered as “defects” of a regular structure - Disclination and Dislocation. A crystal with vanishing disclination and dislocation is called regular, in analogy to a space-time with vanishing curvature and torsion which is called flat (and torsionless)

- In order to visualize curvature, consider the parallel transport of a vector around a closed curve. If, when returning to the initial point, there is an angular deficit, the surface is said to be curved and the curvature of the surface is proportional to this angular deficit

\[ v_i = v_f \]

Fig 11 – Curvature effect on the parallel transport of a vector around a closed curve

A simple example is provided by a spherical surface.

- In order to visualize torsion, as mentioned, one can consider the parallel transport of two vectors. If, when parallel transported one along the other, they do not close a parallelogram, the surface is said to present torsion and the torsion of the surface is proportional to the gap or distance deficit. As mentioned, in crystallography this corresponds to dislocation

Fig 12 – A crystal with a dislocation (torsion) structure
Let us consider both curvature and torsion. Imagine the parallel transport of a vector. If it is not possible to return to the initial point because of a distance deficit and in addition there is an angular deficit, the surface is said to present torsion and curvature.

C.3 Different torsion theories

Introduction

There are many different torsion theories of gravity. All of them presuppose a certain space-time paradigm, that is, a choice of the space-time physical manifold and a description of the dynamical role of space-time geometry in describing gravitational interaction. Some theories admit that torsion is produced by the intrinsic spin of matter while others accept that even usual angular momentum can be a source of torsion. This depends on the choice of the Lagrangian. Besides in some theories one could predict the propagation of torsion waves and in others, such as Einstein-Cartan, the relation between the sources and torsion is done via algebraic equations. The following table, taken from [24], characterizes some of the fundamental aspects of these theories.

<table>
<thead>
<tr>
<th>Theory</th>
<th>Dynamical degrees of freedom</th>
<th>Propagating</th>
<th>Source of torsion</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>U₄ Einstein-Cartan</td>
<td>Metric, affine connection</td>
<td>No</td>
<td>Spin</td>
<td>The first of all gravitational torsion theories</td>
</tr>
<tr>
<td>Pagels</td>
<td>O(5) gauge fields</td>
<td>No</td>
<td>Spin</td>
<td>An O(5) theory of gravity</td>
</tr>
<tr>
<td>Metric-affine gravity</td>
<td>General gauge fields</td>
<td>Yes</td>
<td>Spin</td>
<td>Gauge theory of gravity in the metric affine space</td>
</tr>
<tr>
<td>Stelle-west</td>
<td>SO(3,2) gauge fields</td>
<td>Yes</td>
<td>Spin, grad of Higgs field</td>
<td>An SO(3,2) gauge theory spontaneously broken to SO(3,1)</td>
</tr>
<tr>
<td>Hayashi-Shirafuji</td>
<td>Tetrads</td>
<td>Yes</td>
<td>Spin, rotational</td>
<td>Theory in Weitzenböck space-time</td>
</tr>
<tr>
<td>Einstein-Hayashi-Shirafuji</td>
<td>Tetrads</td>
<td>Yes</td>
<td>Spin, rotational</td>
<td>A class of theories in U₄</td>
</tr>
<tr>
<td>Teleparallel gravity</td>
<td>Tetrads</td>
<td>Yes</td>
<td>Energy-momentum</td>
<td>Theory in Weitzenböck space-time</td>
</tr>
</tbody>
</table>

We will be examining 3 different theories and enhancing their differences.

C.3.1 Riemann-Cartan space-time and Einstein-Cartan theory

The Einstein-Cartan theory or Einstein-Cartan-Sciama-Kibble theory was first proposed by Élie Cartan in 1922 (see [34] and references therein). Dennis Sciama and Tom Kibble independently revisited the theory in the 1950’s and again around 1976. Einstein became familiarized with this theory around 1928, a fact that contributed for his search of a unified field theory of gravity and electromagnetism using teleparallelism theory. Due to the fact that according to this theory a direct observation of torsion is expected to be difficult, there was a slightly loss of interest in it. More recently some revival has taken place, since
theorists have been trying to incorporate torsion into quantum and unifying theories and also exploring its cosmological applications.

The main idea motivating Cartan’s theory is that, at the microscopic level, matter is represented by elementary particles with energy-momentum and spin. Adopting a geometrical reasoning, in the spirit of general relativity, spin should also be a source of gravitation. According to Cartan, just like energy-momentum is a source of curvature, spin is a source of torsion.

Although GR can accommodate particles with spin, including spin \( \frac{1}{2} \), by using the tetrad formalism, it cannot accommodate the coupling between spin and orbital angular momentum. Since this coupling is very familiar in quantum theory, a coherent theory of gravitation that is compatible with quantum physics should be able to incorporate spin (and spin-orbit coupling) directly into its field equations. In this sense, the ideas in Einstein-Cartan theory may be seen as extremely important and relevant for the path into quantum gravity. Cartan’s ideas were also fundamental for the journey of physics generally speaking, since were able to expand the geometries used in physical theories beyond Riemann. Once again the door was opened to rethink the geometrical nature of space-time.

The reason why GR cannot accommodate spin-orbit coupling is rooted in Riemann geometry. The Einstein tensor \( G_{\mu\nu} \) is proportional to the energy-momentum tensor \( \Sigma_{\mu\nu} \) and these are both symmetrical in their indices. Now, when spin and orbital angular momentum are being exchanged, the energy-momentum is known to be non-symmetric, according to the conservation of angular momentum (see [34] and references therein):

\[
\text{(Divergence of spin current)} \propto (\Sigma_{\mu\nu} - \Sigma_{\nu\mu}) \neq 0
\]

In 1922 Élie Cartan suggested that GR should be extended to include affine torsion, which allows the Ricci tensor to be non-symmetric. His theory:

1- Makes clear that an affine (or metric-affine) theory provides a coherent description of gravitation and a clear gauge framework;
2- It suggests an explanation for the meaning of affine torsion, which appears naturally in some theories of quantum gravity;
3- It interprets spin as affine torsion, which geometrically is a continuum approximation in the “space-time medium”.

Another relevant motivation for the Einstein-Cartan theory is that GR is known to be “w-inconsistent” (see [34] and references therein). This means that there is a “limit of the theory” that actually does not fit in the theory. If we consider a “fluid”, or more precisely, a set of simple rotating black holes, these are animated with simple (orbital) angular momentum only and can be considered within a torsionless space-time. However, when we perform the continuum limit the angular momentum of this continuum fluid of rotating black holes turns into intrinsic or spin angular momentum and the space-time acquires contortion characteristics. By assumption, however, in GR the space-time cannot accommodate torsion, so the limit has some property that isn’t allowed by the theory although this limit seems something relatively simple and valid from a physical point of view. Einstein-Cartan theory can be viewed as the minimal w-consistent extension of GR. Petti in 1986 showed that GR plus a (continuum) fluid of rotating black holes generate affine torsion that enters the field equations exactly as in the equations of Einstein-Cartan theory. Therefore, starting from classical GR and (classical) differential geometry without recourse to quantum-mechanical spin concepts such as spinor fields, it has been showed that GR plus spin-orbit coupling implies non-zero torsion and a theory such as Einstein-Cartan. One starts from a Kerr-Newman rotating black hole solution of GR and construct an ensemble of many such black holes with correlated rotations. Then, one computes the translational (an)holonomy (by parallel translating an affine frame) around the equatorial space-like loop of constant radius. In the continuum limit, the translation (an)holonomy reveals affine torsion, the distribution of rotating black holes generate a spin density field, and the torsion and spin density fields are related exactly as in Einstein-Cartan theory.

The Cartan connection, \( \Gamma^\alpha_{\beta\gamma} = \{\alpha^\alpha_{\beta\gamma} + K^\alpha_{\beta\gamma} \} \), presents both curvature and torsion. Similarly to GR, the Lagrangian for the gravitational field is the simplest linear function of the Ricci invariant, although in this case it depends on curvature and torsion:

\[
\mathcal{L}_{EC} = \sqrt{-g} \frac{R}{2k}
\]

\[
R = \tilde{R} + 2 \tilde{\nabla}^\alpha K^\mu_{\alpha\mu} - K^\mu_{\alpha\mu} K^\nu_{\alpha\nu} + K_{\mu\nu\tau} K^\mu_{\tau
u}
\]

Consider the total action

\[
S_{EC\text{tot}} = \int (\mathcal{L}_{EC} + \mathcal{L}_{\text{matter field}}) dx^4
\]

Varying this action with respect to the connection and metric, one obtains the Einstein-Cartan field equations:

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2 \kappa \varepsilon_{\mu\nu}
\]

\[
\varepsilon_{\mu\nu} = \text{Non-symmetric (canonical) energy-momentum}
\]

\[
t^\nu_{\mu} = k \left( s^\nu_{\mu} + \frac{1}{2} \left( \delta^\nu_{\mu} s^\alpha_{\alpha\nu} - \delta^\nu_{\alpha} s^\alpha_{\alpha\mu} \right) \right)
\]

\[
s^\nu_{\mu} = \text{Spin tensor}
\]

Let us see more carefully the context, within the Poincare gauge approach, in which these equations appear.
Basics for Poincaré Gauge approach

- **Global Poincaré invariance**

The isometries of Minkowsky space-time $M_4$ are expressed in the Poincaré group (PG), consisting of all Lorentz transformations SO(3,1) and space-time translations T(3,1) which reflect the assumption that space-time is homogeneous and isotropic (and the space-time interval $ds^2 = \eta_{ij}dx^idx^j$ is invariant for all inertial observers).

In Relativistic quantum mechanics one assumes this unchanging, flat space-time and therefore all relevant physical relations (including field equations) have Poincaré or Lorentz invariance. It is well known that one has to pass from the global gauge invariance of the Schrodinger-like equations (under global gauge transformations of the wave functions) into local gauge invariance, when one considers interaction fields (gauge potentials) such as electromagnetic or Yang-Mills vector fields. Analogously, by taking into account gravity, one has to pass from global Poincaré invariance into local Poincaré invariance. The space-time symmetries become local symmetries completely similar to the local gauge symmetries (on local internal spaces - fibers) of quantum field theories. A global Poincaré transformation can be written as follows [30]

$$x^i \rightarrow x'^i = L^i_j x^j + b^i$$

Here $L^i_j$ and $b^i$ are real parameters and $L^i_k L^k_j = \delta^i_j$. For infinitesimal transformations we have

$$\delta x^i = \epsilon^i_j x^j + \epsilon^i$$

The parameters $\epsilon^i_j$ and $\epsilon^i$ give rise to infinitesimal rotations and translations respectively. These correspond to 6 + 4 degrees of freedom. The Lie algebra of the 10 generators characterizing the PG are [30]:

$$[\mathcal{E}_{ij}, \mathcal{E}_{kl}] = \eta_{ik} \mathcal{E}_{jl} + \eta_{jl} \mathcal{E}_{ik} - \eta_{jk} \mathcal{E}_{il} - \eta_{il} \mathcal{E}_{jk} \quad [\mathcal{E}_{ij}, T_k] = \eta_{jk} T_i - \eta_{ik} T_j \quad [T_i, T_j] = 0$$

where $\mathcal{E}_{ij}$ and $T_i$ are the Lorentz and translation generators respectively. The Lagrangian for a general multiplet field $\chi$ (possibly interacting with other fields) in Minkowsky space-time will be invariant under PG. Therefore, Noether theorem assures 10 independent conserved quantities which consist of the energy-momentum (4), orbital angular momentum (3) and intrinsic/spin angular momentum (3) for the whole system.

- **Local Poincaré invariance**

Consider an infinitesimal local Poincaré transformation

$$\delta x^\mu = \epsilon^\mu_\nu(x) x^\nu + \epsilon^\mu(x)$$

Considering Lorentz transformations, we saw from the tetrad formalism that we may regard infinitesimal (fixed point) local gauge transformations as local rotations of basis vectors belonging to the tangent space of the manifold. For this reason, given a local frame on a tangent plane to the point $x$ on the base manifold, we can obtain all other frames on the same tangent plane by means of local rotations of the original basis vectors. Reversing this argument, we observe that by knowing all frames residing in the horizontal tangent space to a point $x$ on the base manifold enables us to deduce the corresponding gauge group of symmetry transformations. Consider the generators of the (local) Lorentz group $S^{ij}$ satisfying

$$S_{ij} = -S_{ji} \quad [S_{ij}, S_{kl}] = \eta_{ik} S_{jl} + \eta_{jl} S_{ik} - \eta_{jk} S_{il} - \eta_{il} S_{jk}$$

It can be shown that in order to define local Poincaré invariance of the action (and dynamical equations) for some multiplet field $\chi$, one needs to define the gauge fields $A^{ij}_\mu$ and a covariant derivative as follows[30]:

$$\nabla_\mu \chi = \partial_\mu \chi + \frac{1}{2} A^{ij}_\mu S_{ij} \chi$$

where $A^{ij}_\mu = \frac{1}{2} A^{ab}_\mu S_{ab}$ is the previously introduced spin connection. Now consider the case where $\chi = \psi$ is a Dirac (spinor) field. In this case, Lorentz invariance requires that under Lorentz transformations, the Dirac field transforms as

$$\psi(x) \rightarrow \psi'(x) = S(\Lambda) \psi(x)$$

Here $S(\Lambda)$ is an irreducible unitary representation of the Lorentz group and $A^{ij}_\mu$ is a Lorentz transformation matrix satisfying $A_{ij} = -A_{ji}$. It can be shown that the group generators $S^{ij}$ can be given in terms of the Pauli-Dirac matrices (the properties of which are resumed in the box bellow):
The bilinear form $j^k = i\bar{\psi}\gamma^k\psi$ is a vector and therefore, it transforms according to $j^k = A^k_{\mu}\gamma^k$. The invariance of $j^k$ (or the covariance of the Dirac equation) under the transformation $\psi(x) \rightarrow \psi'(x) = S(\Lambda) \psi(x)$ leads to

$$S^{-1}(\Lambda)\gamma^k S(\Lambda) = \Lambda^m_\mu \gamma^m$$

One can show that (assuming $S(\Lambda)$ to be expressed by $S(\Lambda) = e^{A(\lambda) i A^a_{\lambda}}$) after expansion to the first order in the parameters $\epsilon^a_{\lambda}(x)$ we get the following expression for $S(\Lambda)$ [30]:

$$S(\Lambda) = \mathbb{1} + \frac{1}{2}\epsilon_{ij}^{\lambda} \gamma_{ij}$$

The covariant derivatives for spinors, adjoint spinors, are [30]

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{2} A^i_{\mu \gamma} \gamma_{ij} \psi \quad \nabla_\mu \bar{\psi} = \partial_\mu \psi - \frac{1}{2} A^i_{\mu \bar{\lambda}} \bar{\psi} \gamma_{ij}$$

For Lorentz vectors (and co-vectors) we have

$$\nabla_\mu v^k = \partial_\mu v^k + A^k_{\mu \nu} v^\nu \quad \nabla_\mu v_\nu = \partial_\mu v_\nu - A^\nu_{\kappa \mu} v_\kappa \nu$$

We can also (covariantly) derive the tetrad fields with respect to the spin connection, that is, with respect to Lorentz transformations, on their Latin indices, these fields behave as vectors (or co-vectors)

$$\nabla_{ij}^{\text{lorentz}} e^c_{\nu} = \partial_\mu e^c_{\nu} + A^c_{d \mu} e^d_{\nu} \quad \nabla_{ij}^{\text{lorentz}} e^\nu_c = \partial_\mu e^\nu_c - A^d_{\nu \mu} e^d_c$$

For space-time vectors (and co-vectors), the affine connection naturally arises

$$\nabla_\mu v^\beta \equiv e^i_\beta \nabla_\mu v^k = \partial_\mu v^\beta + \Gamma^\beta_{\mu \lambda} v^\lambda \quad \nabla_\mu v_\beta \equiv e^k_\beta \nabla_\mu v_k = \partial_\mu v_\beta - \Gamma^\beta_{\mu \lambda} v_\lambda$$

The affine connection is given by

$$\Gamma^\beta_{\mu \lambda} \equiv e^c_\beta \partial_\mu e^e_c - e^c_\beta \partial_\mu e^c_\lambda + e^c_\lambda \partial_\mu e^e_\beta$$

which can be used to obtain the (previously introduced) total covariant derivatives for the tetrad fields:

$$\nabla_\mu e^\nu_c = \nabla_{ij}^{\text{lorentz}} e^\nu_c + \Gamma^\nu_{\beta \mu} e^\beta_c = \partial_\mu e^\nu_c - A^\nu_{\kappa \mu} e^\kappa_c + \Gamma^\nu_{\beta \mu} e^\beta_c = 0$$

$$\nabla_\mu e_c^\nu = \nabla_{ij}^{\text{lorentz}} e_c^\nu - \Gamma^\nu_{\beta \mu} e_c^\beta = \partial_\mu e_c^\nu + A^\nu_{\kappa \mu} e^\kappa_c - \Gamma^\nu_{\beta \mu} e_c^\beta = 0$$

These 3 expressions are exactly the one that were introduced in (C.2 connections) and we see them emerging from the somehow natural construction of our gauge covariant derivative $D_\mu$. The other relation previously introduced expressing the spin connection in terms of the affine connection can be re-written [30]:

$$\Lambda^a_{\mu \nu} = e^a_\mu \partial_\nu e^\mu_{\nu} + e^a_\mu \Gamma^\nu_{\lambda \mu} e^\mu_\nu - e^a_\beta \partial_\nu e^a_\mu + e^a_\lambda \Gamma^\lambda_{\mu \nu} e^\mu_\nu = -w^a_{\nu \lambda} + e^a_\lambda \Gamma^\lambda_{\mu \nu} e^\mu_\nu$$

$$w_{\mu \lambda \nu} \equiv e^a_\mu (\partial_\nu e^a_{\lambda}) \eta_{\lambda \mu} = e^a_\mu \partial_\nu e^a_{\lambda}$$
Finally, having introduced the objects $w_{bv}^a$, by using our previous expression for the covariant derivative of spinors, we get

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{2} A_{ij}^\mu \gamma_{ij} \psi = (\partial_\mu - \Gamma^{(\text{Fock})}_\mu) \psi$$

Where $\Gamma_\mu$ is the so-called Fock-Ivanenko connection [30]

$$\Gamma^{(\text{Fock})}_\mu \equiv \frac{1}{4} (w_{ij}^\mu - \Gamma^\lambda_\mu e^i_\lambda e^j_\gamma) \gamma_i^j$$

The connection so including the contributions of Christoffel (space-time curvature), spin connection (Poincare local invariance) and contortion (torsion), so that it operates in general spinorial arguments, have the following expression [30]:

$$\Gamma_\mu \equiv \frac{1}{4} B_{AB} (\Delta^\beta_\mu \rho - \Gamma^\beta_\rho \mu - \kappa^\beta_\rho \mu) \gamma^{\lambda \rho} = \frac{1}{4} B_{AB} (\Delta^\beta_\mu \rho - \Gamma^\beta_\rho \mu) \gamma^{\lambda \rho}$$

This is the connection that allows us to define the relevant gauge covariant derivative for spinors in Riemann-Cartan space-time:

$$\mathcal{D}_\mu \psi = (\partial_\mu - \Gamma_\mu) \psi \quad \mathcal{D}_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \bar{\psi} \Gamma_\mu$$

In the connection $\Gamma_\mu$, while $\Gamma^\beta_\rho \mu$ acts on vectors and arbitrary tensors, $\Delta^\beta_\mu \rho$ acts only on multicomponent spinor fields and is defined by the expression:

$$\Delta^\beta_\mu \rho \equiv e_d^a \eta^{ad} e^b_\mu \Delta_{aby} \quad \Delta_{aby} \equiv w_{aby} = \frac{1}{2} e^c_\gamma (\Omega_{cab} + \Omega_{bca} - \Omega_{abc})$$

Where $\Omega_{cab}$ are the so-called objects of anholomicity related to the (local) Lie Algebra of the Tetrads and defined as

$$\Omega_{cab} \equiv e_{vc} (e^\mu_a \partial_\mu e^\gamma_b - e^\mu_b \partial_\mu e^\gamma_a)$$

This said, the most general connection in the Poincaré gauge approach to gravity is given by

$$A_{aby} = \Delta_{aby} - K_{aby} + \Gamma^\beta_\mu \rho e^a d e^\beta_\mu$$

**The Lagrangian density and field equations of Einstein-Cartan theory applied to fermion fields**

Taking into account the definitions considered previously we now address the EC field equations in the presence of fermions (Dirac spinors). The full action,

$$S_{EC \text{ tot}} = \int (\mathcal{L}_{EC} + \mathcal{L}_{\text{matter field}}) dx^4$$

The Lagrangian density for spinor fields is given by [30]

$$\mathcal{L}_{\text{field}} = \mathcal{L}_{\text{field}}(\psi, \partial_\mu \psi, e^\mu_\lambda, A_{ij}^\mu) = \frac{1}{2} (\bar{\psi} e^{\alpha} (D_\alpha \psi) - (D_\alpha \bar{\psi}) e^{\alpha} \psi)$$

The minimal coupling of the spinor fields to Riemann-Cartan geometry (gravity) is done through the covariant derivative which ultimately depends on the tetrads (and their derivatives) and on the spin connection. The role of the metric is implicitly represented by the tetrad fields. The connection and the metric represent the degrees of freedom associated to gravity, in particular to curvature and torsion. Varying this action with respect to the connection and metric, one obtains the Einstein-Cartan field equations:

$$g_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = k e_{\mu \nu} \quad \tau_{\mu \nu} = k \left( s^\rho_{\mu \nu} + \frac{1}{2} \left( \delta^\rho_{\mu} s^\alpha_{\nu \alpha} - \delta^\rho_{\nu} s^\alpha_{\mu \alpha} \right) \right)$$

Where the non-symmetrical (canonical) energy-momentum source is given by

$$\mathcal{E}_{\mu \nu} = \Sigma_{\mu \nu} + S_{\mu \nu}$$

Canonical energy-momentum

Dynamical energy-momentum ($\Sigma_{\mu \nu} = \frac{2}{\sqrt{-g}} \delta L_m$)
The canonical energy-momentum,

\[ E^\mu_\nu = \frac{\partial L_{\text{field}}}{\partial (\nabla^\mu \chi)} \nabla_\nu \chi - \delta^\mu_\nu L_{\text{field}} \]

in the case of spinors, it is given by [30]

\[ E_{\mu \nu} = - (\bar{\psi} \gamma_\mu (\nabla_\nu \psi) - (\nabla_\nu \bar{\psi}) \gamma_\mu \psi + \bar{\psi} \gamma_\nu (\nabla_\mu \psi) - (\nabla_\mu \bar{\psi}) \gamma_\nu \psi) \]

The second term \( S_{\mu \nu} \) (Stress-energy tensor of the non-Riemann geometry) depends quadratically on the torsion tensor and, using the second set of field equations, it can be seen as containing information about the spin tensor. The spin tensor can also be called the spin energy potential and has dimensions of (energy/area). It is usually defined as

\[ S_{\mu \nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta L_m}{\delta \kappa^{\mu \nu}_\rho} \]

For fermions, it is totally antisymmetric and is given explicitly by

\[ s_{\rho \mu \nu} = \bar{\psi} \gamma^{[\rho} \gamma^\mu \gamma^\nu] (\psi) \]

Varying the action with respect to \( \bar{\psi} \), one obtains the space-time dynamical equation for the fermion field:

\[ i\hbar \gamma^\rho \nabla_\rho \psi + \frac{3kQ^2}{8} \gamma^{[\rho} \gamma^\mu \gamma^\nu] \psi - mc^2 \psi = 0 \]

This a general curved space-time Dirac equation which can also be recast into the nonlinear equation of the Heisenberg-Pauli type (or Hehl-Datta equation). [30]

\[ i\hbar \gamma^\rho \nabla_\rho \psi + \frac{3kQ^2}{8} (\bar{\psi} \gamma^5 \gamma_\mu \psi) \gamma^5 \gamma^\mu \psi - mc^2 \psi = 0 \]

The equation of motion of particles is usually obtained by considering the generalized matter energy–momentum covariant conservation law, integrating over a space-like section of the world tube of the particle, and expanding the gravitational field in power series. For particles with spin, in addition to a coupling between the particle’s spin and the Riemann tensor, there is a direct coupling between spin and torsion. For spinless particles, it reduces to the geodesic equation.

