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Smooth representations of Groups associated with Algebras defined over non-archimedean fields

" Documento Definitivo"

Doutoramento em Matemática

Especialidade de Álgebra, lógica e fundamentos

João Miguel Cardoso Dias

Tese orientada por:

Professor Carlos Alberto Martins André

Documento especialmente elaborado para a obtenção do grau de doutor

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Resumo alargado

O objectivo deste trabalho é generalizar resultados sobre representações de certos tipos de grupos associados a álgebras (associativas) definidas sobre corpos finitos a representações suaves de grupos similares associados a álgebras (associativas) definidas sobre corpos locais não-Arquimedianos, isto é, extensões finitas do corpo dos p-ádicos (em característica zero), ou corpos de séries formais sobre um corpo finito (em característica prima).

Os grupos que serão estudados são generalizações de três grupos clássicos:

(1) o grupo unitriangular definido sobre um corpo $\mathbb{k}$, (isto é, o grupo de matrizes triangulares superiores de tamanho n com entradas em $\mathbb{k}$ e com entradas diagonais iguais a 1), cuja generalização é a classe de grupos-álgebra, isto é, grupos da forma $1 + \mathcal{J}$ onde $\mathcal{J}$ é uma $\mathbb{k}$-álgebra nilpotente com dimensão finita e a multiplicação é dada por

$$(1+x)(1+y) = 1 + x + y + xy, \qquad x, y \in \mathcal{J};$$

(2) o grupo das matrizes invertíveis triangulares superiores de tamanho n com entradas em $\mathbb{k}$, cuja generalização é o grupo de unidades de $\mathbb{k}$-álgebras básicas decomponíveis (em inglês, "split basic algebras") que são definidas como $\mathbb{k}$-álgebras $\mathcal{A}$ em que a álgebra quociente pelo radical de Jacobson é isomorfa a uma soma directa de um número finito de cópias do corpo $\mathbb{k}$;

(3) os grupos unipotentes clássicos (isto é, subgrupos unipotentes dos grupos simpléticos, ortogonais e unitários), que generalizam da forma seguinte: dada uma $\mathbb{k}$-álgebra $\mathcal{A}$ equipada com uma (anti-)involução linear σ, define-se o grupo dos pontos fixos

$$C_P(\sigma) = \{g \in P \mid \sigma(g) = g^{-1}\}$$

sendo $P = 1 + \mathcal{J}(\mathcal{A})$ e $\mathcal{J}(\mathcal{A})$ o radical de Jacobson de $\mathcal{A}$.

Diversos autores provaram que, quando $\mathbb{k}$ é um corpo finito, qualquer representação irredutível de um grupo-álgebra, de um grupo de unidades de uma álgebra básica decomponível,

ou de um grupo definido por involução, é induzida por um carácter linear de um subgrupo que é também um grupo-álgebra, um grupo de unidades de uma álgebra básica decomponível, ou de um grupo definido por involução, respectivamente. O objectivo deste trabalho é provar que o mesmo tipo de resultado é verdadeiro quando substituímos o corpo finito com um corpo local não-Arquimediano (com as devidas alterações). Para esse efeito, iremos considerar representações suaves, isto é, homomorfismos de grupos $\pi\colon G \to \mathrm{GL}(V)$ em que, para qualquer $v \in V$, o estabilizador de v para a acção de G (via π) é um subgrupo aberto de G (aqui, G tem uma estrutura natural de grupo topológico); abordaremos ainda, de forma resumida, as representações unitárias (isto é, representações contínuas sobre espaços de Hilbert cujo produto interno é invariante para a acção de G).