In general relativity and (as we will see) in teleparallel gravity, the equations of motion of spinless particles are obtained by replacing the ordinary differential by the corresponding covariant differential. The equation of motion for such particles in Einstein–Cartan theory does not follow from the minimal coupling prescription. Some authors have considered this a drawback of the Einstein–Cartan model, considering the crucial role played by the minimal coupling prescription in the quantum description of the fundamental interactions [32]

Note

For a theory in Riemann-Cartan space where the equation of motion for (spinless) particles is given by the coupling prescription, this gives the auto-parallel geodesic equations:

\[ \frac{du^\alpha}{ds} + \Gamma^\alpha_{\beta \gamma} u^\beta u^\gamma = 0 \]

\[ \frac{d u^\alpha}{ds} + \left\{ \alpha \right\} u^\beta u^\gamma = - \bar{K} (\beta \gamma) u^\beta u^\gamma \]

which, in the absence of torsion turn into extremal geodesics.

Another point that has been pointed as a fundamental limitation is that the coupling of torsion with the electromagnetic field breaks the U(1) gauge invariance of Maxwell’s theory.

Now, it can be shown that torsion leads to a contact spin-spin interaction with the classical potential given by [31]

\[ U_{\text{spin–spin}} \sim \frac{1}{M_{\text{Pl}}^2} S^2 \]

Where S is the spin tensor and \( M_{\text{Pl}} \) is the Planck mass (in energy units). Although this interaction may be important in the Early Universe (see [ref Shapiro]), it can only lead to some extremely weak effects at low energies, since the spin-spin potential contains the \( 1/M_{\text{Pl}}^2 \sim 10^{-35} \text{ GeV}^{-2} \) term. Therefore, the effects of torsion, in the Einstein-Cartan theory, are suppressed by what could be called the torsion mass, which is of the Planck order. An alternative possibility is to suppose that torsion is
massless. This is implicit in theories where the torsion equations are not algebraic as in Einstein-Cartan. In these cases, torsion can propagate, and in principle one could detect a torsion field detached from matter and, naturally, gravitation torsion waves. There are limitations coming from the consistency requirement for an effective quantum field theory for “light” torsion.

In [31], Shapiro reviews many quantum aspects of torsion theory and discuss the possibility of the space-time torsion to exist and to be detected. It considers the classical gravity with torsion and also the renormalization of quantum theory of matter fields and related topics, like renormalization group, effective potentials and anomalies. From the action of spinning and spinless particles in a space-time with torsion, he considers possible physical effects generated by the background torsion reviewing the upper bounds for the magnitude of the background torsion which are known from the literature. The possibility of a theory for the propagating completely antisymmetric torsion field is presented and it is supposed that the propagating field should be quantized, and that its quantum effects must be described by, at least, some effective low-energy quantum field theory. It is shown that the propagating torsion may be consistent with the principles of quantum theory only in the case when the torsion mass is much greater than the mass of the heaviest fermion coupled to torsion. If this is so, according to this author, universality of the fermion-torsion interaction implies that torsion itself has a huge mass, and cannot be observed in realistic experiments. Thus, the theory of quantum matter fields on the classical torsion background can be formulated in a consistent way, while the theory of dynamical torsion meets serious obstacles.

Note on why seems natural to associate torsion with spin

There is an intuitively way to illustrate geometrically why it seems natural to associate torsion with spin [34]. Consider the equation of motion coming from the usual coupling procedure in Riemann-Cartan space-time:

$$\frac{du^\alpha}{ds} + \Gamma_\beta^\alpha u^\beta u^\gamma = 0 \iff \frac{du^\alpha}{ds} + \left( \frac{\alpha}{\beta} + K_\beta^\alpha \right) u^\beta u^\gamma = 0$$

This equation shows clearly the distinction between the metric and torsion influences on the geometrical paths in space-time. Now, given the definition of the contortion tensor, in the case of flat space-time with a completely antisymmetric torsion tensor $\tau_{a\beta\gamma} = \tau_{[a\beta\gamma]}$, one can show that the equation of motion

$$\frac{du^\alpha}{ds} + K_\beta^\alpha u^\beta u^\gamma = 0$$

Describes gives rise to a rotation about the translation axis. The geodesic deviation equation, for example, is modified due to the torsion. Let $\delta^\mu$ represent the components of a deviation vector between two infinitesimally neighboring geodesics, and let $t^\beta$ be a tangent to the geodesics along which $\delta^\mu$ is transported. The modified geodesic deviation equation that expresses the acceleration between neighboring geodesics is

$$a^\mu = R^\mu_{a\beta\gamma} \delta^a t^\beta t^\gamma + t^\rho \nabla_\rho (t^\mu t^\alpha \delta^\beta)$$

It can be shown that whereas the Riemann curvature applies a tidal acceleration, the torsion causes the geodesics to twist around each other. If we imagine a rod made up of a lattice of particles where the relative positions of the particles are fixed to at the spacing of the lattice, but the particles are otherwise non-interacting. As the rod moves along, the rod as a whole does not rotate but rather each of the infinitesimally small particles making up the rod will want to spin about its own axes.

Note the gauge breaking of electromagnetism for a minimal coupling to torsion in Riemann-Cartan space-time

The minimal interaction with torsion breaks the gauge invariance for the vector field, since

$$F_{a\beta} \equiv \nabla_\alpha A_\beta - \nabla_\beta A_\alpha = \tilde{F}_{a\beta} + 2A_\mu K^\mu_{a\beta}$$

is not invariant. Here, $\tilde{F}_{a\beta}$ is the usual electromagnetic field tensor. The possibility to modify the gauge transformation in the theory with torsion has been studied in [Shapiro] and references therein. In other words, for the most interesting case of purely antisymmetric torsion it is not possible to save gauge invariance for the vector coupled to torsion in a minimal way.

C.3.2 Teleparalell equivalent of general relativity

In 1918 Weyl made the first attempt to unify gravitation and electromagnetism [32]. His work, though unsuccessful, introduced for the first time the notions of gauge transformations and gauge invariance, and was the seed which has grown into
today’s gauge theories. About ten years after the introduction of torsion by E. Cartan, a second unification attempt was made by Einstein. It was based on the concept of distant (or absolute) parallelism, also referred to as teleparallelism. The crucial idea was the introduction of the tetrad fields. Since the specification of a tetrad involves sixteen components, and the gravitational field, represented by the space-time metric, requires only ten, the six additional degrees of freedom were related by Einstein to the electromagnetic field. This attempt did not succeed either but it also introduced ideas that remain important to this day. In fact, teleparallel gravity can be considered today a viable theory for gravitation – the teleparallel equivalent of general relativity. This theory is consistently interpreted as a gauge theory for the translation group. According to the gauge structure of teleparallel gravity, to each point of space-time there is attached a Minkowski tangent space, on which the translation (gauge) group acts. Gauge transformations are defined as local translations of the Minkowski tangent space coordinates:

\[ x'^a = x^a + e^a(x^\mu) \]

As a gauge theory, the fundamental field of teleparallel gravity is the translational gauge potential \( B_\mu \), a 1-form assuming values in the Lie algebra of the translation group [32]

\[ B_\mu = B^a_\mu p^a \]

where \( p^a = \partial/\partial x^a \) are the translation generators. Under a gauge transformation \( \delta x^a = e^a(x) \equiv e^a \) the gauge potential transforms according to

\[ B'^a_\mu(x) = B^a_\mu(x) - \partial_\mu e^a(x) \]

This is a consequence of the fact that this gauge potential appears naturally as the nontrivial part of the tetrad field:

\[ e^a_\mu = \partial_\mu x^a + B^a_\mu \]

The fundamental connection of teleparallel gravity is the Weitzenböck Connection defined as [32]

\[ \tilde{\Gamma}^\lambda_{\mu\nu} = e_a^\lambda \partial_\nu e^a_\mu \]

This is a connection with torsion and with zero curvature. Moreover, according to the general definition that we saw earlier,

\[ \Gamma^\lambda_{\mu\nu} = e_a^\lambda \partial_\nu e^a_\mu + e_a^\lambda A^a_{\nu\mu} e_\mu \]

we see that, in teleparallel gravity there is a zero spin connection

\[ \tilde{A}^a_{\nu\mu} = e^a_\mu \partial_\nu e_b^\mu + e^a_\lambda \tilde{\Gamma}^\lambda_{\mu\nu} e_b^\mu = 0 \]

The curvature and torsion are [32]:

\[ R^a_{\nu\mu\beta} = \partial_\nu A^a_{\mu\beta} - \partial_\mu A^a_{\nu\beta} + A^a_{\nu\mu} A^e_{\beta\mu} - A^a_{\mu\nu} A^e_{\beta\mu} = 0 \]

\[ \tau^a_{\nu\mu} = \partial_\nu e^a_\mu - \partial_\mu e^a_\nu + A^a_{\nu\mu} e^e_\mu - A^a_{\mu\nu} e^e_\nu = \partial_\nu e^a_\mu - \partial_\mu e^a_\nu \]

<table>
<thead>
<tr>
<th>Teleparallel – connection (Weitzenböck)</th>
<th>GR – connection (Levi-Civita)</th>
</tr>
</thead>
</table>
| \( \tilde{\Gamma}^\delta_{\beta\gamma} = e_a^\delta \partial_\gamma e^a_\beta \) \( \tilde{\Gamma}^a_{\rho\gamma} \neq 0 \) torsion | \( \left\{ \Gamma^\delta_{\beta\gamma} \right\}_E = \frac{1}{2} g^{\alpha\delta} (\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\gamma\alpha} - \partial_\alpha g_{\beta\gamma}) \equiv \left\{ \delta_{\beta\gamma} \right\} = \Gamma^\delta_{\beta\gamma} \)
| \( \tilde{R}^a_{\rho\gamma\delta} = 0 \) vanishing curvature | \( \tilde{\Gamma}^a_{\rho\gamma} = 0 \) vanishing torsion
| \( \tilde{R}^a_{\rho\gamma\delta} \neq 0 \) curvature |

\[ \tilde{\Gamma}^a_{\beta\gamma} = \left\{ \Gamma^a_{\beta\gamma} \right\} + \tilde{K}^a_{\beta\gamma} \]

The Weitzenböck and the Christoffel connections are related by the fundamental relation
where $\tilde{R}^a_{\beta \gamma}$ is the (teleparallel) contorsion tensor defined in terms of the torsion tensor $\tilde{\tau}_{\beta \gamma} = \frac{1}{2}(\tilde{\tau}_{\beta \gamma} + \tilde{\tau}_{\gamma \beta} - \tilde{\tau}_{\beta \gamma} - \tilde{\tau}_{\gamma \beta})$. The coupling prescription in teleparallel gravity is obtained by requiring consistency with the general covariance principle, an active version of the strong equivalence principle. It is constructed in consistency with the coupling prescription of GR. Acting on a space-time vector field, for example, it is given by [32]

$$\nabla_{\mu} v^\beta = \partial_{\mu} v^\beta + (\tilde{\tau}_{\lambda \mu} - \tilde{\tau}_{\lambda \mu}) v^\lambda$$

Analogously, since

$$\tilde{\tau}^a_{\lambda \mu} = \tilde{\tau}^a_{\lambda \mu} + \tilde{R}^a_{\lambda \mu} = 0$$

the covariant derivative of a Lorentz vector is

$$\nabla_{\mu} v^k = \partial_{\mu} v^k + \tilde{\tau}^k_{\mu \lambda} v^\lambda = \partial_{\mu} v^k - \tilde{R}^k_{\mu \lambda} v^\lambda$$

These relations are exactly the same in GR and this is one of the main reasons for the “formal equivalence” between the two theories. It is interesting to note the strong similarity with electromagnetic gauge theory. The torsion can be expressed explicitly in terms of the gauge potential $B^a_{\mu}$ [32]

$$\tilde{\tau}^a_{\nu \mu} = \partial_{\nu} e^a_{\mu} - \partial_{\mu} e^a_{\nu} = \partial_{\nu} B^a_{\mu} - \partial_{\mu} B^a_{\nu}$$

In this theory the previous expressions for vanishing curvature and for torsion are:

$$\tilde{R} = R + 2\tilde{\tau}^a_{\mu \nu} \tilde{R}^a_{\nu \mu} - \tilde{R}^a_{\mu \nu} \tilde{R}^a_{\nu \mu} + \tilde{R}^a_{\mu \nu} \tilde{R}^a_{\nu \mu} = 0$$

A simple Lagrangian in Weitzenböck space-time, the one corresponding to teleparallel gravity, is naturally quadratic in the field strength (torsion) [32]:

- **Lagrangian for teleparallel gravity**
  
  $$\bar{\mathcal{L}} = \frac{\epsilon}{4k} \tilde{\tau}_{a \beta \gamma} \tilde{\iota}^{a \beta \gamma}$$

  $$\tilde{\iota}^{a \beta \gamma} \equiv \tilde{R}^{a \beta \gamma} - g^{a \beta} \tilde{\tau}^{\gamma \lambda} + g^{a \gamma} \tilde{\tau}^{\beta \lambda}$$

  $$\epsilon \equiv \det(e^i_{\nu})$$

Using $\tilde{R}^a_{\beta \gamma} = \{a \beta \gamma\} + \tilde{R}^a_{\beta \gamma}$, it is easy to see that the lagrangian of teleparallel gravity is equivalent to that of general relativity, up to a divergence [32]

$$\bar{\mathcal{L}} = \mathcal{L} - \partial_{\mu} \left(2\epsilon \tilde{\tau}^{\lambda \mu} \frac{\epsilon}{k}\right)$$

A known result from field theory is that we can always construct equivalent actions, in the sense that the new action gives the same dynamical equations, if we make the following transformation $\mathcal{L} \rightarrow \bar{\mathcal{L}} = \mathcal{L} + \partial_{\mu} Q^\rho(\Phi_{\rho})$. Therefore, the evolution equations of teleparallel gravity are formally equivalent to the Einstein field equations. Variation of $\bar{\mathcal{L}}$ in relation to the gauge field yields the teleparallel field equation

- **Teleparallel (vacuum) field equations**

  $$\partial_{\sigma} \left(\epsilon \tilde{\tau}^a_{\rho \sigma}\right) - k(\epsilon j^a_{\rho}) = 0$$

Where $j^a_{\rho}$ is the energy-momentum pseudo-tensor of the gravitational field and $\epsilon j^a_{\rho}$ is the gravitational energy-momentum current:

$$\epsilon j^a_{\rho} \equiv - \frac{\delta \mathcal{L}}{\delta B^a_{\rho}} = \frac{\epsilon}{k} e^i_{a \sigma} \tilde{\tau}_{\rho \lambda}^{\sigma \epsilon} e^\epsilon_{\mu \lambda} - \epsilon e^a_{\rho} \tilde{\iota}$$

The field equations are exactly equivalent to the general relativistic equations, since

$$\partial_{\sigma} \left(\epsilon \tilde{\tau}^a_{\rho \sigma}\right) - k(\epsilon j^a_{\rho}) \equiv \epsilon \left(\tilde{R}^a_{\rho} - \frac{1}{2} e^a_{\rho} \tilde{R}\right)$$
These equations, allow the propagation of torsion in free space! Considering the total Lagrangian \( L_{\text{tot}} = \bar{L} + L_m \). The field equations, obtained from the variation with respect the gauge field, are

\[
\partial_\alpha (\bar{\epsilon}_{\alpha \mu}^{\mu \rho}) - k(\bar{\epsilon}_f^{\mu \rho}) = k(\bar{e} \Sigma_\mu^\rho)
\]

Here \( \Sigma_\mu^\rho \) represents the symmetric energy-momentum tensor that appears as source of curvature in GR and source of torsion in Teleparallel gravity.

**Motion of test particles in teleparallel gravity and analogy with electromagnetism. Equivalence with GR.**

The action integral for the motion of a spinless particle of mass \( m \) in a gravitational field \( B^a_\mu \) is analogous to electromagnetism [32]

\[
S_{\text{Telep}} = -m \int (u_a dx^a + B^a_\mu u^a dx^\mu) \quad u^a \equiv e^a_\mu u^\mu
\]

In the case of electromagnetism we have

\[
S_{\text{Electromagnetic}} = -m_i \int \left( u_a dx^a + \frac{q}{m_i} A_a dx^a \right)
\]

where \( m_i \) is the inertial mass.

The first term in the teleparallel action, represents the action of a free particle whereas the second represents the coupling of the particle’s mass with the gravitational field. The teleparallel action can be written in the form

\[
S_{\text{Telep}} = -m \int (u_a dx^a + B^a_\mu u^a dx^\mu) = -m \int u_a (\partial_\mu x^a + B^a_\mu) dx^\mu = -m \int u_a e^a_\mu dx^\mu = -m \int u_\mu dx^\mu = -mc \int ds
\]

which is exactly the action for a test particle in GR. From this action of a (spinless) test particle we obtain the teleparallel equation of motion that resembles a force equation:

\[
\frac{du^\alpha}{ds} + (\bar{p}^{\alpha}_{\beta\gamma}) u^\beta u^\gamma = K^{\alpha}_{\beta\gamma} u^\beta u^\gamma
\]

Nevertheless, this is formally equivalent to the geodesic equation of general relativity

\[
\frac{du^\alpha}{ds} + \{\alpha\}_{\beta\gamma} u^\beta u^\gamma = 0
\]

The motion of a (spinless) particle along a geodesic in Einstein-Riemann space-time manifold is equivalent to the motion of the same particle on a Weitzenböck space along an extremal geodesic. In spite of this equivalence, we should take in mind the fundamental difference that the space-time geometry is different in both theories. In GR we cannot perform a coordinate transformation that makes \( \{\alpha\}_{\beta\gamma} = 0 \) everywhere, and the reason for this is the existence of curvature: \( \bar{R}^{\alpha}_{\beta\gamma\delta} \neq 0 \iff \partial_\mu g_{\alpha\beta} \neq 0 \). In teleparallel gravity, on the contrary, the reason why we cannot perform a coordinate transformation that makes \( \{\alpha\}_{\beta\gamma} = 0 \) everywhere, is not the existence of curvature (this is zero by definition), but the fact that the metric is affected by the existence of torsion. To see this, remember that by solving the teleparallel field equations we get the gauge field \( B^a_\mu \), and therefore the tetrads \( e^a_\mu \), consequently we can compute the space-time metric

\[
g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu = \eta_{ab} (\partial_\mu x^a + B^a_\mu)(\partial_\nu x^b + B^b_\nu) = \eta_{ab} \frac{\partial x^a}{\partial x^\mu} \frac{\partial x^b}{\partial x^\nu} + \eta_{ab} (B^b_\nu \partial_\mu x^a + B^a_\mu \partial_\nu x^b + B^a_\mu B^a_\nu)
\]

\[
= \eta_{\mu\nu} + \eta_{ab} (B^b_\nu \partial_\mu x^a + B^a_\mu \partial_\nu x^b + B^a_\mu B^a_\nu)
\]
The metric is a “distortion” of the Minkowsky metric, caused by the existence of the gravitational field $B_\gamma^\nu$, which produces torsion. There are in general, two ways to define geodesics that can be called correspondingly: extremal or auto parallel. Whereas in a Riemann space the extremal geodesics (obtained by requiring $ds$ to be an extreme) coincide to the auto-parallel geodesics (obtained from the parallel transport of the tangent vector), in a general Riemann-Cartan space they are not necessarily the same:

**Extremal geodesics**

$$\frac{du^\alpha}{ds} + \{ \alpha \}_{\beta\gamma} u^\beta u^\gamma = 0$$

**Auto-parallel geodesics**

$$\frac{du^\alpha}{ds} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0$$

Note that for Auto-parallel geodesics we have the following

$$\frac{du^\alpha}{ds} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0 \iff \frac{du^\alpha}{ds} + \Gamma^\alpha_{(\beta\gamma)} u^\beta u^\gamma = 0$$

$$\iff \frac{du^\alpha}{ds} + \{ \alpha \}_{(\beta\gamma)} u^\beta u^\gamma = -K^\alpha_{(\beta\gamma)} u^\beta u^\gamma$$

These auto-parallel geodesics are reminiscent of force equations, whereas the extremal geodesics aren’t. The auto-parallel geodesics are always compatible to the coupling prescription in which the full connection of a given manifold is used to construct covariant derivatives. In this sense, it seems to me somehow unnatural the coupling prescription used in teleparallel gravity that uses not the (full) existing connection but instead $\bar{\Gamma}^\beta_{\lambda\mu} - \bar{\Gamma}^\beta_{\lambda\mu}$. Of course, by doing so it is natural to get a description that is formally equivalent to that of GR, even though this equivalence is exaggerated since they are completely different theories. The curves obtained from the so called “force equation” of teleparallel gravity and from the GR geodesic equation are completely different in a fundamental way, in spite of their formal equivalence.

In [32] torsion is playing the role of gravitational force and it is similar to the Lorentz force equation of electrodynamics, a property related to the fact that, like Maxwell’s theory, teleparallel gravity is also a gauge theory. I think that this idea is essentially incorrect, since a “force equation” would be something of the form

$$\frac{du^\alpha}{ds} + \{ \alpha \}_{\beta\gamma} u^\beta u^\gamma = f^\alpha$$

And this is not what happens in teleparallel gravity where, as we saw, the equation of motion obtained corresponds to an extremal geodesics (in Weitzenböck space-time). These authors say that

“the force equation of teleparallel gravity and the geodesic equation of general relativity, therefore, describe the same physical trajectory. This means that the gravitational interaction has two equivalent descriptions: one in terms of curvature, and another in terms of torsion”. Although equivalent, however, there are conceptual differences between these two descriptions. In general relativity, curvature is used to geometrize the gravitational interaction. In teleparallel gravity, on the other hand, torsion accounts for gravitation, not by geometrizing the interaction, but by acting as a force. As a consequence, there are no geodesics in teleparallel gravity, but only force equations, quite analogous to the Lorentz force equation of electrodynamics”.

This is essentially incorrect for the reasons I have shown. First of all, let us analyze the first part of this sentence. We saw that both in teleparallel gravity and GR, the solutions of the equation of motion for particles correspond to extremal geodesics. Do these describe the same physical trajectory? The answer might require some deeper interpretation. First of all, if one assumes the conjecture that there is some objective reality of space-time, then these theories are proposing very different physical space-times from the point of view of geometry. Therefore, from this alone, the proposed trajectories describe different physical trajectories. One could enhance this idea by remembering that a formal mathematical equivalence not always corresponds to a “physical equivalence” (by this I mean an equivalence where different descriptions describe the same physical phenomenon and assume the same physical ontology). I emphasize this idea that these theories suggest different space-time ontologies. On the other hand, one may relax the notion of space-time physicalism and assume a posture in which ideas related to space and time can be interpreted as nothing but elements of our language used to describe the physical phenomenon. According to this posture, whether or not the space-time has some objective reality is secondary or even irrelevant. But even from this perspective, when the authors in [32] make use of the notion of “physical trajectory”, this presupposes the physical objective reality of “trajectory” and therefore of space-time. Consequently, independently of the philosophical position one may
have about the objective reality (or its absence) of space-time, the sentence quoted is inherently incoherent. On the other hand we already saw that the equation of motion do not represent a truly force equation, it corresponds in fact to the extremal geodesics of the Weitzenböck space-time, and therefore, to a geometrization of gravity similarly to GR (using torsion instead of curvature). I should add that even though one can write the equation of motion in a form that resembles a force equation (although this resemblance isn’t completely valid as we saw) the simple fact that torsion is the mathematical object representing the “causal agency” over the particle, is sufficient to say that the gravitational interaction is explained through a geometrization – wherever the particle is, it feels the space-time torsion!

Related to this, can we say that there is equivalence between GR and teleparallel? I propose that there is a formal equivalence but not a physical equivalence, and it seems to me that there is a fundamental failure in many usual interpretations of this “equivalence”. Remember that the field equations can be shown to be equivalent to those of GR. Let us look with more caution to this aspect. Consider the expressions supporting this equivalence:

\[ \partial_a \left( \epsilon^a_{\nu} \epsilon^\nu_{\rho} \right) - k \left( \epsilon^a_{\mu} \right) = 0 \equiv \epsilon \left( \bar{R}^\rho_a - \frac{1}{2} e^\rho_a \bar{R} \right) \]

\[ \bar{L} = \bar{L} - \partial_\mu \left( 2 \epsilon \frac{\bar{a}_\mu}{k} \right) \]

This expressions come straightforwardly from \( \bar{R}^\alpha_{\beta \gamma} = \left\{ \alpha \right\}_{\beta \gamma} + \bar{R}_{\beta \gamma} \), that is, from the theory itself. Now comes the interesting part: all the quantities \( \bar{R}^\rho_a, \bar{R} \), (and therefore \( \bar{L} \)) are constructed from \( \left\{ \delta \right\}_{\beta \gamma} = \frac{1}{2} g^{\alpha \delta} \left( \partial_\gamma g_{\mu \rho} + \partial_\rho g_{\gamma \mu} - \partial_\mu g_{\gamma \rho} \right) \). The metric on its turn, is given by

\[ g_{\mu \nu} = \eta_{\mu \nu} + \eta_{ab} \left( B^b_\nu \partial_\mu x^a + B^a_\mu \partial_\nu x^b + B^a_\mu B^b_\nu \right) \]

Therefore, the Christoffel symbols of GR and Teleparallel gravity would be different (as well as the Einstein tensor constructed from it) if the metrics in both theories were different!

\[ \Sigma_{a \beta \gamma}^{GR} \neq \Sigma_{a \beta \gamma}^{Teleparallel} \Rightarrow g_{a \beta \gamma}^{GR} \neq g_{a \beta \gamma}^{Teleparallel} \Rightarrow \left\{ a \right\}_{\beta \gamma}^{GR} \neq \left\{ a \right\}_{\beta \gamma}^{Teleparallel} \]

\[ \iff \left( g_{a \beta \gamma}^{\left( \left\{ a \right\}_{\beta \gamma}^{GR} \right)} \right)_{Teleparallel} \neq \left( g_{a \beta \gamma}^{\left( \left\{ a \right\}_{\beta \gamma}^{Teleparallel} \right)} \right)_{Teleparallel} \]

Note that the same energy-momentum tensor is used as a “source” in both theories, therefore the metrics are the same. Therefore, there is in fact a formal equivalence between GR and Teleparallel gravity but one cannot say that curvature and torsion are equivalent to describe gravitation. One should be careful with this. The “lesson of equivalence” that we can conclude out of the teleparallel gravity is that:

**In Weitzenböck space-time there can be an equivalence between a description of gravitation base on the torsion and an (equivalent) description of gravitation based on the torsion of the Weitzenböck-Christoffel connection.**

Addressing the question of why gravitation has two “equivalent descriptions”, In [32] R. Aldrovandi and J. G. Pereira say that this duplicity is related to universality. Like the other fundamental interactions of nature, gravitation can be described in terms of a gauge theory and teleparallel gravity fits into this scheme. On the other hand inspired by the universality of free fall, GR is constructed based on the weak equivalence principle. Considering gravitation, these authors say that

“As the sole universal interaction, it is the only one to allow a geometrical interpretation, and hence two alternative descriptions. From this point of view, curvature and torsion are simply alternative ways of describing the gravitational field, and consequently related to the same degrees of freedom of gravity. If this interpretation is correct, Einstein was right when he did not include torsion in general relativity.”[32]

Although these authors have given numerous of valuable contributions to the study of gravitation and physics, once again, I disagree with the above sentence. First of all, admitting that gravity is the unique universal interaction, this does not imply that “it is the only one to allow a geometrical interpretation”, rather that it is the only one to allow a geometrical interpretation based on the Universality of free fall”. It should be also mentioned that within GR it is easily shown that a (localized) particle moves along a geodesic of a locally curved space-time, the curvature of which is produced by nearby sources and by itself. Therefore, there are no true test particles, and the universality of free fall is only valid for idealized test particles. Nevertheless a geometrization of gravity was done. GR was inspired by the idea of the universality of free fall but in fact, what Galileo and others had no idea is that the practical inexistence of test particles is not due to the fact that these perturb the sources (of the
external gravitational field) but rather because the particles themselves perturb the local space-time, where motion is taking place. Regarding the second part of the above quoted sentence, it should be changed into:

*In Weitzenböck space-time the curvature of the Weitzenböck-Christoffel connection and the torsion of the Weitzenböck (full) connection are simply alternative ways of describing the gravitational field, and consequently related to the same degrees of freedom of gravity.*

Regarding the various theories of gravity with torsion and the different interpretations on torsion, the above authors have an scientific posture and open mind:

“(…) from the theoretical point of view, the teleparallel interpretation presents several conceptual advantages in relation to the Einstein–Cartan theory: it is consistent with the strong equivalence principle, and when applied to describe the interaction of the electromagnetic field with gravitation, it does not violate the U(1) gauge invariance of Maxwell’s theory. From the experimental point of view, on the other hand, at least up to now, there are no evidences for new physics associated with torsion. We could then say that the existing experimental data favor the teleparallel point of view, and consequently general relativity. (…) However, due to the weakness of the gravitational interaction, no experimental data exist on the coupling of the spin of the fundamental particles to gravitation. (…) For this reason, in spite of the conceptual soundness of the teleparallel interpretation we prefer to say once more that a definitive answer can only be achieved by experiments”

In comparison to GR, Teleparallel gravity has the advantage that it does not require the equivalence principle. There are conceptual problems between general relativity and quantum mechanics. Some of the problems are connected to the principles on which these theories take their roots. General relativity: is based on the equivalence principle, whereas – a local principle -, whereas quantum mechanics: is based on the uncertainty principle - non-local. As a consequence, there is no a “quantum equivalence principle”. At the quantum level, general relativity might break down. In particular the equivalence principle might be invalid [32].

Teleparallel gravity survives without the (weak) equivalence principle. For example, the action integral for a spinless particle of inertial and gravitational masses $m_i$ and $m_g$ in a gravitational field $B^a_\mu$ can be written as

$$S_{\text{Telep}} = -m_i \int \left( u_a dx^a + \frac{m_g}{m_i} B^a_\mu u_a dx^\mu \right)$$

$$S_{\text{Electromagnetic}} = -m_i \int \left( u_a dx^a + \frac{q}{m_i} A_\mu dx^\mu \right)$$

Notice the similarity with electromagnetism! This is true only when the inertial and gravitational masses coincide: $m_i = m_g$. The equations of motion for teleparallel gravity in the case where $m_i \neq m_g$ are:

$$p^\rho_\mu \left( \partial_\rho x^a + \frac{m_g}{m_i} B^a_\mu \right) \frac{du^a}{ds} = \frac{m_g}{m_i} q a^a \mu_\rho u^\rho \quad \Rightarrow \quad m_i \left( \frac{du^\mu}{ds} + (\vec{F}_{\rho}^\mu) u^\rho u^\gamma \right) = m_g K_{\rho}^a u^\rho u^\gamma$$

Here $p^\rho_\mu \equiv \delta^\rho_\mu + u_\mu u^\rho$, is a projection tensor. At the quantum level, the geometric description of general relativity might break down, but the gauge description of teleparallel gravity may remain a consistent theory due to this property. TG could eventually present better chances to successfully describe the gravitational interaction at the quantum level

**Note on Einstein’s unified theory**

Between 1928 and 1932, Einstein developed a theory which should unify gravitation and electromagnetism [32]. This theory was based on the teleparallel structure The fundamental field was the tetrad, which has 16 independent components10 for the gravitational field 6 for the electromagnetic field. As is well known, the theory did not work for several reasons. One of the reasons is that the teleparallel Lagrangian is local Lorentz invariant, which reduces the number of degrees of freedom to only 10. If Teleparallel describes Gravity, then it can describe gravitation only!

**Note on electromagnetism as torsion**

Could Electromagnetic force be caused by space-time torsion?

$$\frac{d{u^a}}{ds} + \left\{ a \right\}_{\beta \gamma} u^\beta u^\gamma = -K^a_{(\beta \gamma)} u^\beta u^\gamma$$

$$f^\mu = q F_{\rho}^\mu u^\rho = -m K^\mu_{(\beta \gamma)} u^\beta u^\gamma \Rightarrow \frac{q}{m} F_{\rho}^\mu = -K^\mu_{(\beta \gamma)} u^\beta$$

$$\frac{q}{m} F_{\rho}^\gamma = -K_{\mu(\gamma \beta)} u^\beta$$

In this interpretation: A charged particle, in the presence of a torsion field, manifests an electromagnetic force, through the coupling between the particles-4-velocity and torsion! The interaction is geometrized, but only charged particles can feel it!
C.3.3 New teleparallel gravity

After Einstein’s attempt to unify gravitation and electromagnetism (see [32] and references therein), in the sixties, Moller revived the idea of teleparallelism, but with the sole purpose of describing gravitation. Following this, Pelligrini & Plebanski found a Lagrangian formulation for teleparallel gravity, a problem that was reconsidered later by Moller (see [32] and references therein). In 1967, Hayashi & Nakano formulated a gauge model for the translation group. A few years later, Hayashi pointed out the connection between that theory and teleparallelism, and an attempt to unify these two developments was made by Hayashi & Shirafuji in 1979 (see [32] and references therein). In this approach, general relativity is supplemented with a generalized teleparallel gravity, a theory that involves only torsion, and presents three free parameters, to be determined by experiment.

Like in the teleparallel equivalent of general relativity, the relevant connection of new general relativity is the Weitzenböck connection, but unlike the teleparallel equivalent of general relativity, the coupling prescription, however, is assumed to be given by a covariant derivative in terms of the Weitzenböck connection:

\[
Connection \quad \bar{\Gamma}^\lambda_{\mu\nu} = \bar{\Gamma}^\lambda_{\mu\nu} = e_a^\lambda \partial_\nu e_a^\mu \quad \bar{\Lambda}^a_{b\mu} = \bar{\Lambda}^a_{b\mu} = 0
\]

\[
Coupling\ prescripion \quad \bar{\nabla}_\mu v^\beta = \partial_\mu v^\beta + \Gamma^\beta_{\lambda\mu} v^\lambda \quad \bar{\nabla}_\mu v^k = \partial_\mu v^k + \bar{\Lambda}^k_{\mu\lambda} v^\lambda = \partial_\mu v^k
\]

The equation of motion for spinless particles is not given by the coupling prescription, but rather by the (extremal) geodesic equations:

\[
\frac{du^\alpha}{ds} + \left\{\beta\gamma\right\} u^\beta u^\gamma = 0
\]

The Lagrangian of the gravitational field in new general relativity has the form [32]

\[
\mathcal{L} = \frac{e}{2k} \left( a_1 \tau^\mu_{\lambda\beta} \tau^\lambda_{\mu\beta} + a_2 \tau^\mu_{\lambda\beta} \tau^\lambda_{\mu\beta} + a_3 \tau^\mu_{\lambda\beta} \tau^\lambda_{\mu\beta} \right)
\]

\[
e \equiv \det(e^\mu_\nu)
\]

with \(a_1, a_2, a_3\) arbitrary coefficients. As previously mentioned, torsion can be decomposed in irreducible components under the global Lorentz group [32]: the (trace) vector part \(\tau_\beta\); the (pseudo-trace) axial vector \(v^\delta\); the (trace-free) tensor part \(q^a_{\beta\gamma}\)

We may defined the vector and axial parts by the following expressions (the 1/6 factor in \(v^\delta\) was absent in the definition that I used previously and also, in the definition of \(\tau_\beta\), the upper index of the torsion is now contracted with the last index)

- **Vector**
  \(\tau_\beta \equiv \tau^\mu_{\beta\mu}\)

- **Axial vector**
  \(a^\delta \equiv e^a_{\beta\gamma} \tau_a^{\beta\gamma}\)

With these definitions, the torsion tensor can be expressed by

\[
\tau_{a\beta\gamma} = \frac{2}{3} \left( q_{a\beta\gamma} - q_{a\gamma\beta} \right) + \frac{1}{3} \left( g_{a\beta} \tau_\gamma - g_{a\gamma} \tau_\beta \right) + e_{a\beta\gamma} a^\delta
\]

Where the tensor part is given by

\[
q_{a\beta\gamma} \equiv \frac{1}{2} \left( \tau_{a\beta\gamma} - \tau_{\beta a\gamma} \right) + \frac{1}{6} \left( g_{\gamma a} \tau_\beta - g_{\gamma \beta} \tau_a \right) - \frac{1}{3} g_{a\beta} \tau_\gamma
\]

In terms of these components, the above Lagrangian reads for new teleparallel theory reads:

\[
\mathcal{L} = \frac{e}{2k} \left( b_1 \bar{q}^\mu_{\lambda\beta} \bar{q}^\lambda_{\mu\beta} + b_2 \bar{\tau}_\beta \bar{\tau}^\beta + b_3 \bar{a}_{\beta\gamma} \bar{a}^{\beta\gamma} \right)
\]

with \(b_1, b_2, b_3\) new arbitrary coefficients. Considering the fact that \(R_{\text{tot}} = 0\), one can deduce the following identity
\[ \bar{R} = \frac{2}{3} g^{\mu\lambda} \bar{q}_{\lambda\beta}^{\rho} \bar{q}^{\rho}_{\mu} + \frac{2}{3} \bar{r}^{\lambda}_{\rho} \bar{r}^{\rho}_{\lambda} - \frac{3}{2} \bar{u}^{\rho}_{\rho} \bar{a}^{\rho} \]

Where \( \bar{R} \) is the Ricci tensor constructed from \( \{ \delta \}_{\beta Y} \). Finally, using this relation we get the Lagrangian

\[ \bar{\mathcal{L}} = \frac{e}{2k} \left( \bar{R} + c_1 \bar{q}_{\lambda\beta}^{\rho} \bar{q}^{\rho}_{\mu} + c_2 \bar{r}_{\rho} \bar{r}^{\rho} + c_3 \bar{a}_{\rho} \bar{a}^{\rho} \right) \]

With

\[ c_1 = b_1 - \frac{2}{3}, \quad c_2 = b_2 - \frac{2}{3}, \quad c_3 = b_3 + \frac{3}{2} \]

The first term in \( \bar{\mathcal{L}} \), will naturally give raise to Einstein-like equations. According to this theory, therefore, torsion is assumed to produce deviations from the predictions of general relativity (Einstein-like equations) — or equivalently, from the predictions of the teleparallel equivalent of general relativity. This means that, similarly to Einstein–Cartan theory, torsion represents additional degrees of freedom of gravity beyond the 10 d.o.f associated to \( g_{\mu\nu} \) (or to \( \{ \delta \}_{\beta Y} \)). From the point of view of new teleparallel gravity. It should be remarked that solar system experiments restrict severely the existence of non-vanishing \( c_1 \) and \( c_2 \). Furthermore, as already shown in the literature (see [32] and references therein), the Schwarzschild solution exists only for the case with \( c_1 = c_2 = c_3 = 0 \). It seems that this theory lacks experimental support. A proposal to look for some eventual effects produced by a non-vanishing \( c_3 \) using the Gravity Probe B data [24] was made. The constraints on torsion using the GP-B data will be analysed in this work (C.8). The idea behind such proposal lies on the fact that the axial torsion \( a_B \), which represents the gravitomagnetic component of the gravitational field [32], is responsible for producing the Lens–Thirring effect, which is one of the effects Gravity Probe B was intended to detect.

### C.3.5 Discussion

The following table resumes some of the principal aspects of the above theories:

<table>
<thead>
<tr>
<th>Theory</th>
<th>Covariant derivation of space-time and Lorentz vectors</th>
<th>Lagrangian</th>
<th>Field equations</th>
<th>Action for a particle</th>
<th>Equation of motion for particles</th>
</tr>
</thead>
<tbody>
<tr>
<td>GR</td>
<td>( \bar{\nabla}_a V^b + \bar{\nabla}_b V^a ) ( V^a )</td>
<td>( \mathcal{L}_{EH} = -\frac{\sqrt{-g}}{2k} \bar{R} )</td>
<td>( \bar{R}<em>{\mu\nu} - \frac{1}{2} g</em>{\mu\nu} \bar{R} = k \Sigma_{\mu\nu} )</td>
<td>( S = -mc \int ds )</td>
<td>Geodesics (auto-parallel and extremal length coincide)</td>
</tr>
<tr>
<td>Einstein–Cartan</td>
<td>( \bar{\nabla}_a V^b + \bar{\nabla}_b V^a ) ( V^a )</td>
<td>( \mathcal{L}_{EC} = -\frac{\sqrt{-g}}{2k} \bar{R} )</td>
<td>( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = k \varepsilon_{\mu\nu} )</td>
<td>( \varepsilon_{\mu\nu} = k \left( s_{\mu\nu} + \frac{1}{2} \left( \delta_{\mu}^{\alpha} s_{\alpha\nu} - \delta_{\nu}^{\alpha} s_{\alpha\mu} \right) \right) )</td>
<td>Deviations from the geodesic equations, including the couplings spin/Riemann and spin/torsion.</td>
</tr>
<tr>
<td>Teleparallel</td>
<td>( \bar{\nabla}_a V^b ) + ( \bar{\nabla}_b V^a ) ( V^a )</td>
<td>( \mathcal{L}_{telep} = \frac{e}{4k} \bar{\nabla}_a \bar{\nabla}_b \bar{\nabla} \bar{\nabla} )</td>
<td>( \bar{R}_{\mu\nu} ) – energy-momentum pseudo-tensor of the gravitational field</td>
<td>( S_{telep} = -mc \int \left( u_{\mu} \bar{d}x^{\mu} + B_{\mu}^{\nu} u_{\nu} \bar{d}x^{\mu} \right) )</td>
<td>Force equation with ( f_{\mu} = \bar{\nabla}<em>{\mu} a</em>{\mu} + \lambda )</td>
</tr>
<tr>
<td>New Telepar</td>
<td>( \bar{\nabla}_a V^b ) + ( \bar{\nabla}_b V^a ) ( V^a )</td>
<td>( \bar{\mathcal{L}} = \frac{e}{2k} \left( a_1 \bar{r}^{\lambda}<em>{\mu} \bar{r}^{\mu}</em>{\lambda} + a_2 \bar{a}^{\lambda}<em>{\mu} \bar{a}^{\mu}</em>{\lambda} + a_3 \bar{r}^{\lambda}<em>{\mu} \bar{r}^{\rho}</em>{\lambda} + a_4 \bar{a}^{\lambda}<em>{\mu} \bar{a}^{\rho}</em>{\lambda} \right) )</td>
<td>Parameterized field equations analogous to Teleparallel gravity, allowing the propagation of torsion.</td>
<td>( S = -mc \int \frac{du_{\mu}}{\bar{d}x^{\mu}} + \left( \delta_{\beta Y}^{\nu} \right) u^{\beta} u^{\nu} = 0 )</td>
<td></td>
</tr>
</tbody>
</table>

**Table** – Four different theories of gravity.
In theories such as Einstein–Cartan and gauge theories for the Poincaré groups, curvature (of the Christoffel connection) and torsion are considered as independent fields, related with different degrees of freedom of gravity, and consequently with different physical phenomena. This is a conflicting situation to the case of Teleparallel gravity. Precision experiments either in laboratory or in astrophysical and cosmological tests are expected to be able to distinguish which theories are valid.

In general relativity, curvature represents the gravitational field. In teleparallel gravity, it is torsion that represents the gravitational field. In spite of this fundamental difference, the two theories are found to yield equivalent descriptions when of the gravitational interaction. The symmetric matter energy-momentum tensor appears as source in both theories: as source of curvature in general relativity, as source of torsion in teleparallel gravity. Another point that is worth to enhance is that, as previously mentioned, and contrary to what is commonly said, the coupling prescription in teleparallel gravity does not provide a force equation, since it is equivalent to an extremal geodesic equation.

These differences give rise to a conceptual question concerning the actual role played by torsion. The two points of view seem in complete conflict, but one should keep in mind that, actually the torsion and the Christoffel curvature are both parts of the full curvature. The fact that there are theories in which gravity appears as curvature (Christoffel), others in which torsion represents gravity alone and intermediate theories in which both torsion and the Christoffel curvature represent gravitation, seem to suggest the non-standard interpretation in which torsion and curvature (Christoffel) are interconvertible. Experience should give the answer, but, nevertheless it is worthwhile to search theoretical demonstration that, in principle, it is possible to conceive a conversion of torsion energy into curvature energy and vice-versa. In fact, as we will see the Cartan structure equations show that this is something rather natural to expect.

From all the theories of gravity with torsion and curvature the EC gravity is the closest one to GR. In fact, this theory should not be regarded as an alternative to GR, but rather as its natural extension (to sources with intrinsic spin) because it has the same simple Lagrangian, proportional to the Ricci scalar. In this theory, the torsion tensor is a dynamical variable without any constraints on its form. As a result, the spin of matter is a source of torsion, which has a natural physical interpretation in the context of the Poincaré group. Another advantage of the ECSK theory relative to other theories with torsion (and almost all modified theories of gravity) is that it has no free parameters, like GR.

The EC theory predicts that the torsion tensor is related to the spin density through a linear, algebraic equation, so that torsion does not propagate and vanishes where matter is absent. Nevertheless, there is a fundamental point which is often disregarded in the literature, which is the fact that, although there are some theoretical challenges and difficulties to couple electromagnetic fields with torsion, and therefore usually it is assumed that these fields do not “feel” torsion, this fact doesn’t exclude the possibility that:

**Electromagnetic fields might be a source of torsion.**

In particular, if one computes the spin density of propagating electromagnetic waves, then it is reasonable to expect that torsion exists where this spin density exists and therefore:

*When considering the possibility of propagating electromagnetic fields as source of torsion, even in the Einstein-Cartan theory torsion can therefore propagate!*

Nevertheless there are some challenges in defining spin and orbital angular momenta for electromagnetic fields see []

### C.4 Cartan’s structure equations

In 1854 Riemann suggested that the geometry of space-time might have an active role in physical phenomena and in 1915, through General Relativity, Einstein revealed his geometrical description of gravitation. Eventually, geometrical methods based on an approach to differential geometry trough the calculus of differential forms also emerged, being extremely useful and appropriate to set a common formalism within the various theories of physical interactions [29].

What is the role of the fundamental geometrical structures of the space-time manifold, such as connections, metric, curvature, in the various theories of gravitation? Connections for example are important for the construction of gauge theories and in particular for the unification of gravity with other gauge field theories. As mentioned the gauge potentials act as connections in some fiber bundle and one can construct from it an object acting as the respective curvature in the same manifold. These concepts are important in some gravitational theories and other gauge theories such as electromagnetism and Yang-Mills theories. In theoretical constructions where gravity is described as a gauge theory, the connection is a field variable independent and in some cases one considers as fundamental variables the connection and the tetrads or “vierbein”. The metric is somehow incorporated in the tetrads and in the connection, which has a Lie algebra of the Lorentz group (the gauge symmetry group).

Geometrical structures present in theories of gravity based on post-Riemann geometries (Einstein-Cartan, metric theories with torsion, metric-affine gravity) also affect the coupling between gravity and electromagnetism and are extremely relevant in unifying Yang-Mills theories with gravitation (in the framework of super-gravity or even in string theories) [29]. In the last quarter of XXth century some interesting proposals for unifying theories of gravitation with other physical forces emerged
following the development of super-gravity. The formalism of differential forms developed by Cartan has revealed to be appropriate to deal with all these theories. After Einstein’s theory of gravity in 1915 and Weyl’s attempts to unify gravity and electromagnetism in 1918 (introducing the concept of “gauge” invariance), Cartan started to concern about gauge theories from a geometrical perspective. He introduced a generalized connection (gauge potentials) and after, he generalizes General Relativity to manifolds with torsion (1922) introducing also the formalism of tetrads in the development of the calculus of differential forms (Calculus of Cartan). Within the formalism of calculus of differential forms the relations between geometry, group theory (symmetries) and physical concepts may be explored in a clear way and the formal relations between different theories may also be more easily established. Let us now introduce the formalism developed by Cartan that is also useful for the formulation of gravity as a Gauge invariance theory [29].

Consider the set of 1-forms that represent the connection (potential)

\[ \Gamma^a_{\mu\rho} = \Gamma^a_{\mu\rho} \mathrm{d}x^\mu \]

(The components \( \Gamma^a_{\mu\rho} \) might be seen as Yang-Mills potentials for the gauge group \( \text{LG}(4, \mathbb{R}) \) (general linear group) \((4.1)\))

indices \( ab \) are associated to the respective Lie algebra of the symmetry (invariance) group. By derivation, one obtains the set of curvature 2-forms (field intensity)

\[ \mathcal{R}^a_{b\rho\delta} = \mathrm{df}^a_{b\rho\delta} + \Gamma^a_{b\rho\delta} \Gamma^\rho_{b\gamma} = \frac{1}{2} \mathcal{R}^a_{b\rho\delta} \mathrm{d}x^\gamma \wedge \mathrm{d}x^\delta \]

\((4.2)\)

The components \( R^a_{b\rho\delta} \) correspond to the usual Riemann tensor components if we chose a coordinate basis. Once again \( ab \) are group indices and \( \gamma \delta \) are space-time indices so the gauge transformations act on \( ab \) and coordinate changes (diffeomorphisms) act on \( \gamma \delta \). Incidentally, the gauge transformations are also elements of the general diffeomorphisms. We get general relativity if we identify (the indices of) both groups: \( \text{LG}(4, \mathbb{R}) \) which act in the “internal spaces” – the fibers; and the group of general diffeomorphisms which act on space-time [29]. In general relativity, there is no distinction on the nature of indices \( a\beta \) and \( \gamma \delta \) present in \( R^a_{b\rho\delta} \) and in fact there are mixed symmetries which “disguise” this distinction namely:

\[ R^a_{b\rho\gamma\delta} = R^a_{\gamma\delta a\beta} = R^a_{\beta a\gamma\delta} + R^a_{\gamma \delta \beta} + R^a_{\beta \gamma \delta} = 0 \]

These symmetries are related to the condition for metricity \( \nabla_{a\beta} g_{b\gamma} = 0 \) and the symmetry of the connection (absence of torsion). The antisymmetry on the last pair of indices \((R^a_{b\rho\gamma\delta} = -R^a_{b\rho\delta\gamma})\) reflect the anti-commutativity in the (Grassman) algebra of the differentials, whereas the antisymmetry in the first pair of indices \( (R^a_{b\rho\gamma\delta} = -R^a_{b\rho\delta\gamma}) \) is a consequence of the metricity condition reflecting therefore the invariance of length for vectors that are parallel transported – it is connected to the algebra of the Lie group \( \text{SO}(3,1) \).