O primeiro capítulo da tese é um resumo dos resultados principais àcerca de grupos-álgebra e de grupos de unidades de álgebras básicas decomponíveis. É exposto o desenvolvimento do assunto, desde os anos 60 do século XX, altura em o assunto começou a ser estudado devido a uma conjectura (famosa) de G. Higman; são mencionadas outras conjecturas, incluindo algumas que mais tarde foram refutadas. Certas generalizações foram feitas por C. André e A. Nicolás que estudaram o caso de um grupo-álgebra definido sobre um anel finito ou sobre o anel dos inteiros p-ádicos (ou de uma sua extensão finita); no último caso, o grupo-álgebra é um grupo compacto (para a topologia natural) e, portanto, as suas representações unitárias irredutíveis são de dimensão finita e qualquer representação é semisimples (de modo que a teoria de representação de grupos finitos pode ser usada). No final do capítulo, iremos expor resumidamente a teoria de Kirillov, que fornece uma ferramenta teórica muito útil para a descrição das representações de certos tipos particulares de grupos (nomeadamente, dos grupos-álgebra em que o grau de nilpotência é bastante pequeno em comparação com a característica do corpo de base); em particular, quando pode ser aplicada, a teoria de Kirillov permite uma prova alternativa para concluir que as representações irredutíveis de grupos-álgebra são induzidas por caracteres lineares de subgrupos-álgebra.

O segundo capítulo é uma introdução à representação de grupos topológicos (infinitos), nomeadamente, às representações suaves de grupos topológicos que sejam Hausdorff, localmente compactos, totalmente desconexos e segundo contáveis ("second countable"). Diversos temas são apresentados, nomeadamente, os teoremas de isomorfismo usuais, o teorema órbita-estabilizador (que, nesta situação, é válido), e são apresentados exemplos dos casos dos grupos aditivo e multiplicativo do corpos p-ádicos (cuja estrutura é discutida neste capítulo). É introduzido, também, o conceito de ℓ-grupo e ℓ_c-grupo (que é um ℓ-grupo que é a união dos seus subgrupos compactos abertos). Certas diferenças entre as representações de ℓ-grupos e de ℓ_c-grupos são mencionadas (nomeadamente, provaremos que caracteres contínuos de ℓ_c-grupos são sempre unitários). São introduzidos os functores de indução-suave e de indução-compacta,

assim como são enunciadas as reciprocidades de Frobenius correspondentes e as relações entre os dois tipos de indução. Posteriormente, são discutidos os functores de Jacquet que nos dão a parte co-invariante de uma representação suave; estes functores serão usados frequentemente ao longo da tese, em especial, porque nos permitem relacionar representações suaves de um ℓ-grupo com as de um subgrupo normal fechado. É feita, ainda, uma pequena introdução às álgebras de Hecke associadas a ℓ-grupos; estas álgebras são os substitutos naturais das álgebras de grupo no contexto dos ℓ-grupos (e das suas representações suaves). É salientada, também, a importância dos subgrupos compactos abertos.

O terceiro capítulo é totalmente dedicado à teoria de Rodier. Esta teoria serve como substituto das (bem-conhecidas) teorias de Clifford e de Mackey, no sentido em que relaciona representações suaves de um ℓ-grupo com representações suaves de estabilizadores de caracteres (suaves) de um subgrupo normal fechado. No entanto, a teoria de Rodier tem mais restrições do que as teorias de Clifford e de Mackey, nomeadamente, só pode ser aplicada quando o subgrupo normal é um ℓ_c-grupo (além disso, outras condições têm de ser impostas às representações do ℓ-grupo ambiente). No final do capítulo, é usada a teoria de Rodier para descrever algumas representações suaves irredutíveis do grupo unitriangular sobre corpos locais não-Arquimedianos (que já são conhecidas no caso em que o corpo é finito).

O quarto capítulo é, essencialmente, o resumo de um artigo fundamental de M. Boyarchenko, onde é provado que qualquer representação suave de um grupo-álgebra é induzida (tanto suavemente, como compactamente) de caracteres de subgrupos-álgebra. Ao longo da prova, é definida uma forma bilinear num par de espaços vectoriais (associados à álgebra nilpotente considerada) que será útil nos restantes capítulos. Certos aspectos da prova também irão ser usados nos capítulos posteriores; notamos que a demonstração de M. Boyarchenko é construtiva e pode substituir outras conhecidas na situação de grupos finitos. Uma das secções deste capítulo é dedicada à descrição de todas as representações suaves irredutíveis dos grupos unitriangulares de ordem 3 e 4 sobre um corpo local não-Arquimedianos. Neste capítulo são introduzidos os grupos definidos por involuções e é provado que toda a representação suave irredutível de um grupo deste tipo é induzida de um carácter de um subgrupo definido pela mesma involução; este resultado generaliza um resultado de C. André no caso de corpos finitos. É provado, também, que toda a representação suave irredutível de um grupo definido por involução sobre um corpo local não-Arquimediano é admissivel e unitarizável.