We see that we obtain General relativity if we identify (the indices of) both groups \( \text{LG}(4, \mathbb{R}) - \text{LG}(4, \mathbb{R}) \) - the one that is acting on the internal spaces (gauge group - \( \text{SO}(3,1) \)) with that which is acting in space-time (diffeomorphisms). It is possible to relate these groups in different ways. One possibility is to consider symmetric connections (vanishing torsion) and the other is to contract in the action, constructed from the curvature scalar, the space-time indices with “internal “ indices. In this way, the geometry of the tangent bundle gets connected to the Physics described by the theory of Yang-Mills potentials \( \Gamma^a_{\mu\beta} \) in order to produce general relativity in the gauge formulation [29].

We see that if we use symmetry groups different than \( \text{LG}(4, \mathbb{R}) \), or if we introduce torsion and/or non-metricity, we relax the relation between space-time indices and those of the tangent spaces (internal). That is what happens in some unified theories. Many, interesting variant of general Relativity uses a formalism based on tetrads. We will use this formalism to derive Cartan’s structure equations.

We know that the dual basis \( \{ \hat{e}_a \} \) associated to gravitational fields are non-coordinate basis (or non-holonomic) and are characterized by a non-null Lie bracket at each point \( p \) \( [\hat{e}_a, \hat{e}_a](p) = \hat{f}^c_{ab}(p)\hat{e}_c(p) \), where \( \hat{f}^c_{ab}(p) = e^c_{\nu}(e^\lambda_{\mu})e^{\nu}_a - e^c_{\nu}(e^{\lambda}_{\mu})e^\nu_b \) \( (4.2) \). Now, using the covariant derivative of the metric \( \nabla_{\gamma} g_{\mu\nu} = \eta_{ab}(\nabla_{\gamma} e^a_{\mu})e^b_{\nu} + \eta_{ab} e^{a}_{\mu}(\nabla_{\gamma} e^b_{\nu}) \), one can show that the metricity condition can only be obeyed if

\[ \nabla_{\gamma} e^a_{\mu} = \partial_{\gamma} e^a_{\mu} - e^a_{\sigma}(\Gamma^\gamma_{\mu\sigma} - \Gamma^\gamma_{\nu\mu}) = w^a_{\gamma\mu} = w^a_{\gamma\mu} = w^a_{\gamma\mu} \]

where \( w^a_{\gamma\mu} = -w^a_{\mu\gamma} \)

By antisymetrization of the indices \( \gamma, \mu \) and considering a symmetric connection \( \Gamma^a_{\gamma\mu} \) (no torsion) one arrives at

\[ \partial_{\gamma} e^a_{\mu} - \partial_{\mu} e^a_{\gamma} - w^a_{\mu\beta} e^\beta_{\gamma} + w^a_{\gamma\mu} e^\beta_{\mu} = 0 \]
Introducing the set of 1-forms \( w^a_b \equiv w^a_{\gamma b} dx^\gamma \), we can rewrite this equation in a way known as Cartan’s first structure equation:

\[
D\hat{e}^a = d\hat{e}^a + w^a_b \wedge \hat{e}^b = 0
\]

This equation allows us to introduce the absolute (or covariant) exterior differential \( D \) of any one-parameter set of p-forms (p-forms with vector values). In the general case of a p-form \( z \) with tensor values of the kind \((r, s)\) (a set of p-forms whose elements are specified by \( r \) contra-variant indices and \( s \) co-variant indices), there is also a corresponding \((p+1)\)-form \( Dz \) known as the absolute (or covariant) exterior derivative of \( z \).

The first structure Cartan equation defines a set of Torsion 2-forms \( \tau^a \)

\[
\tau^a \equiv d\hat{e}^a + w^a_b \wedge \hat{e}^b
\]

whose components correspond to those of the Torsion tensor \( \tau(X, Y) \equiv \nabla_X Y - \nabla_Y X - [X, Y] \).

\[
\tau^a = \frac{1}{2} \tau^a_{\alpha\beta} dx^\alpha \wedge dx^\beta = \frac{1}{2} \tau^a_{\alpha\beta} e^\alpha_{\gamma} e^\beta_{\delta} \wedge \hat{e}^c = \frac{1}{2} \tau^a_{\beta\alpha} \hat{e}^b \wedge \hat{e}^c
\]

\[
\tau(X, Y) = \tau^a(X, Y) e_a = \tau^a_{\alpha\beta} X^\alpha Y^\beta \quad (\tau^ \mu_{\alpha\beta} = \tau^a_{\alpha\beta} e^\mu)
\]

We saw that the tetrads link space-time manifold and its tangent fiber at each point. They live in the tangent bundle and since each \( w^a_b \) plays the role of a connection (1-form) it is reasonable to expect curvature to be related to the second (covariant) exterior derivative of the tetrads. This is accomplished by derivation of Cartan’s torsion 2-form:

\[
D\tau^a = D(D\hat{e}^a) \implies \implies d\tau^a + w^a_b \wedge \tau_b = d(d\hat{e}^a + w^a_b \wedge \hat{e}^b) + w^a_c \wedge (d\hat{e}^c + w^c_d \wedge \hat{e}^d) = \implies \implies D\tau^a = (dw^a_b + w^a_c \wedge \hat{e}^c) \wedge \hat{e}^b \equiv R^a_b \wedge \hat{e}^b
\]

In this way we define a set of Curvature 2-forms which corresponds to Cartan’s second structure equation

\[
R^a_b \equiv dw^a_b + w^a_c \wedge \hat{e}^c
\]

The components of the Curvature 2-form correspond to those of the usual Riemann curvature tensor

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z
\]

\[
R^a_b = \frac{1}{2} R^a_{b\gamma\delta} dx^\gamma \wedge dx^\delta = \frac{1}{2} R^a_{\gamma\delta \beta\alpha} e^\gamma_{\epsilon} e^\delta_{\zeta} \hat{e}^\epsilon \wedge \hat{e}^\zeta = \frac{1}{2} R^a_{\beta\gamma \alpha\delta} \hat{e}^\epsilon \wedge \hat{e}^\zeta
\]

\[
R(X, Y) e_a = R^a_b(X, Y) e_a = R^a_{b\gamma\delta} X^\gamma Y^\delta e_a = R^a_{\gamma\delta \beta\alpha} X^\gamma Y^\delta e_a \quad (R^ \mu_{\gamma\delta \beta\alpha} = R^a_{\gamma\delta \beta\alpha} e^\mu_e e^\nu)
\]

The so called Bianchi’s first identity we already obtained and is expressed by the equation

\[
D\tau^a = R^a_b \wedge \hat{e}^b
\]

To obtain Bianchi’s second identity we need to compute the covariant exterior derivative of the Curvature 2-forms,

\[
DR^a_b = dR^a_b + w^a_c \wedge R^c_b = R^a_c \wedge w^c_b \implies dR^a_b + w^a_c \wedge R^c_b - R^a_c \wedge w^c_b = 0
\]

To see how torsion and curvature transform under the gauge transformations, consider the following orthogonal rotation:

\[
\hat{e}^a \rightarrow \hat{e}^a' = \Phi^a_{\mu} \hat{e}^\mu, \quad \text{where} \quad \Phi^a_{\mu}, \Phi^b_{\nu} \eta_{ab} = \eta_{\mu\nu} = \eta_{cd}
\]

Taking into account that \( (d\Phi)^a_b (\Phi^{-1})^b_c = -\Phi_{b(f} (d\Phi^{-1})^b_{c)} \), one can show that the new connection is given by

\[
w^a_{b'} = \Phi^a_{\alpha} w^c_{\delta} (\Phi^{-1})_{b'}^\delta + \Phi^a_{\epsilon} (d\Phi^{-1})_{b'}^\epsilon
\]

And the transformations for torsion and curvature are
Similarly, one can show that the “covariant derivative” transforms in fact in a covariant way: $(DV)^{a'}_{b'} = \Phi^{a}_{b}(DV)^{c}_{d}(\Phi^{-1})^{d}_{a}$

As a summary, the main equations in this formalism are presented in the following box

<table>
<thead>
<tr>
<th>Tetrads, and connection</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{e}^b = e^b_v , dx^v \quad { e^\mu_a } \in \text{GL}(4, \mathbb{R})$</td>
</tr>
<tr>
<td>$\nabla_{\gamma} e^a_{\mu} = \partial_{\gamma} e^a_{\mu} - e^a_{\sigma} \Gamma^\sigma_{\gamma\mu} = w^a_{\gamma\mu} e^b_{\mu} \quad w^a_{b} \equiv w^a_{\gamma\mu} dx^\gamma$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Torsion and curvature</th>
<th>Bianchi’s identities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau^a = D \hat{e}^a = d \hat{e}^a + w^a_b \wedge \hat{e}^b$</td>
<td>$D \tau^a = R^a_b \wedge \hat{e}^b$</td>
</tr>
<tr>
<td>$R^a_{b} \equiv dw^a_{b} + w^a_{c} \wedge w^c_{b}$</td>
<td>$(d\tau^a + w^a_{\gamma} \wedge \tau^\gamma - R^a_{b} \wedge \hat{e}^b = 0)$</td>
</tr>
<tr>
<td>$D R^a_{b} = R^a_{c} \wedge w^c_{b}$</td>
<td>$(dR^a_{b} + w^a_{c} \wedge R^c_{b} - R^a_{c} \wedge w^c_{b} = 0)$</td>
</tr>
</tbody>
</table>

In this formalism one could say that $g_{a\beta}$ and $\Gamma^\gamma_{\delta \epsilon}$ may somehow be represented by $e^a_{\mu}$ and $w^a_{\gamma \mu}$ respectively. We can construct the Riemann tensor components without using the Christoffel connection. On the other hand we can always contract contravariant tensors with $e^a_{\mu}$ to obtain objects which behave as scalars relative to space-time diffeomorphisms. We can work with p-forms which contain exactly the same information of covariant tensors completely antisymmetric. The most general object with which we can deal in this formalism is a p-form with values in some vectorial space, or more precisely, with values in the Lie algebra of a certain representation of the Lorentz group, or possibly of groups which have SO(3,1) as sub-group. We see that the space of p-forms is extremely rich. One advantage of it is the facility with which we can deal with spinor-like quantities – objects which transform as representations of the group SL(2, $\mathbb{C}$). Note that theories based in GL(4, $\mathbb{R}$) symmetry can only deal with spinors with the introduction of the tetrads. Concerning the connection, in general relativity we have the following

<table>
<thead>
<tr>
<th>Levi-Civita connection</th>
</tr>
</thead>
<tbody>
<tr>
<td>Metricity: $\nabla_a g_{\beta \gamma} = \partial_a g_{\beta \gamma} - \Gamma^\lambda_{a \beta} g_{\lambda \gamma} - \Gamma^\lambda_{a \gamma} g_{\beta \lambda} = 0$</td>
</tr>
<tr>
<td>No torsion: $\tau^a_{\beta \gamma} = \frac{1}{2} (\Gamma^\alpha_{\gamma \beta} - \Gamma^\alpha_{\beta \gamma}) = 0$</td>
</tr>
</tbody>
</table>

In Cartan’s formalism, the so called Levi-Civita spin connection is obtained restricting the affine spin connection in a similar way [29]:

<table>
<thead>
<tr>
<th>Levi-Civita spin connection</th>
</tr>
</thead>
<tbody>
<tr>
<td>Metricity: $w_{ab} = -w_{ba}$</td>
</tr>
<tr>
<td>No torsion: $\tau^a = d \hat{e}^a + w^a_b \wedge \hat{e}^b = 0$</td>
</tr>
</tbody>
</table>

C.5 Analogies with electromagnetism

The tetrads might be seen as fundamental gauge fields, being members of the (local) Lorentz group $\{ e^\mu_a \} \in \text{GL}(4, \mathbb{R})$ - the gauge group of the theory - in accordance to the equivalence principle. Therefore, since

$$\nabla_{\gamma} e^a_{\mu} = \partial_{\gamma} e^a_{\mu} - e^a_{\sigma} \Gamma^\sigma_{\gamma \mu} = w^a_{\gamma \mu} e^b_{\mu}$$
We see that
\[ w^a \gamma_b e^b_{\mu} e^\mu_c = w^a \gamma_c = (\nabla \gamma \mu) e^\mu_c \Rightarrow w^a_b \equiv w^a \gamma_b d\gamma = (\nabla \gamma \mu) e^\mu_c d\gamma \]
Therefore if we know the connections \( \Gamma^a_{\gamma \mu} \) (or \( w^a \gamma_b \)) and the tetrads \( e^b_{\mu} \), we can obtain torsion and curvature. The equations defining torsion and curvature are analogous to the definition of the Maxwell 2-form from the electromagnetic potential

\[
\begin{align*}
\tau^a &= D \tilde{e}^a = d \tilde{e}^a + w^a_b \wedge \tilde{e}^b \\
R^a_b &= D w^a_b = d w^a_b + w^c_a \wedge w^b_c
\end{align*}
\]

Similarly, the (Bianchi’s) equations are analogous to the homogeneous equations

\[
\begin{align*}
D \tau^a &= R^a_b \wedge \tilde{e}^b \\
D R^a_b &= R^a_c \wedge w^c_b
\end{align*}
\]

So the field equations for gravity should be analogous to

\[ dG = kJ \quad \text{where} \quad G \mapsto F \]

In fact it should be possible to construct a single object (form) that contains both “potentials” \( \tilde{A} \mapsto (\tilde{e}^a; \tilde{w}^a) \), another single object that contains both curvature and torsion \( \tilde{F} \mapsto (\tilde{R}^a_b; \tilde{\tau}) \) and finally a single form \( \tilde{\tau} \) that contains both the usual energy-momentum tensor and the “spin” tensor (in the most general case this tensor should contain information not only on the intrinsic spin but also of “normal” angular momentum). The fact that there is an equivalent alternative to General relativity (teleparallel equivalent) that assumes torsion but no curvature, and the idea that torsion should somehow be analogue to magnetism [ref], may also suggest that it should be possible to obtain (differential) equations relating torsion and curvature, having as a consequence the possibility to interconvert these geometrical and physical quantities. In fact the equation \( D \tau^a = R^a_b \wedge \tilde{e}^b \) already suggests part of this interpretation, namely, that curvature should act as source for torsion.

There is an alternative way to explore the analogies with electromagnetism using torsion and this is related to the Einstein-Cartan theory.

Exploring the analogies using Einstein-Cartan theory

Hehl and collaborators suggested a metric-affine geometry, with torsion and nonmetricity of the connection field, representing the micro structure of space-time, with Riemannian geometry emerging as some sort of macroscopic average over this metric-affine microstructure (see [35] and references Therein). They generalized therefore the earlier approach to the Einstein-Cartan formalism of Hehl et al based on a metric connection with torsion: “We claim that the [Einstein-Cartan] field equations… are, at a classical level, the correct microscopic gravitational field equations. Einstein’s field equation ought to be considered a macroscopic phenomenological equation of limited validity, obtained by averaging [the Einstein-Cartan field equations]”[35]. Adamowicz has an alternative approach [35] suggesting that “the relation between the Einstein-Cartan theory and general relativity is similar to that between the Maxwell theory of continuous media and the classical microscopic electrodynamics”. In fact Adamowicz treats the spin density that enters in the Einstein-Cartan theory as the macroscopic average of microscopic angular momenta, but he does this only within the linear approximation. Therefore he doesn’t make explicit the relation he suggests by developing a formal analogy between quantities in macroscopic electrodynamics and in Einstein-Cartan theory. Such analogy is developed by John Stachel [35] in a paper dedicated to Engelbert Schucking (on his seventieth birthday as a recognition for his many contributions to science as a scholar and as a human being). In this paper, the analogy with macroscopic electrodynamics is developed for the exact, nonlinear version of the Einstein-Cartan theory. I now present these analogies in [35].

Concerning the macroscopic theory it is possible to introduce a microscopic model for the material system and derive the form of the polarization tensor using the methods of statistical physics. For Dielectrics, usually one considers the electric and magnetic dipoles, neglecting intrinsic magnetic dipoles and moments of higher order which are (term by term) gradually decreasing corrections.

The ideas of microscopic and macroscopic electrodynamics can be expressed using forms in the following way:
It is possible to treat gravitation theory in a similar, analogous way. For the “microscopic” theory we introduce two fields, the Christoffel symbols of the first kind $[\alpha \beta, \tau]$ and a symmetric connection $\left( \Gamma^\mu_{a\beta} \right)_{(E)} = \left( \Gamma^\mu_{\beta a} \right)_{(E)}$, which can be called the Einstein connection. The symbols $[\alpha \beta, \tau]$ are derived from a set of potentials (the metric functions)

$$[\alpha \beta, \tau] = \frac{1}{2} (\partial_a g_{\beta \tau} + \partial_\beta g_{\tau a} - \partial_\tau g_{a\beta})$$

Just as in the electromagnetic case, these equations can be interpreted as asserting the absence of (gravito) magnetic monopoles. Using the commutativity of the second derivatives of the metric, one can easily derive from this expression the following relation, analogous to the Homogeneous Maxwell equations:

$$\partial_\delta ([\alpha \beta, \tau] + [\alpha \tau, \beta]) - \partial_\alpha ([\beta \delta, \tau] - [\delta \tau, \beta]) = 0$$

With the Einstein connection one constructs the Einstein tensor:

$$G_{a\beta} = G_{a\beta} \left[ \left( \Gamma^\mu_{a\beta} \right)_{(E)} \right] \equiv R_{a\beta} - \frac{1}{2} g_{a\beta} R$$

The Einstein field equations are analogous to the other set of Maxwell’s equations

$$G_{a\beta} \left[ \left( \Gamma^\mu_{a\beta} \right)_{(E)} \right] = k T_{a\beta}$$

The Einstein connection and Christoffel symbols (of the first kind) are related by the metric tensor in analogy to the constitutive relations of electromagnetism

$$\left( \Gamma^\mu_{a\beta} \right)_{(E)} = g^\mu \nu [\alpha \beta, \tau] = \left\{ \frac{\mu}{a\beta} \right\}$$

Here $\left\{ \frac{\mu}{a\beta} \right\}$ are the Christoffel symbols of the second kind, so the Einstein connection is the symmetric, metric connection for which $\nabla^E_\mu g^a_\beta = 0$ (where the covariant derivative $\nabla^E_\mu$ is constructed with the Einstein connection). In general relativity the tensor field $g_{a\beta}$ is both the metric tensor and the potentials for the Christoffel symbols of the first kind. Therefore, the treatment of gauge transformations is slightly different from electromagnetism. Here, diffeomorphisms play the role of gauge transformations. Any vector field $\nu^\mu$ can generate a one-parameter family of such diffeomorphisms and for infinitesimal transformations (specified by the parameter $\lambda$), the tensor field is dragged into $g^a_{\beta'} = g_{a\beta} + \lambda \nu^\mu g_{a\beta}$. It is easy to see that the Christoffel symbols of the first kind are also dragged by the diffeomorphism, which makes sense because in order to main the same values at each physical point (since $g_{a\beta}$ is not only potential for the Christoffel symbols but also a metric field). Let us
now consider the “macroscopic” Einstein-Cartan theory. Within the Einstein-Cartan theory we introduce another connection \( \Gamma_{(C)} \) which can be called the Cartan connection and will be the gravitational analogue of the “macroscopic electromagnetic field” \( G_M \).

It is assumed that this connection is metric, so the non-metricity tensor (the covariant derivative of the metric with respect to this connection), vanishes. As a consequence, the difference between the Cartan and Einstein connections depends merely on the torsion tensor with components (using a coordinate basis)

\[
\tau^\mu_{\alpha\beta} = \frac{1}{2} \left( \left( \tau^\mu_{\alpha\beta} \right)_{(C)} - \left( \tau^\mu_{\beta\alpha} \right)_{(C)} \right)
\]

We define the so called contortion tensor,

\[
K^\mu_{\alpha\beta} \equiv \tau^\mu_{\alpha\beta} - \tau^\mu_{\alpha\beta} - \tau^\mu_{\beta\alpha}
\]

where the indices are lowered or raised with the metric tensor (and its inverse), then it follows that

\[
\left( \tau^\mu_{\alpha\beta} \right)_{(C)} = \left\{ \frac{\mu}{\alpha\beta} \right\} - K^\mu_{\alpha\beta}
\]

The stress-energy tensor is assumed to be separated into free and bound portions. Inside matter the torsion and contortion tensors do not vanish. The latter is assumed to depend on the properties of matter as well as on the gravitational field to which matter is subjected. Its form may be postulated within the macroscopic theory or an attempt may be made to derive it from a microscopic model of the medium. Stachel followed an intermediate route, postulating its form but motivating the \textit{Ansatz} by a microscopic argument. In gravity there is no evidence for the existence of negative mass and therefore we cannot expect the elementary constituents of matter to have an invariant (irrespective to the origin chosen for the evaluation) mass dipole moment. In fact, by choosing the evaluation point for each such constituent as its center of mass, we can make the mass dipole moment to vanish. In principle there will be mass quadrupole moments (evaluated at the center of mass), which can contribute to the “macroscopic” contortion tensor. Let us assume for now that this can be neglected.

This said, we shall consider only the gravitational analogue of the magnetic dipole moment, treated within the macroscopic theory. We consider a velocity field \( U^\mu \) throughout the matter and at each point a spin vector which has only spatial (“magnetic type”) components. This corresponds in fact to the so called Weynhoff \textit{Ansatz} for the form of the spin tensor density field \( s^\mu_{\alpha\beta} \):

\[
s^\mu_{\alpha\beta} = U^\mu s^\mu_{\alpha\beta} \quad s^\alpha_{\beta} = -s^\beta_{\alpha} \quad s_{\alpha\beta} U^\mu = 0
\]

In the 3-space orthogonal to the 4-velocity, at each point, the spin tensor is formally equivalent to a spin vector. In Einstein-Cartan theory it is assumed that inside matter, the spin-tensor density field is essentially given by the so called modified torsion tensor (ref); but since this only differs from the torsion tensor by its trace, which is zero in Weynhoff \textit{Ansatz}, it follows that

\[
\tau^\mu_{\alpha\beta} = \sigma s^\mu_{\alpha\beta} \quad \sigma = \text{const.} \quad \Rightarrow \quad K^\mu_{\alpha\beta} = \sigma \left( U^\alpha s^\mu_{\beta} - U^\mu s^\alpha_{\beta} - U^\beta s^\mu_{\alpha} \right) \quad \Rightarrow \quad K^\beta_{\alpha\beta} = 0 \quad g^a^bK^\mu_{\alpha\beta} = 0
\]

We now introduce the (macroscopic) field equations inside matter in the form

\[
G_{\alpha\beta} \left( \left( \tau^\mu_{\alpha\beta} \right)_{(C)} \right) = k \left( T_{\alpha\beta} \right)_{\text{free}}
\]

The Cartan connection is not torsion free, so the tensors \( G_{\alpha\beta} \left( \left( \tau^\mu_{\alpha\beta} \right)_{(C)} \right) \cdot \left( T_{\alpha\beta} \right)_{\text{free}} \) are not symmetric. Using equation (4.38) in the definition of the Ricci tensor for the Cartan connection, and using (4.43), we get

\[
R_{\alpha\beta} \left( \Gamma_{(C)} \right) = R_{\alpha\beta} \left( \Gamma_{(E)} \right) - V^\mu_{(E)_{\alpha\beta}} K^\mu_{\alpha\beta} - s_{\alpha\tau\tau} s_{\beta\alpha} U^\mu U^\beta
\]

Taking the trace and using (4.43) again we find

\[
R \left( \Gamma_{(C)} \right) = R \left( \Gamma_{(E)} \right) - s_{\alpha\tau\tau} s_{\beta\alpha}
\]
And therefore

\[ G_{\alpha\beta}(\Gamma^{(G)}) = G_{\alpha\beta}(\Gamma^{(E)}) - \mathcal{V}_{\mu}^{(E)\nu} K_{\alpha\beta}^{\mu} - s_{\alpha\tau} s^{\tau\tau} (U_{\alpha} U_{\beta} - \frac{1}{2} g_{\alpha\beta}) \]

In this way, defining the “bound” and “free” energy-momentum tensors by

\[ (T_{\alpha\beta})_{\text{bound}} = \mathcal{V}_{\mu}^{(E)\nu} K_{\alpha\beta}^{\mu} + s_{\alpha\tau} s^{\tau\tau} (U_{\alpha} U_{\beta} - \frac{1}{2} g_{\alpha\beta}) \]

\[ (T_{\alpha\beta})_{\text{free}} = T_{\alpha\beta} - (T_{\alpha\beta})_{\text{bound}} \]

Note that these tensors are not symmetric. We see that the “macroscopic” Einstein–Cartan field equations follow from the microscopic Einstein equations and these definitions. The following tables resume what was exposed.