O quinto e último capítulo é dedicado aos grupos das unidades de álgebras básicas decomponíveis e à prova de que qualquer representação suave irredutível de um grupo deste tipo é induzida compactamente de um carácter linear de um subgrupo que é, também, o grupo das unidades de uma subálgebra básica decomponível. São dadas certas condições para que uma representação suave irredutível seja admissível e é provado, ainda, que qualquer repre-

sentação suave irredutível pode ser factorizada numa parte unitarizável e numa parte linear não-unitarizável, o que significa que qualquer representação suave irredutível é unitarizável a menos da multiplicação por um carácter. A última secção do capítulo é dedicada à descrição das representações suaves irredutíveis dos grupos das unidades das álgebras das matrizes triangulares superiores de ordem 2 e 3.
Palavras chaves: Álgebra grupo; ; grupo de unidades; representações suaves; indução com suporte compact; conjectura de Gutkin.

Abstract

In this thesis, we study smooth representations of algebra groups, involutive algebra groups and unit groups of split basic algebras. We prove that every smooth irreducible representation of such a group is induced by a smooth representation of dimension one, which correspond to a continuous character of a subgroup of the same type. We also prove results about admissibility and unitarisability. This work generalises work of C. André and Z. Halasi who proved similar results in the case of finite fields, and is based on a method introduced by M. Boyarchenko for the case of algebra groups over local non-Archimedean fields.

Keywords: Algebra group; unit group; smooth representation; induction with compact supports; Gutkin's conjecture.

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Notation

$\mathbb{C}[q]$	ring of polynomials over q
$\mathbb{F}_q$	finite field with q elements
$\mathbb{F}_q^\times$	multiplicative group of a finite field with q elements
$U_n(q)$	Unitriangular group over $\mathbb{F}_q$
$U_n(\Bbbk)$	Unitriangular group over a field $\Bbbk$
$\mathfrak{u}_n(q)$	Unitriangular algebra over $\mathbb{F}_q$
$\mathfrak{u}_n(\Bbbk)$	Unitriangular algebra over a field $\Bbbk$
$\mathcal{J}(\mathcal{A})$	Jacobson radical of an algebra $\mathcal{A}$
$\mathbb{Q}_p$	field of p-adic rationals
$\mathbb{Z}_p$	ring of p-adic integers
$\mathrm{GL}_n(q)$	general linear group over $\mathbb{F}_q$
$\mathrm{GL}_n(\Bbbk)$	general linear group over a field $\Bbbk$
$e_{i,j}$	(i,j)-th elementary matrix (with 1 in the (i,j)-th entry, and 0 elsewhere
$U_\mathcal{P}$	pattern group associated to a partially ordered set $\mathcal{P}$
Res_H^G	restriction functor from a group G to a subgroup H
a-Ind_H^G	abstract induction functor from the subgroup H to the group G
Ind_H^G	smooth induction functor from the subgroup H to the group G
c-Ind_H^G	compact induction functor from the subgroup H to the group G
u-Ind_H^G	unitary induction functor from the subgroup H to the group G
$\mathcal{M}_n(\Bbbk)$	algebra of matrices with size n with over a field $\Bbbk$
$\mathrm{Rep}(G)$	category of smooth representations of G
V^∞	smooth submodule of a module V
$C_c^\infty(G)$	algebra of the locally constant functions with compact support
$\mathrm{Spec}(V)$	spectrum of the module V
$\mathrm{Spec}_A(V)$	spectrum of V with respect to A

Introduction

In [Gut73], A. Gutkin defined the class of *admissible groups* over a field $\Bbbk$: if R is a nilpotent associative algebra over a locally compact self-dual field $\Bbbk$, then R becomes a group (referred to as an *admissible group*) with respect to the multiplication defined by

$$x \cdot y = x + y + xy, \qquad x, y \in R.$$

Gutkin claimed that every irreducible unitary representation of an admissible group over a self dual field is induced from a unitary character of an admissible subgroup. However, I.M. Isaacs in [Isa95] showed that Gutkin's proof is defective, replaced the expression "admissible group" by the more suggestive expression *"algebra group"*, and proved that every irreducible representation of an algebra group over a finite field with q elements has dimension equal to a power of q. Later, Z. Halasi in [Hal04] used the previous result to prove that every irreducible representation of an algebra group over a finite field is induced from a (linear) character of an algebra subgroup (hence, proving the result by Gutkin in the particular case where $\Bbbk$ is a finite field).