Some notes

Adamowicz has proved within the linear approximations for both theories, a correspondence theorem between solutions of the macroscopic Einstein-Cartan equations and the microscopic Einstein equations. He showed that, in the linear approximation, to a static solution to the Einstein-Cartan equations for an axially symmetric body with suitable Weyssenhoff spin-density field, corresponds the same external gravitational field as the one for a stationary solution of the Einstein equations for the same body, without any spin density, but in rigid rotation about its axis. This is analogous to electromagnetism, where the external fields of a charged magnet, treated by macroscopic Maxwell equations, and of a rotating charged body, treated by the microscopic equations can be the same.

As we saw, the analogy between Einstein-Cartan and macroscopic Maxwell equations on one hand, and between Einstein and microscopic Maxwell equations on the other, is already expressed within the full gravitational theories, not only in the linear approximations. Stachel says asserts that “It should be possible to find exact static interior solutions of the macroscopic

<table>
<thead>
<tr>
<th>&quot;Microscopic&quot; gravitation (Einstein)</th>
<th>&quot;Macroscopic&quot; gravitation (Einstein-Cartan)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (F = dA) )</td>
<td>( (F = dA) )</td>
</tr>
<tr>
<td>[ [\alpha\beta, \tau] = \frac{1}{2} (\partial_{\alpha} g_{\beta\tau} + \partial_{\beta} g_{\tau\alpha} - \partial_{\tau} g_{\alpha\beta}) ]</td>
<td>[ [\alpha\beta, \tau] = \frac{1}{2} (\partial_{\alpha} g_{\beta\tau} + \partial_{\beta} g_{\tau\alpha} - \partial_{\tau} g_{\alpha\beta}) ]</td>
</tr>
<tr>
<td>( (dF = 0) )</td>
<td>( (dF = 0) )</td>
</tr>
<tr>
<td>[ \partial_{\alpha} ([\alpha\beta, \tau] + [\alpha\tau, \beta]) - \partial_{\beta} ([\beta\tau, \alpha] + [\beta\alpha, \tau]) = 0 ]</td>
<td>[ \partial_{\alpha} ([\alpha\beta, \tau] + [\alpha\tau, \beta]) - \partial_{\beta} ([\beta\tau, \alpha] + [\beta\alpha, \tau]) = 0 ]</td>
</tr>
<tr>
<td>( (dG = J) )</td>
<td>( (dG_{M} = J_{f}) )</td>
</tr>
<tr>
<td>[ G_{\alpha\beta} \left( (\Gamma_{\alpha\beta})^{\mu}<em>{(E)} \right) = kT</em>{\alpha\beta} ]</td>
<td>[ G_{\alpha\beta} \left( (\Gamma_{\alpha\beta})^{\mu}<em>{(C)} \right) = k(T</em>{\alpha\beta})_{\text{free}} ]</td>
</tr>
<tr>
<td>( (G = F) )</td>
<td>( (G_{M} = G - P) )</td>
</tr>
<tr>
<td>[ (\Gamma_{\alpha\beta})^{\mu}_{(E)} = g^{\mu\nu} [\alpha\beta, \tau] = { \frac{\mu}{\alpha\beta} } ]</td>
<td>[ (\Gamma_{\alpha\beta})^{\mu}<em>{(C)} = { \frac{\mu}{\alpha\beta} } - K</em>{\alpha\beta}^{\mu} ]</td>
</tr>
<tr>
<td>( (J = I_{f} + I_{b}) )</td>
<td>[ \text{Contortion and torsion} ]</td>
</tr>
<tr>
<td>[ T_{\alpha\beta} = (T_{\alpha\beta})<em>{\text{free}} + (T</em>{\alpha\beta})_{\text{bound}} ]</td>
<td>[ \text{Torsion and spin} ]</td>
</tr>
<tr>
<td></td>
<td>[ \tau_{\mu}^{\alpha\beta} = \sigma s_{\alpha\beta}^{\mu} ]</td>
</tr>
<tr>
<td>&quot;bound&quot; and “free” sources</td>
<td>[ (T_{\alpha\beta})<em>{\text{bound}} = \mathcal{V}</em>{\mu}^{(E)\nu} K_{\alpha\beta}^{\mu} + s_{\alpha\tau} s^{\tau\tau} (U_{\alpha} U_{\beta} - \frac{1}{2} g_{\alpha\beta}) ]</td>
</tr>
<tr>
<td></td>
<td>[ (T_{\alpha\beta})<em>{\text{free}} = T</em>{\alpha\beta} - (T_{\alpha\beta})_{\text{bound}} ]</td>
</tr>
<tr>
<td></td>
<td>[ \text{Weynhoff Ansatz} ]</td>
</tr>
<tr>
<td></td>
<td>[ s_{\alpha\beta} = U_{\alpha} s_{\beta\beta} ]</td>
</tr>
<tr>
<td></td>
<td>[ s_{\alpha\beta} = -s_{\beta\alpha} ]</td>
</tr>
<tr>
<td></td>
<td>[ s_{\alpha\beta} U_{\beta} = 0 ]</td>
</tr>
</tbody>
</table>
Einstein-Cartan equations that have the same stationary external fields as corresponding exact stationary solutions of the microscopic Einstein equations. It should even be possible to prove theorems relating entire classes of exact stationary solutions to the Einstein-Cartan equations for rigidly rotating sources to classes of exact solutions to the Einstein-Cartan equations with the same exterior metric but a static interior solution representing a nonrotating source with corresponding spin tensor density distribution.” (1). This type of analysis may even be generalized from the Einstein-Cartan macroscopic approach to the case of metric-affine geometries discussed by Hehl and collaborators (1), in which the connection is no longer metric.

The Einstein-Cartan theory with the Weyssenhoff Ansatz seems adequate to handle the gravitational analogue of magnetic polarization \( M \), but not the gravitational analogue of electric polarization \( P \). This author argued that no gravitational analogue of electric dipole moment should exist, but it should exist definitely the gravitational analogue of the electric quadrupole moment. In the electric theory, the divergence of the electric quadrupole moment contributes to the polarization, and something similar is expected in the gravitational theory. Stachel suspects that something like the covariant derivative of the quadrupole moment should be related to the nonmetricity tensor.

Stachel and Adamowicz believe that the macroscopic approach to the Einstein-Cartan Theory “may be used effectively for solving certain cosmological or astrophysical problems” (1), and “the extension of the analogy to matter with intrinsic or induced quadrupole moments would considerably extend its range of applicability, in particular to problems involving interactions of gravitational radiation with matter”.

C.7 Einstein-Cartan and Gauge theories of gravity in differential forms

Translational gauge theory of gravity

Cartan’s investigations started by analyzing Einstein’s GR and by taking recourse to the theory of Cosserat continua. The points of these continua carry independent translational and rotational degrees of freedom. Torsion is related to translations in a similar way as curvature is to rotations. Besides ordinary (forces) stresses, in these structures additionally there are spin moment stresses. In a 3-dimensional (continuum) crystal with dislocation lines, a linear connection can be introduced that takes the crystal lattice structure as a basis for parallelism. Such a continuum has similar properties as a Cosserat continuum, and the dislocation density is equal to the torsion of this connection. Subsequently, these ideas were applied to the 4-dimensional spacetime. A translational gauge theory of gravity can be made in a Weitzenböck or teleparallel space-time, as well as in a Riemann-Cartan space-time.

The construction of the gauge theory for the group of translations is quite nontrivial because the local space-time translations look very similar to the diffeomorphisms of space-time. Nevertheless, they are different. The underlying geometrical structure of the theory is the affine tangent bundle \( \Lambda(M) \) over the base space \( M \), which arises when one replaces at every point of \( M \) the usual tangent space by an affine tangent space. In the affine space, one can perform translations of the points and vectors, and in this way, the translation group is realized as an internal symmetry.

In accordance with the general gauge-theoretic scheme, associated with the generators \( P_a \) of the translation group there is a Lie algebra-valued 1-form

\[
\Gamma^{(T)} = \Gamma^{(T)a}_\beta P_a dx^\beta
\]

It is the previously mentioned translational gauge field potential. Here \( (T) \) stands for “translation”. Under translations \( y^a \rightarrow y^a + \epsilon^a \) in the affine tangent space, it transforms like a connection

\[
\delta \Gamma^{(T)a}_\beta = -\partial_\beta \epsilon^a
\]

Since the translational group is Abelian, i.e., translations commute with each other, there is no homogeneous term in this transformation law and therefore it resembles the “phase transformation” of an electromagnetic potential. For the same reason, the gauge field strength

\[
F^{(T)a} = d\Gamma^{(T)a} = \frac{1}{2} F_{a\beta} \, dx^a \wedge dx^\beta
\]

is formally reminiscent of a generalized electromagnetic field strength. In addition to the translational gauge field, another important structure is a field \( \xi^a \) defined as a local section of the affine tangent bundle. Geometrically, this field determines the “origin” of the affine spaces; it is known as Cartan’s “radius vector”. Under the gauge transformation (translation) it changes as \( \xi^a \rightarrow \xi^a + \epsilon^a \). However, the combination

\[
e^a_\beta = \partial_\beta \xi^a + \Gamma^{(T)a}_\beta
\]
is obviously gauge invariant, see. In a rigorous gauge-theoretic framework, the 1-form
\[ \hat{e}^a = e^a_\beta \, dx^\beta = d\xi^a + \Gamma^{(T)a} \]
translational gauge field
arises as the non-linear translational gauge field with \( \xi^a \) interpreted as the "Goldstone field" describing the spontaneous breaking of the translational symmetry (see Hehl, ‘Élie Cartan’s torsion in geometry and in field theory”). We can consistently treat \( \hat{e}^a = e^a_\beta \, dx^\beta \) as the coframe of our manifold. Then the translational gauge field strength is actually the 2-form:
\[ F^{(T)a} = d\Gamma^{(T)a} = d\hat{e}^a \]
(2-form)
With the help of the Goldstone type field \( \xi^a \), the translational gauge field gives rise to the coframe \( \hat{e}^a = e^a_\beta \, dx^\beta \). The "anholonomy 2-form" \( F^{(T)a} \) is the corresponding translational gauge field strength.

The gravitational theories based on the coframe as the fundamental field have a long history. The early coframe (vierbein, or tetrad, or teleparallel) gravity models were developed by Møller, Pellegrini and Plebánski, Kaempfer, Hayashi and Shirafuji, to mention but a few, see (see Hehl, “Élie Cartan’s torsion in geometry and in field theory” and references therein). The first fiber bundle formulation was provided by Cho. More recent advances can be found in the works of Aldrovandi and Pereira, Andrade and Pereira, Gronwald, Itin, Maluf and da Rocha-Neto, Muench, Obukhov and Pereira, and Schucking and Surowitz (see Hehl, “Élie Cartan’s torsion in geometry and in field theory” and references therein).

The Yang-Mills type Lagrangian 4-form for the translational gauge field \( \hat{e}^a \) is constructed as the sum of the quadratic invariants of the field strength:
\[ \tilde{L}(\hat{e}^a, d\hat{e}^a) = -\frac{1}{2k} F^{(T)a} \wedge \left( \sum_3 a_i \left( F^{(T)}_i \right)^3 \right) \]
Lagrangian 4-form
Here, \( k = 8\pi G/c^4 \) and \( \ast \) denotes the Hodge dual of the Minkowski flat metric, that is used also to raise and lower the local frame indices (Latin). We can decompose the field strength \( F^{(T)a} \) into the three irreducible pieces of the field strength, the tensor part, the trace, and the axial trace, respectively.

\[ \left( F^{(T)}_a \right)^1 \equiv F^{(T)}_a - \left( F^{(T)}_a \right)^2 - \left( F^{(T)}_a \right)^3 \]
\[ \left( F^{(T)}_a \right)^2 \equiv \frac{1}{3} \hat{e}^a \wedge (\hat{e}^b \wedge F^{(T)b}) \]
\[ \left( F^{(T)}_a \right)^3 = \eta^{ba} \left( F^{(T)}_b \right)^3 \left( F^{(T)}_c \right)^3 \equiv \frac{1}{3} \hat{e}^b \wedge (\hat{e}^c \wedge F^{(T)c}) \]

There are three coupling constants, \( a_1, a_2, a_3 \). In accordance with the general Lagrange scheme [ref] one derives from the Lagrangian the translational excitation 2-form \( \tilde{H}_a \) and the canonical energy momentum 3-form \( \tilde{T}_a \):
\[ \tilde{H}_a = -\frac{\delta \tilde{L}}{\delta F^{(T)a}} = \frac{1}{k} \ast \left( \sum_3 a_i \left( F^{(T)}_i \right)^3 \right) \]
translational excitation 2-form
\[ \tilde{T}^{(gravity)}_a = \frac{\delta \tilde{L}}{\delta \hat{e}^a} = \hat{e}_b : \tilde{L} + \left( \hat{e}_c : F^{(T)c} \right) \wedge \tilde{H}_a \]
(gravitational) energy momentum 3-form
Accordingly, the variation of the total Lagrangian \( \tilde{L}_{\text{tot}} = \tilde{L} + \tilde{L}_{\text{matter}} \) with respect to the tetrad results in the gravitational field equations:
\[ d\tilde{H}_a - \tilde{T}^{(gravity)}_a = \tilde{T}^{(source)}_a \]
with the canonical energy-momentum current 3-form of matter \( \tilde{T}^{(source)}_a \) as the source
\[ \frac{\delta \tilde{L}_{\text{matter}}}{\delta \hat{e}^a} = \frac{\delta \tilde{L}_{\text{matter}}}{\delta \hat{e}^a} \]
The tetrad models do not possess any other symmetry except the diffeomorphism invariance and the invariance under the rigid Lorentz rotations of the tetrads. However, for a special choice of the coupling constants
\[ a_1 = 1 \quad a_2 = -2 \quad a_3 = -1/2 \]
the field equations turn out to be invariant under the local Lorentz transformations \( \hat{e}_a \rightarrow L^a_b(x) \hat{e}_b \), with \( L^a_b(x) \) arbitrary functions of the space-time coordinates. It can be shown that the tetrad field equations take the form of Einstein’s equations
Here $\bar{R}^{bc}$ is the Riemannian curvature of the Christoffel connection which in differential forms is can be expressed by

$$\bar{R}^b_a = d\bar{\Gamma}^b_c + \bar{\Gamma}^b_c \wedge \bar{\Gamma}^c_a$$

Remember that the quantities with (~) refer to Riemann geometry with a Levi-Civita (Christoffel) connection.

One can show that the Einstein-Cartan field equations in differential forms arise from the variations of the Lagrangian of the theory with respect to the coframe $\bar{\omega}^c$ and the connection $\bar{\Gamma}^a_b$, leading to

\[
\text{Einstein-Cartan equations}
\]

\[
\frac{1}{2} \eta^{abc} \wedge \bar{R}^{bc} = k T^{(\text{energ.-mom.})}_a
\]

\[
\frac{1}{2} \eta^{abc} \wedge \tau^c = k S^{(\text{spin})}_{ab}
\]

The gravitational field sources are: the canonical energy-momentum current (3-form) $T^{(\text{energ.-mom.})}_a \equiv \frac{\delta L_{\text{matter}}}{\delta \bar{\omega}^a}$ and the canonical spin current 3-form of matter $(S^a_{\ b})^{(\text{spin})} \equiv \frac{\delta L_{\text{matter}}}{\delta \bar{\Gamma}^a_b}$.

C.8 On the coupling and unification of electromagnetism and gravity

- **Electromagnetism**

  The axiomatic structure of electromagnetism presented in Part A is equivalent to the 4-dimentional formalism in which the general Maxwell equations on arbitrary (pseudo) Riemann manifolds are:

  \[
dF = 0 \quad dG = J
\]

  Remember that these constitute a set of 8 independent equations for the 12 unknowns (both $F$ and $G$ are regarded as 2-forms). Since $F$ and $G$ couple to each other (and to space-time geometry), according to the 6 (local, linear, homogeneous and isotropic) space-time constitutive relations, $G = * F$, we have 12-6 = 6 unknown variables. Therefore since there are 8 Maxwell equations, we end up with 2 independent degrees of freedom for the Electromagnetic interaction field. Another way of saying this is that out of the 6 degrees of freedom, 4 are absorbed by the gauge transformations of the 4-potential, resulting on the 2 independent polarizations of $A$, characteristic of electromagnetic waves.

  Since $A$ is a 1-form and the local gauge transformations (subjacent to the structure of Maxwell equations) $A \rightarrow A + d\lambda$, are characterized by the 0-form $\lambda$, they form a 1-parameter gauge group: Just the unitary group $U(1)$!

  Therefore,

  Maxwell Electromagnetism with local, linear, homogeneous and isotropic space-time constitutive relations on a 4-dimentional manifold (with $E;H$ and $D;B$ defined on the 3d hypersurfaces)

  \[
  \Rightarrow \quad 2 \text{ (external) degrees of freedom} + 1 \text{ (internal) gauge degree of freedom}
  \]

  One can try to generalize this to the case of a $n$-dimensional manifold (assuming 1 time-like dimension):
On the other hand, 
\[ \mathbf{dF} = 0 ; \quad \mathbf{dG} = \mathbf{J} \quad \Rightarrow \quad 2 \times C_n^{n-1} = 2n \text{ equations} \]
In this case, 
\[ 2n - \frac{n(n-1)}{2} = \frac{n(5-n)}{2} , \text{ therefore:} \]
\[ \frac{n(5-n)}{2} = \frac{5}{2} n - \frac{n^2}{2} \quad (\text{external) degrees of freedom} \]

Now, since \( \mathbf{F} = \mathbf{dA} \) is invariant under \( \mathbf{A} \rightarrow \mathbf{A} + \mathbf{dA} \), we see that

\begin{align*}
\mathbf{A} & \quad (\text{n-3) - form} \\
\lambda & \quad (\text{n-4) - form} \\
\end{align*}

Therefore we conclude that there are:

\[ \frac{n(n-1)(n-2)(n-3)}{24} \quad (\text{internal) gauge symmetries} \]

In summary, we have:

Maxwell Electromagnetism with local, linear, homogeneous and isotropic space-time constitutive relations on a \( n \)-dimentional manifold

\[ \Rightarrow \quad \frac{n(5-n)}{2} = \frac{5}{2} n - \frac{n^2}{2} \quad (\text{external) degrees of freedom} \]

\[ + \]

\[ \frac{n(n-1)(n-2)(n-3)}{24} \quad (\text{internal) gauge symmetries} \]

The following table resumes the external degrees of freedom and internal symmetries for several different space-time dimensions.
At each point \( x \) in space-time, one can regard the local “internal” (gauge) degrees of freedom as representing the dimension of the local fiber \( \mathcal{G}_{\text{int}} \) – the space of representation for the gauge group. This dimension is on its turn dependent on the space-time dimension. The total number of different gauge potentials, compatible with a certain field configuration, could be interpreted as the total number of some sort of local microscopic degrees of freedom. All these “microscopic degrees of freedom are compatible with the (local) “macroscopic” field configuration \( F(x) \). Taking this analogy with thermodynamics a little further and considering the definition of entropy \( S \), we can write \( S \) at each point, as a function of space-time dimension:

\[
S(x, n) = S(\Omega_{\text{int}}(x, n)) \propto \ln(\Omega_{\text{int}}(x, n))
\]

The number of all possible “microscopic” states (all possible potentials) for a given field configuration \( F(x) \) should be proportional to the total measure (in the language of integration theory) of the local fiber:

\[
\Omega_{\text{int}}(x, n) \propto \int_{\mathcal{G}_{\text{int}}(x, n)} dg
\]

Here \( dg \) is the elementary measure of \( \mathcal{G}_{\text{int}}(x, n) \). In the case of \( n = 4 \), we have the U(1) circle group

\[
\Omega_{\text{int}}(x, 4) \propto \int_{U(1)\text{ repres. space}} dl = \text{"perimeter"}
\]

In this case, \( dl \) is the line element of the U(1) representation space. Might this analysis suggest some guiding principle, inspired by a generalization of the second law of thermodynamics, which could put into evidence (from the point of view of electromagnetism) some preferable number of space-time dimensions \( n \)?

We see for \( n=4 \) there is no “entropy” \( S = 0 \), whereas for \( n \geq 4 \) \( S \) is growing since \( \dim(\mathcal{G}_{\text{int}}) \) is growing and, therefore, \( \Omega_{\text{int}}(x, n) \) should be increasing also. Curiously, a local maximum of \( \dim(\mathcal{G}_{\text{int}}) \), for the case where \( n \) is generalized into a continuum variable, appears precisely between \( n=1 \) and \( n=2 \). For \( n > 5 \)

On the other hand, for \( n > 5 \), we see that \( 2n - \frac{n(n-1)}{2} = \frac{n(14-n)}{2} < 0 \), and therefore, we have an insufficient number of equations to solve for the \( \frac{n(n-1)}{2} \) components of \( F \) and \( G \) taken together (taking into account that these are (n-2)-forms related by the constitutive relations). Not all of the \( \frac{n(n-1)}{2} \) components are necessarily independent. They are \( \text{"priori} \), but one has to consider the Maxwell equations and according to this (simplistic) procedure, for \( n > 5 \) Maxwell equations seem incomplete.

Throughout this analysis we assumed:

1 - A space-time manifold with \( n \) dimensions and 1 time-like variable;
2 - The fields \( E, D, H, B \) are all defined on the (local) spatial hyper-surfaces of constant time, with \( (n-1) \) dimensions;
3 - \( F \) and \( G \) are (n-2) forms
3 - A space-time manifold coupled to the electromagnetic field via the local, linear, homogeneous and isotropic constitutive relations.

The analysis can be generalized to other cases where any of these 4 assumptions may be changed. Regarding the space-time constitutive relations, if one considers a Riemann-Cartan manifold for example, in principle, it would be somehow natural to assume a coupling via the metric and the torsion (or alternatively via the tetrads and the connection). This raises some “difficulties at least in Einstein –Cartan theory, since the coupling between the electromagnetic field and torsion breaks the

<table>
<thead>
<tr>
<th>Space-time dimension</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>degrees of freedom of electromagnetic field</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>(internal) gauge symmetries</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>U(1)</td>
<td>5</td>
<td>15</td>
</tr>
</tbody>
</table>
gauge invariance of electromagnetism (a fact that could be useful to explain the anomalous London magnetic moments observed in superconductors!).

- **Gravity**

Let us consider the gravitational field from a Poincare gauge approach. The metric has \( \frac{n(n+1)}{2} \) degrees of freedom and the torsion (or the part of the connection independent from the metric) has \( n \times \frac{n(n-1)}{2} \) degrees of freedom. One gets the following table:

<table>
<thead>
<tr>
<th>Space-time dimension</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>(external) degrees of freedom of</strong> ( g_{\alpha \beta} )</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>15</td>
<td>21</td>
<td>28</td>
</tr>
<tr>
<td><strong>(external) degrees of freedom of</strong> ( \Gamma^\alpha_{[\beta \gamma]} )</td>
<td>0</td>
<td>2</td>
<td>9</td>
<td>24</td>
<td>50</td>
<td>90</td>
<td>147</td>
</tr>
<tr>
<td><strong>Total (external) degrees of freedom</strong></td>
<td>1</td>
<td>5</td>
<td>15</td>
<td>34</td>
<td>65</td>
<td>111</td>
<td>175</td>
</tr>
</tbody>
</table>

The excess of degrees of freedom could also represent gravity propagating through the manifold (or a sub-manifold) with \( n>4 \) which could explain why it is the weakest of all known interactions. These kinds of ideas were in fact at the heart of Kaluza-Klein type theories. On their original idea, the 5-dimentional metric with 15 degrees of freedom could describe the 4d gravity plus electromagnetism (under some conditions impoised on the remaining d.o.f).