The representation theory of the (upper) unitriangular group is closely linked (via Clifford Theory) to the representation theory of the group of invertible uppertriangular matrices (indeed, the unitriangular group is normal in the group of invertible uppertriangular matrices). In spite of this, B. Szegedy in [Sze96] considered the class of split basic algebras over finite fields. More generally, these are defined as finite-dimensional (associative) algebras $\mathcal{A}$ over an arbitrary field $\Bbbk$ such that the quotient algebra $\mathcal{A}/\mathcal{J}(\mathcal{A})$ is isomorphic to a finite direct sum of the field $\Bbbk$; here, $\mathcal{J}(\mathcal{A})$ denotes the Jacobson radical of $\mathcal{A}$. If $\mathcal{A}$ is a split basic algebra, then we can consider the algebra group $P = 1 + \mathcal{J}(\mathcal{A})$ which is a normal subgroup of the unit group of $\mathcal{A}$. We note that, in particular, the algebra of uppertriangular matrices is a split basic algebra, and so the context of split basic algebras generalises that of the algebra of uppertriangular matrices. Given the relation between algebra groups and unit groups of split basic algebras, it is worth to study this type of groups; indeed, any description of the irreducible characters of one would describe the irreducible characters of the other. In this direction, Z. Halasi proved in [Hal06] that every irreducible representation of the unit group of a split basic algebra over a finite field is induced

by a one-dimensional representation of a subgroup which is also the unit group of some split basic subalgebra.

The unitriangular group over a finite field $\Bbbk$ with q elements is a p-Sylow subgroup of the general linear group over $\Bbbk$. Therefore, we can also consider other finite classical groups, over the finite field $\Bbbk$ (such as the symplectic, the orthogonal or the unitary groups), and study the representation of their Sylow p-groups. The description of the irreducible representations of such groups is as difficult as the description of the irreducible representations of the unitriangular group (and of other algebra groups). However, as in the case of the unitriangular group, some advances have been made. In [And10], C. André considered a more general class of groups which are subgroups of algebra groups consisting of elements that are fixed by a given involution of the algebra. Additionally, André proved that, for this class of finite groups, the irreducible representations are in fact induced by one-dimensional representations of subgroups of the same type (that is, subgroups that are fixed by the given involution).

The proofs of the results mentioned so far use the usual tools of the theory of finite-dimensional representations of finite groups. However, these tools cannot be extended to the context of an arbitrary field. One of the constrains of using these usual tools of representation theory is the fact that representations of infinite groups may fail to have irreducible sub-representations, and so Clifford theory (for example) cannot be applied. However, the representation of compact topological groups is very similar to that of finite groups, since a unitary representation can usually be reduced to the finite case (by taking appropriate quotients). In this case, C. André and A. Nicolás in [AN08] generalised some of the results mentioned above to the case of algebra groups over finite rings and over the ring of integers of a p-adic field (that is, a finite extension of $\mathbb{Q}_p$).

In the case where the base field is a p-adic field (in characteristic zero), or a field of formal Laurent series over a finite field (in prime characteristic), we can study three types of representations: abstract representations (where no topology is involved), smooth representations and unitary representations. (If the field is finite, these three types of representations are the same.) We will study mainly the case of smooth representations, and will also mention the case of unitary representations (although not in much detail). In the case of smooth representations, F. Rodier in [Rod77] developed a method to relate smooth representations of a certain type of topological group with a particular type of subgroups, and this method will serve as a replacement to Clifford Theory (or, more precisely, of Mackey theory). Using this theory, M. Boyarchenko in [Boy11] generalised the result of Z. Halasi to the case where the algebra group is defined over a p-adic field or over the field of formal Laurent series over a finite field (Boyarchenko's method may also be applied to unitary representations). More concretely, Boyarchenko proved that every smooth representation (resp., unitary representation) of an algebra

group over such a field is smoothly induced (resp., unitarily induced) from a one-dimensional smooth representation (which is necessarily unitary) of an algebra subgroup. The present work aims to generalise Boyarchenko's result on smooth representations to the context of unit groups of split basic algebras and of groups defined by involutions. Additionally, some results concerning admissibility and unitarisability of smooth representations are proven. The case of algebra groups and groups defined by involutions differ from the case of the group of units of split basic algebras, mainly because the first two kind of groups are unions of their open compact subgroups (these are called ℓ_c-groups, while the other type is the semidirect product of an ℓ_c-group with a discrete group.