On the other hand, in order to geometrize electromagnetism through space-time structures, the geometry of the gauge fibers (the geometry of the U(1) circle group in the case of n=4) must be incorporated into the space-time manifold. As a result, what were previously different gauge potentials (linked by gauge transformations) at the same space-time point x, would no longer be defined at the same point, but rather on different space-time points. It seems therefore inevitable to consider extra space-time dimensions.

A simple formula can be sketched:

\[
Num (extra \ dim) = Num (Symmetries (G_{int})_{elect})
\]

Consider the following table:
The functions describing the dependence of the number of local fiber symmetries with space-time dimension are:

\[
\text{Symmetries (G_{int})_{elect}} = \dim(G_{int})_{elect} = \frac{n(n - 1)(n - 2)(n - 3)}{24} \quad \text{Symmetries (G_{int})_{gravity}} = \frac{n(n - 1)}{2} + n
\]

Each gravitational-electromagnetic “macro state”, defined locally by the metric, the torsion and the electromagnetic field, has a total of \(\Omega_{int \ electromagnatic}(n) \times \Omega_{int \ gravity}(n)\) internal or “microscopic” configurations. Analogously to what was previously done, we may consider the “entropy” of such field configuration

\[
S(n) = S(\Omega_{int \ electromagnatic}(n) \times \Omega_{int \ gravity}(n))
\]

It is interesting to check which number of space-time dimensions is compatible to the following simultaneous requirements

- **Maximize** \(S(\Omega_{int \ electromagnatic}(n) \Omega_{int \ gravity}(n))\)
- **Minimize** \(\left(\Omega_{ext \ electromagnatic}(n) + \Omega_{ext \ gravity}(n)\right)\)

In order to do this one needs to know \(\Omega_{int \ gravity}(n)\), that is, the number of all different combinations of metric and connection that result from the application of all the Poincare transformations. All of these gauge fields configurations correspond to the same curvature and torsion field strengths. An immediate obstacle appears, for example, the tangent space provides the so called vector representation of the Lorentz group, but this fiber, if extended in an unlimited way give raise to a divergent integral

\[
\Omega_{int}(x, n) \propto \int_{\text{over } G_{int}(x,n)} dg
\]
C.9 Cosmology with torsion – some examples

During the last decade or so there has been a wide interest in cosmological applications of gravity with torsion. This is part of an extensive increase in the efforts to solve the major difficulties in relativistic cosmology that make use of alternative/extended theories of gravity beyond GR. All these efforts should be taken into serious consideration and the resulting implications and predictions should encourage and stimulate observational cosmology. In this way, theory and observation can work together and offer the invaluable gift of scientific cosmology, nurturing the cultural sphere with this amazing link between the Cosmos and the human capability to observe and think. The understanding of simple principles that coherently describe the cosmological dynamics is one of the major achievements born out of the marriage between mathematics and physics. Nevertheless, the astonishment that one experiences when is looking at today’s modern images of the realm of the galaxies, or when is simply reflecting about the Universe’s deep mysteries, suggests “equations” and observations still to be found and gets us more and more inspired to continue on this adventure of physical cosmology.

The application of GR to spatially homogeneous and isotropic cosmological fluids, gives raise to Friedman-Lemaître-Robertson-Walker (FLRW) models supporting the Big Bang paradigm. The present $\Lambda$CDM model of cosmology successfully describes the current cosmological observations by the introduction of the hypothetical fluids of dark matter and dark energy. The big-bang cosmology successfully describes the expansion of the universe, primordial nucleosynthesis and predicts the cosmic microwave background radiation. The details of this radiation provide an enormous amount of relevant physical information and consistently validate the current model of cosmology. The large scale structure formation arises in this scenario thanks to the presence of dark matter, whereas the present apparent acceleration in the cosmic expansion is explained trough the repulsive properties of dark energy. Nevertheless, the big-bang cosmology requires an inflationary scenario with additional scalar fields to explain why the present Universe appears spatially flat, and homogeneous (flatness and horizon problems). In fact, this $\Lambda$CDM model does not address some fundamental questions: What is the origin of the rapid expansion of the Universe from an initial, extremely hot and dense state? What is the nature of dark energy and dark matter? What caused the observed asymmetry between matter and antimatter in the Universe?

The validity of GR also breaks inside black holes and at the beginning of the big bang, where the matter is predicted to compress indefinitely to curvature singularities.

To face the challenge of explaining the current cosmological observations there are 4 major possibilities:

1. $GR$ + assumption of spatial homogeneity and isotropy + additional hypothetical fluids
2. $GR$ + change the assumption of spatial homogeneity and isotropy (without additional hypothetical fluids)
3. Change the Gravity theory + assumption of spatial homogeneity and isotropy (without additional hypothetical fluids)
4. Change the Gravity theory + change the assumption of spatial homogeneity and isotropy (without additional hypothetical fluids)

Naturally, there other possibilities resulting from the simultaneous relaxation of homogeneity and isotropy along with the addition of dark (or “unknown”) fluids”, or even the possibility of changing the gravitational theory and also include “dark” fluids. The requirements of simplicity and consistency in scientific theories raise then the following questions: 1) which of these approaches requires less “à priori” assumptions? 2) Are there any theoretical motivations for such assumptions in each approach? From this point of view, the approaches 3 and 4 on the previous list seem more natural, since there are well known motivations to search for extended/alternative gravitational theories and the assumptions used are typically well rooted in theoretical reasons.

Regarding the cosmological applications of gravitational theories with torsion, I present a list with different contributions reflecting the relevance of the issue of space-time torsion to the field of cosmology, as well as the diversity of approaches. I will not develop all these matters here, but I will start by briefly resume one of the contributions that seem very stimulating (not disregarding the others at all)

Space-time torsion as a possible remedy to many cosmological problems

In [36], Nikodem Poplawsky, explains how the solution to the above questions of cosmology and problems may come from the Einstein-Cartan-Sciama-Kibble (ECSK) theory of gravity. I will briefly resume his approach (see [36] and references therein).

We saw that ECKS gravity is based on the gravitational Lagrangian density proportional to the curvature scalar ($\mathcal{L}_{EC} = \frac{\sqrt{-g}}{2k}$), as in GR, with torsion being regarded as a dynamical variable like the metric. Varying the total action for the gravitational field and matter with respect to the metric gives the Einstein equations that relate the curvature to the dynamical
energy-momentum tensor $\Sigma_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta E_{\mu\nu}}{\delta g^{\rho\sigma}}$. Remember that these equations can be written in a GR form as $G_{\mu\nu} = kE_{\mu\nu} = k(\Sigma_{\mu\nu} + S_{\mu\nu})$, where the source is the modified energy-momentum tensor with an additional term $S_{\mu\nu}$ quadratic in the torsion tensor. Varying the total action with respect to the torsion gives the Cartan equations $\tau_{\mu\nu}^\rho = k\left(s_{\mu\nu}^\rho + \frac{1}{2}(\delta_{\mu}^\rho s^\alpha_{\alpha\nu} - \delta_{\nu}^\rho s^\alpha_{\alpha\mu})\right)$, relating the torsion tensor to the spin tensor $s_{\mu\nu}^\rho \equiv \frac{2}{\sqrt{g}} \frac{\delta L}{\delta \dot{\gamma}_{\mu\nu}}$ of matter. As previously mentioned, because they are algebraic, the torsion tensor vanishes outside material bodies where the spin density is zero. Using the Cartan equation relating torsion and spin, one can show that the expression for $S_{\mu\nu}$ as a function of the spin tensor is proportional to $k$. Therefore, the contribution to the curvature from the spin is proportional to $k^2$, so it is significant only at densities of matter much larger than the nuclear density. Such extremely high densities existed in the very early Universe and exist inside black holes. In other physical situations, the ECSK gravity effectively reduces to GR, passing its experimental and observational tests.

Quarks and leptons, which carry the 1/2 intrinsic spin, are the most relevant source of torsion. The torsion tensor appears in the Dirac Lagrangian via the covariant derivative of a spinor with respect to the affine connection, so that the spin tensor for a Dirac spinor $\psi$ is totally antisymmetric and it is given by $s_{\mu\nu} = \bar{\psi}D_{\mu\nu}\psi = \bar{\psi}(\gamma^\mu\gamma^\nu - \frac{1}{4}\gamma^\rho\gamma_{\rho\mu\nu})\psi$. Therefore, because of the Cartan equations, the torsion tensor is quadratic in spinor fields and its substitution into the Dirac equation gives the previously mentioned cubic Hehl-Datta equation for $\psi$.

$$i\hbar \gamma^\mu \nabla_\mu \psi + \frac{3kch^2}{8}(\bar{\psi}\gamma^5\gamma_\mu\gamma_\nu)\gamma^5\gamma^\mu\psi - mc\psi = 0$$

For a spinor with electric charge $q$ in the presence of the electromagnetic potential $A_\mu$, $\nabla_\mu \psi$ is generalized to $\nabla_\mu \psi - \frac{iq}{\hbar} A_\mu \psi$, so that we get

$$i\hbar \gamma^\mu \nabla_\mu \psi + qA_\mu \gamma^\mu \psi + \frac{3kch^2}{8}(\bar{\psi}\gamma^5\gamma_\mu\gamma_\nu)\gamma^5\gamma^\mu\psi - mc\psi = 0$$

In these equations, the term $\frac{3kch^2}{8}(\bar{\psi}\gamma^5\gamma_\mu\gamma_\nu)\gamma^5\gamma^\mu\psi$ corresponds to an effective axial-axial, four-fermion interaction.

$$L_{\text{effect}}^{\text{spin-spin}} = \frac{3kch^2}{16}\sqrt{-g}(\bar{\psi}\gamma^5\gamma_{\mu\nu}\gamma^5\gamma^\mu\psi$$

This is the already mentioned spin-spin interaction. In this effective Lagrangian density only the metric tensor and spinor fields are dynamical variables. The original Lagrangian density for a Dirac field, in which the torsion tensor is also a dynamical variable (leading to the Hehl-Datta equation), is quadratic in spinor fields (and hence renormalizable).

At macroscopic scales, the contributions to torsion from fermions must be averaged. From the Bianchi identities in the EC one obtains the conservation law for the spin density.

$$\partial_\mu s_{\alpha\beta}^\mu - \Gamma_\alpha^\beta \gamma_{\rho\sigma} s_{\alpha\beta}^\rho + \Gamma_\beta^\rho \gamma_{\rho\sigma} s_{\alpha\beta}^\sigma - 2\Xi_{[\alpha\beta]} = 0$$

Applying a method of multipole expansion in the Riemann-Cartan space-time, called Papapetrou-Nomura-Shirafuji-Hayashi method, to this conservation law for the spin density, we get the conditions describing Weyssenhoff spin fluid mentioned previously in C.5

$$s_{\alpha\beta}^\mu = U^\mu s_{\alpha\beta} \quad s_{\alpha\beta} U^\beta = 0 \quad \text{Weyssenhoff spin fluid approximation}$$

where $U^\mu$ is the four-velocity of the macroscopic matter and $s_{\alpha\beta}$ is an antisymmetric tensor describing (intrinsic) angular momentum-like degrees of freedom. One can compute the macroscopic canonical energy-momentum tensor of a spin fluid is

$$E_{\mu\nu} = cP_\mu U_\nu - p(g_{\mu\nu} - U_\mu U_\nu)$$

where $P_\mu$ is the four-momentum density and $p$ is its pressure, and $cP_\mu U^\mu \equiv \epsilon$ is the rest energy density. Even if the spin orientation of particles is random, the macroscopic space-time average of terms that are quadratic in the spin tensor does not vanish. The average square of the spin density is

$$s^2 = \frac{s_{\alpha\beta} s_{\alpha\beta}}{2} = \frac{(nhc)^2}{8} > 0$$

Where $n$ is the fermion number density. In this case, after an averaging procedure, the Einstein-Cartan equations $(G_{\mu\nu} = kE_{\mu\nu})$ give

$$G_{\mu\nu} = k\left((e - \frac{1}{4}k^2)U_\mu U_\nu - (p - \frac{1}{4}k^2)(g_{\mu\nu} - U_\mu U_\nu)\right)$$
We see that Einstein-Cartan equations for a spin fluid are equivalent to the Einstein equations for a perfect fluid with the effective energy density and effective pressure given by
\[ \varepsilon_{\text{eff}} \equiv \varepsilon - \frac{1}{4} ks^2 \quad \text{and} \quad p_{\text{eff}} \equiv p - \frac{1}{4} ks^2. \]

Finally we are in conditions to apply the theory to cosmology. The Friedman equations for a closed Friedman-Lemaître-Robertson-Walker (FLRW) universe filled with a spin-fluid relativistic matter are given by
\[
\begin{align*}
\frac{\dot{a}}{a} + 1 - \frac{1}{3} k \left( \varepsilon - \frac{1}{4} ks^2 \right) a^2 &= 0 \\
\frac{d}{dt} \left( \left( \varepsilon - \frac{1}{4} ks^2 \right) a^3 \right) + \left( p - \frac{1}{4} ks^2 \right) \frac{d}{dt} (a^3) &= 0
\end{align*}
\]
where \( a \) is the cosmic scale factor. As long as \( \varepsilon_{\text{eff}} \equiv \varepsilon - \frac{1}{4} ks^2 < 0 \), the effect of torsion (namely the spin-spin interaction) manifests itself as a force that counters gravitational attraction. The second equation can be seen as a conservation equation implying that \( s^2 \) scales with the scale factor as \( s^2 \sim a^{-6} \), whereas \( \varepsilon \) (in the early, ultra relativistic Universe) scales as \( \sim a^{-4} \). Therefore, when we look at the far past, torsion prevents the collapsing spin-fluid matter from reaching a singularity since the (average) spin density contribution to \( \varepsilon_{\text{eff}} \) is stronger than \( \varepsilon \).

The avoidance of singularities for matter composed of oriented spins has been shown (see [36] and references therein). The Universe has a finite minimum scale factor at which
\[ \varepsilon_{\text{max}} = \frac{1}{4} ks^2 \]

*The singular big bang is replaced by a nonsingular big bounce from a (previously) contracting universe [ref]*.

It can be shown that result agrees with Hawking-Penrose singularity theorems because the contribution from torsion violates energy conditions at high densities [ref]. It is also interesting that loop quantum gravity also predicts a cosmic bounce.

The Friedman equation can be rewritten as the evolution equation for the total density parameter \( \Omega \):
\[ \dot{\Omega}_t = 1 + \frac{\Omega_0^{\text{tot}} - 1}{\Omega_0^{\text{rad}} a^2 - \Omega_0^{\text{spin}} a^4} \]
Where \( \dot{a} \equiv a/a_0 \) (\( a_0 \) present scale factor), \( \Omega_0^{\text{tot}} \) is the present-day total density parameter, \( \Omega_0^{\text{rad}} \) is the present day radiation density parameter, and \( \Omega_0^{\text{spin}} = -ks_0^2/4\varepsilon_c \) is the present-day torsion density parameter (\( s_0^2 \) being the present value of \( s^2 \) and \( \varepsilon_c/e^2 \) being the present critical density).

- **Flatness problem:**

  In GR, \( \Omega_0^{\text{spin}} \) does not appear in the Friedman equation, so \( \Omega_t \) tends to 1 as \( a \to 0 \), introducing the flatness problem in big-bang cosmology because \( \Omega_t \) at the GUT epoch must have been tuned to 1 to a precision of more than 52 decimal places in order for \( \Omega_t \) to be near 1 today.

- **Horizon problem:**

  The horizon problem is related to the above flatness problem. In the ECSK gravity, the function \( \Omega_t(a) \) tends to infinity as \( a \) tends to its minimum value, and has a local minimum where it is equal to
\[ \Omega_{\text{local min}}^{\text{tot}} = 1 - \frac{4\Omega_0^{\text{spin}}(\Omega_0^{\text{tot}} - 1)}{\Omega_0^{\text{rad}}(\Omega_0^{\text{tot}} - 1)} \sim 1 + 10^{-63} \]

As the Universe expands, \( \Omega_t \) rapidly decreases from infinity to the local minimum and then increases according to the standard dynamics. Thus the apparent fine tuning of \( \Omega_t \) in the very early Universe is naturally caused by the extremely small (in magnitude), negative torsion density parameter \( \Omega_0^{\text{spin}} \). This solves the flatness problem without cosmic inflation. To understand the solution to the horizon problem, note that the velocity of the point that is antipodal to the coordinate origin is given by \( \dot{a} = \pi n (\Omega_0^{\text{tot}} - 1)^{-1/2} \). At the point where \( \Omega_t \) has a minimum, such a velocity has a maximum given by
\[ \pi \Omega_0^{\text{rad}} \left( \Omega_0^{\text{spin}}(1 - \Omega_0^{\text{tot}}) \right)^{-1/2} \approx 10^{32}c \]

Therefore, given the value of \( \Omega_0^{\text{spin}} \), one concludes that the Universe, expanding from
its minimum size, forms approximately $10^{32}$ causally disconnected volumes from a single causally connected region of space-time, solving the horizon problem without cosmic inflation.

- **Origin of the (observed) Universe:**

Before the bounce at the minimum size, the Universe was contracting. Also, the matter inside a black hole, instead of compressing to a singularity, rebounds and begins to expand. Consequently, a natural scenario follows according to which every black hole produces a new, closed nonsingular universe “inside”. Extremely strong gravitational fields near the bounce cause an intense pair production, which generates the observed amount of mass and increases the energy density. Nevertheless, such a particle production does not change the total (matter plus gravitational field) energy of the resulting FLRW universe. The equation of state of the matter remains “stiff”, $\varepsilon = p$, until the bounce [ref]. After the bounce, the matter begins to expand as a new universe, with mass

$$M_{\text{Univ}} \approx \frac{m_{\text{neutron}} M_{\text{Black Hole}}^2}{m_{\text{Plank}}^2}$$

Here, $M_{\text{Black Hole}}$ is the mass of the black hole for observers outside it. Such an expansion is not visible for those observers, in whose frame of reference the horizon’s formation and all subsequent processes occur after infinite time. The new universe is thus a separate space-time branch with its own timeline.

Such a universe can grow infinitely large if the cosmological constant $\Lambda$ is present and if its total mass obeys $M_{\text{Univ}} > \frac{c^2}{3\sqrt{\Lambda}}$, otherwise it would expand to a finite maximum size and then contract to the next bounce. If we substitute in this condition the estimated mass of our Universe, which obeys the above inequality, we conclude that

$$M_{\text{Black Hole}}^{\text{primordial}} \sim 10^3 M_{\text{Sun}}$$

$=>$ *The Universe may therefore have originated from the interior of an intermediate-mass black hole.*

- **Unidirectional flow of time:**

Although the ECKS equations are time-symmetric, the boundary conditions of a universe within a black hole are not, because the motion of matter through the event horizon of a black hole is unidirectional. This gives rise to a time-asymmetric collapse of matter through the event horizon and therefore defines the *arrow of time* of such a universe before the subsequent expansion.

- **Nature of dark energy, origin of dark matter and observed matter-antimatter asymmetry:**

Torsion may explain the origin of dark matter, the nature of dark energy and the observed matter-antimatter imbalance in the Universe the via the Hehl-Datta equation.

-- **Cosmological constant**

If fermion fields have a nonzero vacuum expectation value then, the corresponding fermion interaction term $L_{\text{eff}}^{\text{spin-spin}}$ in the action, acts like a cosmological constant, given by

$$\Lambda = \frac{3k c h^2}{16} \langle 0 | (\bar{\psi} i \gamma^\mu \gamma^\nu \psi) (\bar{\psi} i \gamma^\nu \gamma^\mu \psi) | 0 \rangle$$

For condensing quarks, such a torsion-induced cosmological constant is positive and its energy scale is only about 8 times larger than the observed value:

$$\Lambda = \frac{k^2 c h^2}{3} \langle 0 | \bar{\psi} \psi | 0 \rangle^2 \propto \frac{\lambda_{QCD}^6}{m_{\text{Plank}}^2}$$

where $\lambda_{QCD}$ is the QCD scale parameter of the SU(3) gauge coupling constant.

-- **Matter-antimatter asymmetry and nature of dark matter**

The classical Hehl-Datta equation also turns out to be asymmetric under a charge-conjugation transformation. For the charge-conjugate spinor field $\psi^{ch} = -i \gamma^2 \psi^*$, this equation becomes
\[ i\hbar \gamma^\mu \nabla_\mu \psi^{ch} - qA_\mu \gamma^\mu \psi^{ch} = mc\psi^{ch} + \frac{3kch^2}{8} (\bar{\psi}^{ch} \gamma^5 \gamma_\mu \psi^{ch}) \gamma^5 \gamma^\mu \psi^{ch} \]

Therefore, a classical Dirac spinor and its charge-conjugate satisfy different field equations. This leads to a decay symmetry at extremely high densities in the early Universe. This asymmetry may be responsible for a scenario, according to which dark matter is composed of antimatter.

- **Quantum divergences:**

  Quantum field theory based on the Hehl-Datta equation may avoid divergent integrals in calculating radiative corrections. It has been shown in that a nonlinear spinor theory resulting from a cubic equation, may exhibit self-regulation of its ultraviolet behavior, avoiding “ultra-violet” catastrophes. This fact suggests that similar self-regulation should therefore occur for the cubic Hehl-Datta equation. In addition, the multipole expansion applied to Dirac fields in the ECSK gravity shows that such fields cannot form singular, point-like configurations because these configurations would violate the conservation law for the spin density and thus the Bianchi identities. Instead, they describe nonsingular particles whose spatial dimensions are at least on the order of their Cartan radii \( r_c \), defined by a condition \( \epsilon \sim k s^2 \) \[\text{ref}\], where \( \epsilon \sim mc^2 \bar{\psi}\psi \) and the wave function \( \psi \sim r_c^{-3/2} \psi \sim r^{-3/2} \). Accordingly, the de Broglie energy associated with the Cartan radius of a fermion (~ \( 10^{-27} \) m for an electron) should introduce an effective ultraviolet cutoff for such a fermion in quantum field theory in the ECKS space-time.

The avoidance of divergences in radiative corrections in quantum field theory may thus come from the same mechanism that can prevent the formation of singularities formed from matter composed of quarks and leptons:

- space-time torsion coupled to intrinsic spin

Now follows a list with different contributions to the cosmological applications of gravity with torsion.

**a) Cosmological restrictions to Gauge theories of gravity**

1983 - In [37] F. I. Fedorov, V. I. Kudin and A.V. Minkevich show that applications to cosmology within the gauge approach to gravitation where the Lagrangian includes terms quadratic in the curvature and torsion, allows one to obtain restrictions on the indefinite parameters of the Lagrangian.

**b) Space-time torsion and cosmological inflation in Einstein-Cartan theory**

2005 - In [38] Christian G. Böhmer points out that, although there are models that adopt the viewpoint of particle physicists to construct models that incorporate new physics beyond the minimal standard model of particle physics (like the “The new Minimal Standard Model” (NMSM) of H. Davoudiasl, R. Kitano, T. Li and H. Murayama), one can show that a generalization of the geometric structure of space-time can also be used to explain physics beyond the MSM. It is explicitly shown that for example inflation can easily be explained within the framework of Einstein-Cartan theory.

**c) PGTG and cosmic acceleration**

2007 - In [39] A.V. Minkevich, A.S. Garkun and V.I. Kudin, investigate the "dark energy” problem in the framework of the Poincare gauge theory of gravity in 4-dimensional Riemann-Cartan space-time. These authors analyze homogenous isotropic cosmological models with a pseudo-scalar torsion function. It is shown that under certain restrictions on indefinite parameters of the gravitational Lagrangian, the asymptotic behavior of the cosmological equations contain an effective cosmological constant and can lead to the observed cosmic acceleration. The nature of this effect is purely geometrical and related to space-time torsion.

**d) f(R) gravity with torsion and cosmological acceleration**

2008 - In [40], S. Capozziello, R. Cianci, C. Stornaiolo and S. Vignolo study cosmological applications of f(R)-gravity with torsion. In their model torsion is not related to any spin fluid. These authors derive the field equations in vacuum and in the presence of matter represented by a perfect fluid and discuss the related cosmological models. Torsion vanishes in vacuum for almost all arbitrary functions f(R), leading to standard general relativistic behavior. Only for \( f(R) = R^2 \), torsion gives a contribution in the vacuum equations leading to an accelerated behavior. According to the authors their results are clear indications.
that space-time torsion can be an effective source for inflation and, in general, of accelerated expansion. When material sources are considered, the torsion tensor is different from zero even with spinless material sources. This tensor is related to the logarithmic derivative of \( df/d\tau \), which can be expressed also as a nonlinear function of the trace of the matter energy-momentum tensor. The resulting equations for the metric can always be arranged to yield effective Einstein equations. When the homogeneous and isotropic cosmological models are considered, terms originated by torsion can lead to accelerated expansion. It seems that beyond doubt, in f(R) gravity, torsion can be a geometric source for acceleration.

e) Einstein-Cartan theory and cosmic acceleration

2008 - In [41], Christian G. Böhmer and James Burnett claim to have solved one of the open problems in Einstein-Cartan theory, namely to find a natural matter source whose spin angular momentum tensor is compatible with the cosmological principle. They analyze the resulting evolution equations and find that an epoch of accelerated expansion is an attractor. The torsion field quickly decays in that period.

f) PGTG in homogeneous isotropic models

2009 - In [42], A.V. Minkevich analyses the solutions to some problems of general relativity theory in the frame of PGTG. The investigations of homogeneous and isotropic models with two torsion functions seem to suggest that space-time torsion in PGTG can lead to gravitational repulsion effect not only at extreme conditions, but also at very small energy densities. It is also enhanced that torsion can be important in the Newtonian approximation and that Newton’s law of gravitational attraction has limits of its applicability in the case of usual gravitating systems with sufficiently small energy densities. The law of gravitational interaction for such gravitating systems can include corrections corresponding to additional attraction. This means that the investigation of not homogeneous gravitating systems at galactic scales, in particular, of spherically-symmetric systems in the frame of PGTG is of direct physical interest in connection with the problem of dark matter with the purpose of its solution.

g) PGTG and the coupling of gravity to matter and anti-matter

2010 - In [43], Peter Baekler, Friedrich W. Hehl and James M. Nester, propose a cosmological model in the framework of the PGTG. The gravitational Lagrangian is quadratic in curvature and torsion, containing (i) the curvature scalar \( R \) and the curvature pseudo-scalar \( X \) linearly and quadratically (including an \( RX \) term) and (ii) terms quadratic in the torsion (trace) vector \( \tau \) and the torsion axial vector \( a \) (including a \( \tau a \) term). It is shown that generally, in quadratic PGTG models the number of “parity conserving terms” and of “parity violating terms” is nearly the same. The authors claim that this offers new perspectives in cosmology for the coupling of gravity to matter and anti-matter. Their specific model generalizes the fairly realistic “torsion cosmologies” of Shie-Nester-Yo (2008) and Chen et al. (2009). Using a Friedman type ansatz for a tetrad field and a Lorentz connection, they derive the two field equations of PGTG in an explicit form and discuss their general structure in detail.

h) Space-time torsion as dark matter

2011 - In [44] Andre Tilquin and Thomas Schücker confront the Einstein-Cartan's theory with the Hubble diagram and claim that torsion can play the role of dark matter in a way compatible with today's supernovae data. The analysis of the Hubble diagram that these author did shows that, although curvature is not an alternative to dark matter, Einstein-Cartan torsion allows to fit the data within the error bars of the supernova luminosities. This fit is done with a matter contribution of only 5%, corresponding to visible matter only. However the best fit occurs for a 9% contribution from matter implying still a few percent of dark matter. The authors conclude that their result is encouraging enough to reconsider the rotation curves in galaxies and the CMB analysis in the light of the Einstein-Cartan theory.

i) Torsion in Hermitian gravity and cosmological applications

2008 - In [45], Christiaan Mantz and Tomislav Prokopec describe a generalization of GR based on a Hermitian theory of gravity. In this approach, space-time is extended/generalized to space-time-momentum-energy and both the principles of general covariance and equivalence are extended. The theory is endowed with a hermitian metric on a complex manifold. The hermitian metric contains, apart from the symmetric metric, an anti-symmetric part, which describes dynamical torsion. The causality structure is changed in a way such that there is a minimal time for events to be in causal contact and a maximal radius for a non-local instantaneous causally related volume. The speed of light can exceed the conventional speed of light in non-inertial frames and accelerations are bounded. There are indications that the theory of Hermitian gravity yields general relativity at large scales and a theory equivalent to general relativity at very small scales, where the momenta and energies are very large. As an application, the authors study cosmology in hermitian gravity, where matter is described by two scalar fields. While at
late times hermitian gravity reproduces the standard cosmological FLRW models, at early times it differs significantly: quite generically the Universe of hermitian cosmology exhibits a bounce where a maximal expansion rate (Ricci curvature) is attained. Moreover, the authors prove that no cosmological constant is permitted at the classical level within this model.

j) Cosmological bounce with f(T) gravity

2011 - In [46], Yi-Fu Cai, Shih-Hung Chen, James B. Dent, Sourish Dutta and Emmanuel N. Saridakis show that the f(T) gravitational paradigm (analogous to f(R)), in which gravity is described by an arbitrary function of the torsion scalar, can provide a mechanism for realizing bouncing cosmologies, thereby avoiding the Big Bang singularity. After constructing the simplest version of an f(T) matter bounce, these authors investigate the scalar and tensor modes of cosmological perturbations. Their results show that metric perturbations in the scalar sector lead to a background-dependent sound speed, which is a distinguishable feature from Einstein gravity. Additionally, they obtain a scale-invariant primordial power spectrum, which is consistent with cosmological observations, which nevertheless has the drawback of a large tensor-to-scalar ratio.

C.10 Testing space-time torsion

According to the Einstein–Cartan theory, as well as to other generalizations of general relativity, torsion represents additional degrees of freedom of gravity. As a consequence, new physical phenomena associated to its presence are predicted to exist. There has been many contributions for the issue of testing space-time torsion. One interesting proposal by Yi Mao, Max Tegmark, Alan H. Guth, and Serkan Cabi which cached much attention was presented in [24], suggesting that the data from the Gravity Probe B experiment could be used to constrain space-time torsion. I will briefly describe this proposal since it introduced a model independent way to parameterize torsion around spherically symmetric astrophysical objects. The method introduced was also used in [47] to constrain torsion using LAGEOS (laser geodynamical satellites), in a way independent from the GP-B results. I will also briefly present their work (see [24] and references therein)

Constraining Torsion with Gravity Probe B

There is a more or less accepted idea that all torsion gravity theories predict observationally negligible torsion in the solar system, since torsion (if it exists), according to theories like the Einstein-Cartan, couples only to the intrinsic spin of elementary particles, not to rotational angular momentum. Yi Mao, Max Tegmark, Alan H. Guth advertise that this assumption can and should be tested experimentally, and they consider non-standard torsion theories in which torsion can be generated by macroscopic rotating objects. I enhance the fact that this is a very scientific posture.

If a rotating mass like a planet can generate torsion, then the intuitive idea of an action-reaction correspondence suggests that a gyroscope would be expected to feel torsion. Gravity Probe B (GPB) used gyroscopes without intrinsic spin and can test theories where this is the case. It cannot be used to test torsion theories where (intrinsic) spin is the only source of torsion. These authors used symmetry arguments to show that to lowest order, any torsion field around a uniformly rotating spherical mass is determined by seven dimensionless parameters. These parameters effectively generalize the PPN formalism and provide a concrete framework for further testing GR.

They constructed a parameterized Lagrangian that includes both standard torsion-free GR and Hayashi-Shirafuji (new teleparallel gravity) maximal torsion gravity as special cases. They demonstrate that classical solar system tests rule out the latter and constrain two observable parameters. They show that Gravity Probe B is an ideal experiment for further constraining non-standard torsion theories, and work out the most general torsion-induced precession of its gyroscope in terms of the torsion parameters.

Einstein’s General Theory of Relativity (GR) has emerged as the best candidate for a relativistic theory of gravitation, owing both to its elegant structure and to its impressive agreement with many experimental tests since it was first proposed almost 100 years ago. As the authors in [24] say, it remains worthwhile to subject GR to further tests whenever possible, since these can either build further confidence in the theory or uncover new physics. Early efforts in this regard focused on weak-field solar system tests, and efforts to test GR have since been extended to probe stronger gravitational fields involved in binary compact objects, black hole accretion and cosmology. Embedding GR in a broader parameterized class of theories allowing non-vanishing torsion and non-metricity, and experimentally constraining these parameters provide a natural generalization of the highly successful parameterized post-Newtonian (PPN) program for GR testing, which assumes vanishing torsion.

In [24] a special attention was given to Riemann-Cartan Space-time (also known as U4), which, as previously mentioned, retains the metricity condition but has non-vanishing torsion. In U4, torsion can be dynamical and consequently play a role in gravitation alongside the metric. Note that gravitation theories including torsion retain what are often regarded as the principal or most beautiful aspects of General Relativity: general covariance and the idea that “gravity is geometry”. Torsion is just as geometrical an entity as curvature, and torsion theories can be consistent with the Weak Equivalence Principle (WEP).
Experimental searches for torsion have so far been rather limited, in part because most published torsion theories predict a negligible amount of torsion in the solar system. First of all, many torsion Lagrangians imply that torsion is related to its source via an algebraic equation rather than via a differential equation, so that (as opposed to curvature), torsion must vanish in vacuum. Second, even within the subset of torsion theories where torsion propagates and can exist in vacuum, it is usually assumed that it couples only to intrinsic spin, not to rotational angular momentum, and is therefore negligibly small far from extreme objects such as neutron stars. This second assumption also implies that even if torsion were present in the solar system, it would only affect particles with intrinsic spin (e.g. a gyroscope with net magnetic polarization), while having no influence on the precession of a gyroscope without nuclear spin such as a gyroscope in Gravity Probe B. Whether torsion does or does not satisfy these assumptions depends on what the Lagrangian is, which is of course one of the things that should be tested experimentally rather than assumed. Taken at face value, the Hayashi-Shirafuji Lagrangian provides an explicit counterexample to both assumptions, with even a static massive body generating a torsion field — indeed, such a strong one that the gravitational forces are due entirely to torsion, not to curvature.

As an illustrative example, the authors in [24] developed a family of tetrad theories in Riemann-Cartan space which linearly interpolate between GR and the Hayashi-Shirafuji theory. These particular Lagrangians come with some important problems but, nevertheless, they show that one cannot totally dismiss out the possibility that angular momentum can be a source for non-local torsion. These authors reinforce the fact that the proof of the often repeated assertion that a gyroscope without (macroscopic) intrinsic spin cannot feel torsion crucially relies on the assumption that orbital angular momentum cannot be the source of torsion. This proof is therefore not generally valid in the context of non-standard torsion theories. Since, as the authors say, in the spirit of action=reaction, if a (non-rotating or rotating) mass like a planet or a star can generate torsion, then a gyroscope without intrinsic spin could be expected feel torsion, so the question of whether a nonstandard gravitational Lagrangian causes torsion in the solar system is one which can and should be addressed experimentally.

As we saw the Stanford-led gyroscope satellite experiment, Gravity Probe B (GP-B), was launched in April 2004 and has successfully collected the data and presented the results. At the time Yi Mao, et al. made their proposal, only the preliminary GPB results, released in April 2007, were available, confirming the geodetic precession to better than 1%. According to these authors the GP-B containing the four extremely spherical gyroscopes and flying in a circular polar orbit with an altitude of \(~640\) kilometers, gathers the conditions to potentially constrain a broad class of allowed torsion theories. Of particular interest to these authors was the fact that the GP-B team announced they could reach a precision of \(0.005\%\) for the geodetic precession, which, as the authors showed, enables a precision discrimination between GR and a class of torsion theories.

In general, torsion has 24 independent components, each being a function of time and position. Fortunately, symmetry arguments and a perturbative expansion allow to greatly simplify the possible form of any torsion field of Earth - a nearly spherical slowly rotating massive object. It is shown that the most general possibility can be elegantly parameterized by merely seven numerical constants to be constrained experimentally. In [24], these authors derived the effect of torsion on the precession rate of a gyroscope in Earth orbit and worked out how the anomalous precession that GP-B would register depends on these seven parameters.

These authors also worked out the Hayashi-Shirafuji theory (in Weitzenböck space-time) and obtained the torsion-equivalent of the linearized Kerr solution. Interestingly they, generalized the Hayashi-Shirafuji theory to a two-parameter family of gravity theories, which they termed Einstein-Hayashi-Shirafuji (EHS) theories, interpolating between torsion free GR and the Hayashi-Shirafuji maximal torsion theory. They applied the precession rate results to the EHS theories and discussed the observational constraints that GPB, alongside other solar system tests, could be able to place on the parameter space of the family of EHS theories.

Very important is the fact that these authors called attention to the fact that Flanagan and Rosenthal showed that the Einstein-Hayashi-Shirafuji Lagrangian has serious defects. Nevertheless, the possibility for other viable Lagrangians in the same class (where spinning objects generate and feel propagating torsion), is left open. The EHS Lagrangian should therefore not be viewed as a viable physical model, but as a pedagogical toy model giving concrete illustrations of the various effects and constraints that they discussed. The idea is that, if torsion is able to couple to spin, consistency arguments require that it might also be able to couple to rotation — that is, to orbital angular momentum. Now I will briefly resume their work (see [24])

**Approach**

- Generalization of the PPN formalism to Riemann-Cartan space-time;
- The use of adimensional parameters:

  - 7 related to torsion: \(t_1, t_2, w_1, w_2, w_3, w_4, w_5\) \(\tau^a_{\beta\gamma} = \tau^a_{\beta\gamma}(r, \theta, \varphi, t_1, t_2, w_1, w_2, w_3, w_4, w_5)\)
  - 3 related to the usual PPN parameters \(\mathcal{H}, \mathcal{F}, \mathcal{G}\) \(g_{\beta\gamma} = g_{\beta\gamma}(\mathcal{H}, \mathcal{F}, \mathcal{G})\)
- Parameterization of the instantaneous precession rate of a gyroscope around Earth (~ spherical body with slow rotation);
- The analysis of the particular orbit of GP-B.

A general connection has 64 components, but the metricity condition $\nabla_\mu g_{\nu\rho} = 0$ provides 40 constraints and therefore, we are left with 24 independent components which describe torsion $\gamma^a_{\rho\nu}$. How to parameterize these components?

**Method**

- Symmetry arguments;
- Perturbative method.

Using units in which $G = c = 1$, the relevant quantities are expanded with respect to the following adimensional parameters: $\epsilon_m \equiv \frac{m}{r}$ (mass parameter), $\epsilon_a \equiv \frac{a}{r}$ (angular momentum parameter, where $\equiv \frac{J}{m}$ and $J$ is the orbital angular momentum). For the case of interest $\epsilon_m, \epsilon_a \ll 1$, so the perturbative method is done. In fact, for the GP-B orbit we have $\epsilon_m \approx 6.3 \times 10^{-10}$, $\epsilon_a \approx 5.6 \times 10^{-7}$

**Parameterization**

The results they obtained for the parameterization around Earth (taken from []) gives:

### Connection parameterization

\[
\begin{align*}
\Gamma^t_{tr} &= \left( t_1 - \frac{H}{2} \right) \frac{m}{r^2} , \\
\Gamma^t_{rt} &= -
\frac{H m}{2} \frac{1}{r^2} , \\
\Gamma^t_{r\phi} &= \left( 3G + w_1 - w_3 - w_5 \right) \frac{ma}{2r^2} \sin^2 \theta , \\
\Gamma^t_{\phi r} &= \left( 3G - w_1 - w_3 - w_5 \right) \frac{ma}{2r^2} \sin^2 \theta , \\
\Gamma^t_{\phi \theta} &= w_2 \frac{ma}{2r} \sin \theta \cos \theta , \\
\Gamma^t_{\theta \phi} &= -w_2 \frac{ma}{2r} \sin \theta \cos \theta , \\
\Gamma^t_{tt} &= \left( t_1 - \frac{H}{2} \right) \frac{m}{r^2} , \\
\Gamma^t_{rr} &= -\frac{F m}{2} \frac{1}{r^2} , \\
\Gamma^t_{\theta \theta} &= -r + (F + t_2)m , \\
\Gamma^t_{\phi \phi} &= -r \sin^2 \theta + (F + t_2)m \sin^2 \theta , \\
\Gamma^t_{t \phi} &= \left( G - w_1 + w_3 - w_5 \right) \frac{ma}{2r^2} \sin^2 \theta , \\
\Gamma^t_{r \phi} &= \left( G - w_1 - w_3 - w_5 \right) \frac{ma}{2r^2} \sin^2 \theta , \\
\Gamma^t_{\phi \theta} &= \left( -2G - w_2 + 2w_4 \right) \frac{ma}{2r^2} \sin \theta \cos \theta , \\
\Gamma^t_{\theta \phi} &= \left( -2G - w_2 \right) \frac{ma}{2r^2} \sin \theta \cos \theta , \\
\Gamma^t_{\theta \rho} &= \Gamma^\rho_{\phi \phi} = \frac{1}{r} , \\
\Gamma^t_{\phi \rho} &= \Gamma^\rho_{t \phi} = \frac{1}{r} , \\
\Gamma^t_{\rho \rho} &= \Gamma^\rho_{t t} = \frac{1}{r} .
\end{align*}
\]

### Connection parameterization (cont)

\[
\begin{align*}
\Gamma^\phi_{\phi \phi} &= -\sin \theta \cos \theta , \\
\Gamma^\phi_{t r} &= \left( -G + w_1 - w_3 + w_5 \right) \frac{ma}{2r^2} , \\
\Gamma^\phi_{r t} &= \left( -G + w_1 - w_3 - w_5 \right) \frac{ma}{2r^2} , \\
\Gamma^\phi_{r \theta} &= \left( 2G + w_2 - 2w_4 \right) \frac{ma}{2r^2} \cot \theta , \\
\Gamma^\phi_{\theta t} &= \left( 2G + w_2 \right) \frac{ma}{2r^2} \cot \theta , \\
\Gamma^\phi_{\phi \theta} &= \Gamma^\phi_{\theta \phi} = \cot \theta .
\end{align*}
\]

### Torsion ($\tau^a_{\rho \nu} \equiv S^a_{\rho \nu} \gamma^\rho \nu$) parameterization

\[
\begin{align*}
S^t_{tr} &= t_1 \frac{m}{2r^2} , \\
S^t_{t \phi} &= S^\phi_{t \phi} = t_2 \frac{m}{2r^2} , \\
S^t_{r \phi} &= w_1 \frac{ma}{2r^2} \sin^2 \theta , \\
S^t_{t \theta} &= w_2 \frac{ma}{2r^2} \sin \theta \cos \theta , \\
S^t_{r \theta} &= \frac{w_1}{2r^2} \sin^2 \theta , \\
S^t_{\phi \theta} &= \frac{w_1}{2r^3} \sin \theta \cos \theta , \\
S^t_{\rho \phi} &= \frac{w_2 m}{2r^2} , \\
S^t_{\rho \theta} &= -w_4 \frac{ma}{2r^2} \cot \theta .
\end{align*}
\]
For the metric we have

\[ ds^2 = -\left[1 + \mathcal{H} \frac{m}{r}\right] dt^2 + \left[1 + \mathcal{F} \frac{m}{r}\right] dr^2 + r^2 d\Omega^2 + 2 \frac{m a}{r} \sin^2 \theta dt d\phi, \]

**Parameterization of the gyroscope instantaneous precession rate**

The relevant information to compute the precession rate of gyroscopes is the following:

\[
\frac{DS^\theta}{d\tau} \equiv \frac{dx^\mu}{d\tau} \nabla_\mu S^\theta \quad \text{"spin" (not intrinsic) 4-vector}
\]

*The line universe of the gyro is a geodesic*

\[
\begin{align*}
\text{extremal} & \quad \text{auto-parallel} \\
\end{align*}
\]

The expressions they obtained for the frame dragging (\(\Omega_F\)) and geodetic (\(\Omega_G\)) instantaneous precessions rates (given on the gyroscopes coordinates reference system) are the following:

- **Parameterization for auto-parallel geodesics**

\[
\frac{d\vec{S}_0}{dt} = \vec{S}_0 \times \vec{S}_0 \quad \vec{\Omega} = \vec{\Omega}_F + \vec{\Omega}_G
\]

\[
\vec{\Omega}_F = \frac{Gl}{r^3} \left( -\frac{3}{2} (1 + \mu_1)(w_E \cdot \vec{e}_r) \vec{e}_r + \frac{1}{2} (1 + \mu_2) w_E \right) \quad \vec{\Omega}_G = \frac{m}{2r^3} (1 + \mathcal{F} + 2t_2)(\hat{\vec{r}} \times \vec{v})
\]

Where \(\mu_1 \equiv -(w_1 - w_2 - w_3 + 2w_4 + w_5)/(3\mathcal{G})\) and \(\mu_2 \equiv -(w_1 - w_3 + w_5)/\mathcal{G}\). In these expressions \(I\) is the moment of inertia… is …, \(w_E\) is the orbit west-east angular momentum (see the fig. below)….and \(lw_E = ma\) (in \(G = c = 1\) units).

- **Parameterization for extremal geodesics**

\[
\frac{d\vec{S}_0}{dt} = (\vec{S}_0 \times \vec{S}_0) - t_1 \frac{m}{r^3} (\vec{S}_0 \cdot \vec{v}) \hat{\vec{r}} \quad \vec{\Omega} = \vec{\Omega}_F + \vec{\Omega}_G
\]

\[
\vec{\Omega}_F = \frac{G\ell}{r^3} \left( -\frac{3}{2} (1 + \mu_1)(w_E \cdot \vec{e}_r) \vec{e}_r + \frac{1}{2} (1 + \mu_2) w_E \right) \quad \vec{\Omega}_G = \frac{m}{2r^3} (\frac{\mathcal{F}}{2} - \frac{\mathcal{H}}{4} t_2)(\hat{\vec{r}} \times \vec{v})
\]

*Fig – GP-B Polar orbit (taken from [24])*
The analysis of the polar orbit allows the Fourier expansion \( \frac{d\mathbf{s}_n}{dt}(\varphi) = \tilde{a}_0 + 2\sum_{n=1}^{\infty} (\tilde{a}_n \cos n\varphi + \tilde{b}_n \sin n\varphi) \), where \( \tilde{a}_0 \) gives the following average \( \tilde{a}_0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\mathbf{s}_n}{dt}(\varphi)d\varphi = \frac{dS_n}{dt}(\varphi) \). As a result, the average precession describing the frame-dragging and geodetic effects is given by

\[
\langle \frac{dS_n}{dt}(\varphi) \rangle = \Omega_{\text{eff}} \times \mathbf{S}_0 \quad \tilde{\Omega}_{\text{eff}} = b_t \frac{3m}{2r_o^2} \mathbf{w}_o + b_\mu \frac{l}{2r_o^3} \mathbf{w}_E
\]

Where the parameters \( b_t \) and \( b_\mu \), given by

\[
b_t \equiv \frac{1}{3}(1 + F + 2t_2 + |\eta|t_1) \quad b_\mu \equiv -\frac{G}{2} \left(1 + 3\mu_1 - \mu_2\right) = -\frac{G}{2} \frac{(w_1 + w_2 - w_3 - 2w_4 + w_5)}{G}
\]

Are equal to 1 in GR. In the parameter \( \eta \) depends on the geodesic used

\[
\eta = \begin{cases} 0 & \text{autoparallel geodesics} \\ -1 & \text{extremal geodesics} \end{cases}
\]

The parameters \( b_t \) and \( b_\mu \) describe the deviations from GR, including possible effects due to torsion. As previously mentioned the parameters \( \mathcal{HFG} \) are related to the more familiar PPN parameters \( \alpha_4 \) and \( \gamma \). In fact

\[
\mathcal{H} = -2 \quad F = 2\gamma \quad G = -\left(1 + \gamma + \frac{1}{4}\alpha_4\right)
\]

**Note**

In [], the authors assumed that there is no coupling between torsion and electromagnetic fields, since as was previously mentioned, the coupling procedure \( \partial_a \mathbf{v} \rightarrow \nabla_a \) breaks the gauge invariance of electromagnetism, giving rise to a quadratic term in \( A_a \) (mass term) and to anomalous magnetic forces. Therefore, assuming that light is unaffected by torsion, all solar system gravitational tests that use light (Shapiro effect, light deflection, gravitational redshift), can only constrain the metric via the PPN formalism.