The thesis is structured as follows. In the first chapter, we discuss some of the main results in the case where the base field is finite. The second chapter presents the theory of locally compact, totally disconnected groups and their representations. In the third chapter, the theory of F. Rodier will be discussed, and some applications will be presented (namely, some irreducible representations of the unitriangular group are presented). The fourth chapter focus on the work of M. Boyarchenko which generalises the theorem of Z. Halasi to the case where the algebra group is defined over a p-adic field or over the field of formal Laurent series over a finite field. Additionally, some examples are discussed (using the methods of M. Boyarchenko) and the description of the irreducible smooth representations of the unitriangular group of order 3 and 4 is presented. Furthermore, in this fourth chapter, we describe the groups defined by involutions and extend the result of C. André to the case of an involutive algebra group defined over a p-adic field or over the field of formal Laurent series over a finite field (in this situation we require that the field has odd characteristic); we also prove that every irreducible smooth representation is admissible and unitarisable. In the fifth chapter, we consider unit groups of split basic algebras, and generalise the result by Z. Halasi to the case where the split basic algebra is defined over a p-adic field, or a field of formal Laurent series over a finite field. Additionally, some results are given concerning the admissibility and unitarisability of smooth irreducible representations, and we conclude the chapter with a description of the smooth irreducible representations of the group of invertible uppertriangular matrices of small sizes (orders 2 and 3).

Chapter 1

Representations of finite algebra groups and related groups

Throughout this thesis, we will only consider representations over the complex field; hence, a representation of a group G is a group homomorphism $\pi\colon G \to \mathrm{GL}(V)$ where V is a complex vector space (not necessarily finite-dimensional) and $\mathrm{GL}(V)$ denotes the group of automorphisms of V. A character of a finite group will always be understood as the trace of some finite-dimensional (complex) representation of the given group; thus, the character of a finite-dimensional representation $\pi\colon G \to \mathrm{GL}(V)$ is the map $\chi\colon G \to \mathbb{C}$ given by $\chi(g) = \operatorname{trace} \pi(g)$ for all $g \in G$.

We can divide the representation theory of linear algebraic groups over finite fields into the study of two distinct classes of algebraic groups: on the one hand, the reductive algebraic groups, such as the general linear group $\mathrm{GL}_n(q)$ consisting of all $n \times n$ invertible matrices over the finite field $\mathbb{F}_q$, and on the other hand the unipotent algebraic groups, such as the unitriangular group $U_n(q)$ over $\mathbb{F}_q$ consisting of all $n \times n$ uppertriangular matrices with 1 in the main diagonal. [Throughout the thesis, we will use the (standard) notation $\mathbb{F}_q$ to denote the finite field with q elements, where q is a power of a prime number p; thus, $\mathbb{F}_q$ is a finite extension of the (prime) field $\mathbb{F}_p$, and has characteristic p.] In fact, every (sufficiently well-behaved) algebraic group is the semidirect product of a unipotent group with a reductive group (see for example [Hum12]).

The representation theory of (finite) reductive groups is a well-studied subject; in particular, in [Gre55], J.A. Green constructed all the irreducible characters of $\mathrm{GL}_n(q)$. From Green's construction, one has a good description of how the irreducible characters behave as we consider finite field extensions of $\mathbb{F}_q$. For example, for $n = 2$, we have the following table (which illustrates that the degree and the number of irreducible characters do not depend on the size q of the field, and that both are given by polynomial expressions on q):

degree	1	q	$q-1$	$q+1$
number	$q-1$	$q-1$	$\frac{q}{2}(q-1)$	$\frac{q}{2}(q-1)(q-2)$

However, for unipotent groups, we do not know yet if it is possible to give a similar description. In [Hig60], G. Higman conjectured the following.