Using the above definitions, we can rewrite the relevant parameters describing the deviations from GR related to the gravitational influence of a fairly spherical body (such as Earth) on orbiting gyroscopes:

**Two independent combinations of torsion and PPN parameters**

\[
b_t \equiv \frac{1}{3}(1 + 2\gamma + \frac{1}{3}(2t_2 + |\eta|t_1)) \quad b_\mu \equiv \frac{1}{2} \left(1 + \gamma + \frac{1}{4}\alpha_4\right) - \frac{1}{2}(w_1 + w_2 - w_3 - 2w_4 + w_5)
\]

**Constraints on PPN and torsion parameters from Solar system tests, including the GP-B experiment**

Using this procedure, the authors in [24] presented a table and diagram (shown below) characterizing some solar system constraints on the PPN parameters as well as the constraints on two independent combinations of the torsion parameters assuming a result from GP-B that is in agreement with GR with a precision of \( 0.5 \times 10^{-3}\text{arc sec} \).
Constraining Torsion with LAGEOS

In [47], Riccardo March, Roberto Tauraso, Giovanni Bellettini and Simone Dell'Agnello Analysed the motion of a test body in orbit in the gravitational field of a axially symmetric rotating body, using the parameterization method of Yi Mao, Max Tegmark, Alan H. Guth and Serkan Cabi. They computed the corrections to the orbital Lens-Thirring effect in the presence of space-time torsion. Analyzing the secular variations of the longitudinal coordinate of the node and pericenter of the orbit, they conclude that experiences such as LAGEOS may be used to constrain the space-time torsion parameters. This constrain is complementary to the one resulting from the analysis of the GP-B data since it uses a different (independent) combination of the torsion parameters.

The LAGEOS experiment uses the SLR (Satellite Laser Ranging) technique and the authors in [24] used the data from the ILRS (International Laser Ranging Service) available on the NASA website.

The calculations they made results on the following constraint:

\[ 0.22 < 0.1w_1 - 0.20w_2 - 0.06w_3 + 0.20w_4 + 0.06w_5 < 0.42 \]

Diagram – Constraints on combinations of torsion parameters as a function of PPN parameters from LAGEOS experiment. GR is represented by a dot [47].
Now follows a list with other contributions related to the issue of testing experimentally space-time torsion.

a) Constraining torsion parameters using the Moon (LLR) and Mercury (planetary radar ranging)

2011 - In [48], Riccardo March, Roberto Tauraso, Giovanni Bellettini and Simone Dell'Agnello used again the parameterized torsion framework of Mao, Tegmark, Guth and Cabi and we analyzed the motion of test bodies in the presence of torsion. They compute the corrections to the perihelion advance and to the orbital geodetic precession of a satellite. They considered the motion of a test body in a spherically symmetric field, and the motion of a satellite in the gravitational field of the Sun and the Earth. Describing torsion by means of three parameters and making use of the autoparallel trajectories, which, as previously mentioned, in general are different from geodesics, when torsion is present. The computed secular variations show how the corrections to the perihelion advance and to the orbital de Sitter effect depend on the torsion parameters. All computations are performed under the assumptions of weak field and slow motion. To test their predictions, they used the measurements of the Moon's geodetic precession from lunar laser ranging data and the measurements of Mercury's perihelion advance from planetary radar ranging data. These measurements were then used to constrain suitable linear combinations of the torsion parameters.

b) Dirac equation in Riemann-Cartan space-time : effects on atomic energy levels

1997 - In [49], Claus Lämmerzahl explains in detail how the coupling of space–time torsion to the Dirac equation leads to effects on the energy levels of atoms which can be tested by the so called Hughes–Drever type experiments. Reanalysis of these experiments carried out for testing the anisotropy of mass and anomalous spin couplings can lead to a constraint on the axial torsion by \( a \leq 1.5 \times 10^{-15} \text{m}^{-1} \)

c) Testing torsion from Lorentz symmetry violation

2008 - In [50] V. Alan Kostelecký, Neil Russell, and Jay D. Tasson shows that it is possible to obtain exceptional sensitivity to space-time torsion by investigating its coupling fermions. He assumes that the physical gravitational field is described by a theory predicting a nonzero torsion field in the vicinity of the Earth, and he seeks model-independent constraints. This author explores “recent” experimental searches for the Lorentz violation that are able to provide constraints involving 19 of the 24 independent torsion components with a sensitivity down to levels of order \( 10^{-31} \text{GeV} \).

d) Detecting torsion waves using LISA according to teleparallel gravity

2002 - In [51] L.C. Garcia de Andrade discusses the issue of testing torsion from torsion waves in the context of teleparallel gravity. A “thought” (gedanken) experiment to detect Cartan’s contortion based on a circle of particles not necessarily spinning is proposed. It is shown that by making use of previous value of contortion at the surface of the Earth computed by Nitsch, the author shows that LISA GW detector could be used for an indirect detection of torsion (in teleparallel gravity).

e) Corrections to the Gravitomagnetic effect due to space-time torsion

2003 - In [Gravitomagnetic effect and spin-torsion coupling], A. A. Sousa and J. W. Maluf study the gravitomagnetic effect in the context of Weitzenböck space-time with the use of a modified geodesic equation via a free parameter. We calculate the time difference in two atomic clocks orbiting the Earth in opposite directions and find a small correction due the coupling between the torsion of space time and the internal structure of atomic clocks measured by the free parameter.

f) Testing Lorentz symmetry and torsion

2008 - In [52], Neil Russell, explores similarities between the structures of theories with Lorentz violation and theories with constant torsion in flat space-time in order to establish bounds on torsion components. For example, it is shown that analysis based on a dual-maser experiment can place bounds on the axial-vector and mixed-symmetry components of torsion.

g) The Pound-Rebka experiment and torsion in the Schwarzschild space-time

2009 - In [53], J. W. Maluf, S. C. Ulhoa and F. F. Faria develop some ideas discussed by E. Schucking [Gravitation is torsion] concerning the geometry of the gravitational field. First, they address the concept according to which the gravitational acceleration is a manifestation of the space-time torsion, not of the curvature tensor. It is possible to show that there are situations in which the geodesic acceleration of a particle may acquire arbitrary values, whereas the curvature tensor approaches zero. From their analysis, these authors conclude that the space-time curvature does not affect the geodesic acceleration. They
considered the Pound-Rebka experiment, which relates the time interval of two light signals, in a Schwarzschild type gravitational field. The experiment is determined by four space-time events and the infinitesimal vectors formed by these events could, in the absence of torsion, form a closed parallelogram. The failure in the closure of the parallelogram implies that the space-time has torsion. The authors found the explicit form of the torsion tensor that explains the non-closure of the parallelogram.

h) Probing non-Riemannian space-time geometry

2009 - In [54] Dirk Puetzfeld and Yuri N. Obukhov explore the equations of motion for matter in non-Riemannian space-times derived via a multipole method. According to these authors only test bodies with microstructure couple to the non-Riemannian space-time geometry. They claim that it is impossible to detect space-time torsion with the satellite experiment Gravity Probe B, contrary to the ideas in [5]. Regarding this controversy I quote Riccardo March et al. in [54]

“Puetzfeld and Obukhov derive the equations of motion in the framework of metric gravity theories, which includes the HS theory, and show that only test bodies with microstructure (such as spin) can couple to torsion. In conclusion the EHS theory does not yield a torsion signal detectable for GPB. For these reasons, (in y. Mao et al) the EHS Lagrangian is proposed not as a viable physical model, but as a pedagogical toy model fitting in the parameterized framework, and giving an illustration of the constraints that can be imposed on torsion by the GPB experiment. (...). As also remarked by Flanagan and Rosenthal, the failure of constructing the specific EHS Lagrangian does not rule out the possibility that there may exist other torsion theories which could be usefully constrained by solar system experiments. Such torsion models should fit in the above mentioned theory-independent framework, similarly to a parameterized post-Newtonian framework including torsion.”

i) Solar system constraints to torsion in f(T) gravity

In [55] Lorenzo Iorio and Emmanuel N. Saridakis use recent observations from solar system orbital motions in order to constrain f(T) gravity. These authors impose a quadratic f(T) and extract the spherical solutions of the theory. Using them to describe the Sun’s gravitational field, they were able to infer upper bounds on the allowed f(T) corrections to the linear dependence on T. They found that the maximal allowed deviation from the teleparallel equivalent of General Relativity is of the order of $6.2 \times 10^{-10}$, within the applicability region of their analysis.
D - Open questions and final remarks. Physics and geometry

The aim of the present dissertation was to explore the relations between electromagnetism, gravity and space-time, using geometrical methods. From this perspective, the parts A, B and C are interrelated and there are some interpretations that are useful to elucidate the underlying transversal ideas. As will become clear, one of the principal common ideas is the hypothesis of space-time physicalism (space-time as a dynamical physical entity). Part A addressed the geometrical properties of the electromagnetic theory (and the coupling between electromagnetism and space-time). In part B, the main ideas were derived from the coupling between gravity and the mass-energy according to the Einstein’s space-time paradigm (GR). As a linear approximation, one obtains the gravitoelectromagnetic formalism almost in complete analogy to Maxwell’s theory. Within this formalism, the coupling between gravity and electromagnetic fields were explored for the specific case of plane electromagnetic waves as source of gravitational waves. In part C the torsion of space-time was explored within gravitational theories, comparing some torsion theories, addressing analogies with electromagnetism, cosmological applications and experimental tests to space-time torsion. Throughout this adventure some personal interpretations were proposed and this part aims at clarifying these interpretations and their relations as well as enhancing many open questions that naturally arise.

Electromagnetism and gravity are intimately connected with space-time. The geometrical methods explored in part A revealed, not only, the appropriate mathematical objects and geometrical concepts associated to the fields (E,B), the excitations (D,H), the “sources” (ρ,J) and their relations, but also the inevitable link between space-time geometry and electromagnetic fields (EM). In fact, the axiomatic structure based on charge conservation, magnetic flux conservation and Lorentz force, is only complete when one considers the space-time constitutive relations. This relation between the fields (E,B) and the excitations (D,H) is made through the use of the geometrical structure of space-time. One needs to postulate the form of this relation and a fairly natural procedure is to assume local, linear, homogeneous and isotropic relations. The usual interpretation of the words “homogeneous” and “isotropic” in this context (when we are considering electromagnetic fields in regions detached of matter), is that they refer to the electromagnetic properties of vacuum. In fact, this relations can be regarded as “constitutive relations” of the space-time structure itself, suggesting that electromagnetic fields couple to the space-time geometry in such a way that they can determine geometrical properties of the space-time structure, namely, its causal or conformal structure. Since this (local) causal structure (compatible with the equivalence principle) is determined by the light cone, it seems natural that one finds a derivation (mentioned in A.2.1) of this conformal part of the metric, from the properties of electromagnetic propagation (via the constitutive relations).

These ideas come from electromagnetic theory only but are fundamentally in agreement with the Einstein’s gravitational paradigm, since the electromagnetic energy-momentum is a source of gravity determining the local geometry of space-time. Gravity, in its turn, determines the geometry of the geodesics that light follows. In this sense, one can say that the electromagnetic fields non-linearly interact with themselves via the coupled gravitational and Maxwell equations. The electromagnetic theory invites gravity and the gravitation theory invites electromagnetism.

At the very heart of this relation between electromagnetism and gravity, lies the space-time structure. Both fields can exist and propagate within regions without matter. Nevertheless, although gravity can be seen as a direct manifestation of the geometry, up to now (in spite of the known historical efforts), there is no generally accepted geometrization of electromagnetism based on the space-time manifold. Several approaches to unify gravity and electromagnetism usually try to incorporate the EM degrees of freedom within the space-time structures.

The axiomatic structure of EM field theory, described in part A, revealed a geometrization of EM in the sense that one obtains, through integration theory and the calculus of differential forms, the geometrical relations between the relevant physical quantities (for example, (E,H) are connected to lines, whereas (B,D) are associated with areas). Although this is different than a representation of EM solely in terms of space-time structures, the (space-time) constitutive relations obtained helps to clarify the conceptual and mathematical challenges inherent to a unification of gravity and electromagnetism, through geometry.

Physics progresses via theoretical and experimental investigations, but it is also driven from the interpretations of the physical theories (these theories can only be claimed “physical” when some “coherent” interpretation takes place). In a very deep sense the same questions that pursued Faraday and many others, such as, What are the fields? What is their nature?, are still very actual. It is a fact that we can only experience the fields through their effects on (other) physical systems – we can never “see” light itself. Yet, modern physics is fundamentally structured on the concept of field and a physical ontology is assumed, meaning that it is assumed that the fields are physical entities with measurable physical properties, such as energy (as is well known, in gauge theories of the fundamental interactions, the gauge fields are regarded as non-physical quantities, whereas the “field strengths” are associated to measurable phenomena – I will return to this point later). Regarding EM fields,
we don’t know exactly “what they are”, but we have a huge amount of experimentally validated ideas about “how” they “relate”, or “interact”, with charges, between themselves, with the space-time structure, etc…

One possible conclusion taken from the constitutive relations is simply that the fields \((E, B)\) and charges relate to each other through space-time, rather than in space-time. Why do we need at all the excitations \((D, H)\)? What do they represent? These appear within the non-homogeneous equations (derived from charge conservation alone), being therefore related to the charges and currents. Therefore, the only way we can relate the fields \((E, B)\) with charges and currents, is via the constitutive relations, i.e., through the geometrical structure of space-time. The concept of “vacuum” in this context can be seen as unnecessary. One can say that the a physical ontology of fields is connected to the (physical) ontology of charges, via the (physical) ontology of space-time.

\[
\begin{align*}
\text{(Physical) Ontology of field} & \quad \text{(Physical) Ontology of space-time} & \quad \text{(Physical) Ontology of charge} \\
(E, B) & \quad & (\rho, J)
\end{align*}
\]

On the one hand, \((D, H)\) express the fact that \((\rho, J)\) are “linked” to space-time, as if these “material” objects were in some sense extended (via \(D\) and \(H\)) throughout space-time. On the other hand, the relations 
\[
D = * (\epsilon_0(E)) \quad B = * (\mu_0(H))
\]
express the fact that \((E, B)\) are linked to (the geometry of) space-time.

\[
\begin{align*}
D &= * (\epsilon_0(E)) \\
B &= * (\mu_0(H)) \\
dD &= \rho \\
dH &= J + \partial_t D
\end{align*}
\]

From dimensional analysis, it is exactly when we ought to relate \((E, B)\) with \((D, H)\), through space-time, that the quantities \(\epsilon_{ij}\) \(\mu_{ij}\) appear – There is no need for “vacuum”. Since this interpretation doesn’t require the vacuum, it is natural to expect that \(\epsilon_{ij}\) \(\mu_{ij}\) should characterize the geometrical properties of space-time such as its symmetries.

Each of the above expressions: “(physical) ontology of fields”, “(physical) ontology of space-time”, “(physical) ontology of charge”, represents a huge challenge for physics and the philosophy of physics. Nevertheless we see that the mathematical structure of EM provides an extremely rich background to develop fundamental physical ideas. This development was partially driven by geometrical methods.

If we ought to pursue a physical unification of these three “ontological realms”, several basic ideas naturally arise. First of all, a physical ontology for fields requires the physical ontology of energy and the (physical) ontology of charge is directly related to the physical ontology of matter. On its turn, a “physicalism” for matter is implicitly connected to mass which is also connected to energy. This said, one possible route to unification (the one inspired by space-time physicalism) consists in the recognition that a physical ontology of space-time has to include or incorporate the (physical) ontology of energy (and vice-versa).

\[
\begin{align*}
\text{(Physical) Ontology of space-time} & \quad \text{Mutual inclusion} & \quad \text{Physical ontology of mass-energy}
\end{align*}
\]

Now, interestingly, the Einsteinian paradigm of gravitation establishes precisely a bi-directional link between mass-energy and space-time, since mass-energy sources gravitation and the gravitational field contains energy, that is, the space-time structure contains energy in its geometrical dynamics.
From this logical reasoning, one is driven to conclude that EM and gravity should be seen as different perspectives of the same unified physical ontology.

Nevertheless, charge and gravitational mass are different. Contrary to charge, mass can be associated or identified with inertia. In spite of this difference, both in EM and gravity there is the common problem of how to physically relate charge (or mass) with the interaction field. Previously, I claimed that in EM this link is via the space-time structure. But how does a charge leave an imprint on the physical object of space-time? The same open question is raised in GR: mass affects space-time geometry, and this is expressed via the field equations, but how does this influence take place? The field equations merely establish a causal link between $T_{\mu\nu}$ and geometry ($G_{\mu\nu}$). If these are both representing physical entities (space-time and mass energy), how do they interact? Is there a frontier between both? There are many visual illustrations to represent this interaction, such as the curvature of water below the water bugs or the curvature of elastic nets due to the weight of rolling balls. All of these require a physical interaction between two material systems and therefore, although one can recognize their frontiers, both systems have the same underlining physical ontology – matter.

This said, space-time and energy-mass should be seen as concepts intrinsically inherent to the same physical entity. The unification of matter-energy and space-time may be theoretically investigated with recourse to geometrical methods and one of the underlying unifying concepts can be precisely the concept of a manifold. A unified physicalism of space-time and mass-energy (or matter) seems to suggest a microstructure in space-time and therefore, such procedure, might inevitably include a quantum theory of gravitation. On the other hand, the issue of mass is linked to the Higgs field concept which can be a preliminary candidate for the fundamental structure out of which mass particles arise, but it can also provide a quantum “micro structure” of space-time. An important idea is that the Higgs field itself might be represented by a manifold. Just like the inertial mass of particles is seen as the propagation of “dense configurations of this fundamental field”, gravitational mass can be also the expression of some sort of “agglomeration” or “densification” of the fundamental field Higgs structure around a certain field region. This could distort the geometry of the “Higgs manifold”. If this is so, then it might be the case that geometrical structures like curvature and torsion are essential to represent all the fundamental interactions.

**Physicalism of space-time and physicalism of “internal spaces”**

As a natural consequence of this line of thought, it seems plausible to raise the following question: a unified physicalism of space-time-energy-mass, through geometrical concepts, in which a unifies the physical interactions takes place, presupposes a physicalism of the “internal” spaces related to the “internal” gauge symmetries?

These “inner spaces” do not need necessary to be incorporated in an extra-dimensional space-time, they might be seen as other physical manifolds not related to space or time. Nevertheless, geometrical methods are applicable and it seems to me that one should not disregard the possibility of assuming that the internal degrees of freedom (the number of all possible “gauge configurations” that result in the same field strength) represent physical degrees of freedom, on the contrary of what is commonly said in literature.

It is interesting to note that according to this interpretation, the gauge symmetry breaking of electromagnetism that results from the coupling prescription $\partial_{\mu} \rightarrow \nabla_{\mu}$ in Riemann-Cartan space-time, should not be necessarily regarded as a drawback or something that should not happen. One could interpret this gauge breaking simply by saying that the interaction with the space-time torsion could cause a change according to which what were previously many different microstates compatible to a given field configuration (“macro state”), would simply correspond to different field configurations. If the gauge symmetry reflects the existence of internal physical d.o.f. (“microscopic” d.o.f.), it is reasonable to assume that there might be processes that
result in a change on the number of “internal states” or even on the physical characteristics of the set of all microstates at each space-time point.

The interpretation in part A suggests that \((\varepsilon, \mu)\) should characterize the symmetry properties of space-time with respect to the propagation of electromagnetic waves. As mentioned in A.2.2, the background independence of the Einsteinian paradigm – the idea that the space-time manifold is not given a priori, independently from the matter energy -, suggests that one should consider the symmetries implicit in \((\varepsilon, \mu)\) as being dependent on each physical situation in particular. One has to consider first the space-time structure that is solution to the gravitational field equations.

The velocity of EM waves is determined by this information in \((\varepsilon, \mu)\) and the most general case doesn’t necessarily correspond to an isotropic and homogeneous situation. As already mentioned, the propagation of these waves depends on the geometrical symmetries of space-time which have to be computed from the field equations in each given particular situation. According to this interpretation, at different space-time points one can have a local equivalence to a Minkowsky-like manifold with different (local) light cone structures. Non-homogeneity may imply different values for the speed of light in vacuum \(c\) at different positions and non-isotropy may imply local symmetries that doesn’t include the SO (3). As claimed in A.2.2, the time symmetry in the propagation of EM fields might be also expressed in terms of the properties of the extended tensors \(\varepsilon_{\alpha\beta} \mu_{\alpha\beta}\) \((\alpha, \beta = 0,1,2,3)\), assuming that \(\varepsilon_{00} \mu_{00}\) is a decreasing function of time.

In part B, using GR and, in particular the relation \(R = kT\), a simple expression shows that accompanying any EM wave there is a positive curvature wave propagation \(R = R(E, B)\). Gravitational waves and EM waves are intimately connected. The analogies between gravity and EM were explicitly explored using gravitoelectromagnetism. The equations show that linear gravitational phenomena (weak fields) are expressed through Maxwell-like equations and this formalism helps to visualize the gravitational field, via gravitoelectric and gravitomagnetic field lines. The section describing the GP-B experiment serves to enhance the detection of gravitoelectromagnetism (in according to GR), illustrating the physical effects of the strong analogies between gravity and EM. As explained in C.8, this experiment can also constrain some space-time torsion theories.

In B.3.2, the coupling between gravity an EM using gravitoelectromagnetism showed, once again, that EM waves can generate gravitational waves, and some solutions regarding plane EM waves as sources, were explicitly obtained. Connected to the ideas developed in part A.2.2 regarding the coupling between space-time and EM fields, I suggested that not only EM waves sources gravitational waves, but also gravitational waves are a source of EM waves (something that remains completely hypothetical).

In part C, the issue of torsion in gravitational theories was explored. Most questions remain open until further significant progress comes out from experiment. It is clear that torsion might be crucial to include spin in gravitational theories and that it may be fundamental to unifying theories and quantum gravity. Regarding the source of torsion and its propagation or non-propagation feature, no conclusions are offered until experimental physics undoubtedly detects torsion waves or other effects of torsion caused by material systems without macroscopic spin. Nevertheless, as enhanced, although it is usually assumed that EM and torsion do not couple (for theories within \(U_\lambda\)), because this breaks the EM gauge invariance, the fact that EM carry spin may provide a mechanism for the propagation of torsion, even in the Einstein-Cartan theory.

When considering the question of the role of torsion in gravitation, I introduced the proposal in which, torsion and the Christoffel curvature, being part of the full curvature, may be regarded as being fundamentally interconvertible. This idea that curvature and torsion can be mutually creating each other is partially invoked in Cartan’s structure equations. We saw also, in this context a great analogy between EM and gravity with torsion. In fact, by noticing the analogies between EM and gravity, using differential forms (assuming curvature and torsion) (C.4, C.5), and given the fact that teleparallelism has an equivalence to GR, this hypothesis is strengthen: torsion can generate curvature and curvature can generate torsion. What is the meaning behind the equivalence between teleparallel gravity and GR? These theories assume different manifolds. This equivalence, although being in itself a formal equivalence, as stated in C.3, it might suggest an interconversion between torsion and (part of) the curvature.

In some sense, this resembles the relativity of EM fields in Maxwell’s theory, where electric and magnetic fields are seen as different aspects of the electromagnetic field.

Regarding its cosmological application, the examples given in C.9 sufficiently illustrate that torsion fields might solve some of the fundamental problems in cosmology and therefore its testing is extremely relevant. For the reasons expressed in the
beginning of C.9, I consider this approach preferable than the assumption of extra unknown fields in $\Lambda$CDM cosmology, inspite of the fact that the validity of such ideas is based on the astrophysical and cosmological observations. One should be open to many other forms of matter than those that the physics community has knowledge of. Nevertheless, the idea that there is a space-time physicalism affecting and being affected by the cosmological fluids is sufficiently rich enough, so that it is interesting and convenient to fully explore this link.
References


[34] http://en.wikipedia.org/wiki/Einstein%E2%80%93Cartan_Theory


Bibliography


The 7th International Conference on Gravitation and Cosmology, International Center for Theoretical Sciences, Tata Institute of Fundamental Research, Indian Association for General Relativity and Gravitation, Goa, 2011


Richard S. Palais, “The Geometrization of Physics. Lecture Notes in Mathematics”, Institute of mathematics of the National Tsing Hua University, Taiwan, 1981


Yishi Duan and Ying Jiang, “Can torsion play a role in angular momentum conservation law?”, arXiv:gr-qc/9809009v2, 24 Nov 1998


Friedrich W. Hehl and Yuri N. Obukhov, “How does the electromagnetic field couple to gravity, in particular to metric, nonmetricity, torsion, and curvature?”, arXiv:gr-qc/0001010v2, 3 May 2000


Stephen M. Barnett, Rotation of electromagnetic fields and the nature of optical angular momentum, Journal of Modern Optics, Vol. 00, No. 00, DD Month 200x,