Conjecture (Higman)**.** For every natural number n the number of irreducible characters (and the number of conjugacy classes) of $U_n(q)$ is given by a polynomial function in q with integer coefficients.

We do not know whether the conjecture is true or false; however, in [VLA03], A. Vera-López and J.M. Arregi used some computational methods and verified the conjecture for $n \leq 13$. Later G. Lehrer made a stronger conjecture (see [Leh74]).

Conjecture (Lehrer)**.** For every natural number n, and every natural number c, the number of irreducible characters of $U_n(q)$ with degree q^c is given by a polynomial function in q with integer coefficients.

In particular, if Lehrer's conjecture is true, then Higman's conjecture is also true. It has been proven by I.M. Isaacs in [Isa95, Theorem A], that every irreducible character of $U_n(q)$ has degree a power of q. Indeed, being a p- group every irreducible character of $U_n(q)$ is monomial, and its degree is a power of p (see for example [CR81, Theorem 11.3]).

Later, I.M. Isaacs proposed an even stronger conjecture.

Conjecture (Isaacs)**.** For every natural number n, and for each natural number c, the number of irreducible characters with degree q^c of $U_n(q)$ is a polynomial function in $q-1$ with non-negative integers coefficients.

The description of the irreducible characters of the unitriangular group over a finite field is known to be a wild problem in the sense that it is equivalent to the classification of pairs of square matrices up to conjugacy. In fact, in [GKP$^+$90, Corollary 2], P.M. Gudivok *et al.* proved that the problem of describing all classes of conjugate matrices of size n in the unitriangular group over a field $\Bbbk$ includes the problem of classifying all matrices of order $[n/8]$ (where $[m]$ denotes the integer part of the rational number m) over $\Bbbk$ up to similarity by a unitriangular matrix.

We conclude this introductory section with a table showing the degrees of the irreducible characters of $U_n(q)$ for $n = 2, 3, 4, 5$, and the number of irreducible characters for each degree:

n	$\lvert \mathrm{Irr}(U_n(q)) \rvert$	degree 1	degree q	degree q^2	degree q^3	degree q^4
2	q	q	0	0	0	0
3	q^2+q-1	q^2	$q-1$	0	0	0
4	$2q^3+q^2-2q$	q^3	q^3-q	q^2-q	0	0
5	$5q^4-5q^2+1$	q^4	$2q^4-q^3-q^2$	$2q^4-q^3-2q^2+q$	$2q^3-3q^2+q$	q^2-2q+1

It is easy to see that the number of linear characters (that is, characters of degree 1) must always be equal to q^{n-1} because the commutator group consists of all matrices $u \in U_n(q)$ satisfying $u_{i,i+1} = 0$ for all $1 \leq i < n$. For $n \leq 5$ we see that the number of irreducible characters of a given degree is also a polynomial in q; as we mentioned above, it is unknown whether this is true or not in general.

1.1 Irreducible characters of algebra groups

A family of groups that behave very similarly to the unitriangular group is the family of algebra groups (or, more generally, of adjoint groups of radical rings). Let R be a commutative ring with identity, and let $\mathcal{J}$ be a finitely generated R-algebra (we do not require $\mathcal{J}$ to have an identity). For every positive integer m, we define $\mathcal{J}^m$ to be the subalgebra of $\mathcal{J}$ generated by all products of m elements of $\mathcal{J}$, and say that $\mathcal{J}$ is nilpotent if $\mathcal{J}^n = 0$ for some positive integer n; in particular, $\mathcal{J}$ cannot have an identity. It is well-known that, for every finitely generated R-algebra $\mathcal{A}$, the Jacobson radical $\mathcal{J} = \mathcal{J}(\mathcal{A})$ of $\mathcal{A}$ is a nilpotent R-algebra; on the other hand, every nilpotent finitely generated R-algebra $\mathcal{J}$ may be naturally embedded as a subalgebra of $\mathcal{A} = R \bigoplus \mathcal{J}$ (notice that $\mathcal{J}(\mathcal{A})$ includes $\mathcal{J}$, but may be larger than $\mathcal{J}$). By an *algebra group over* R we mean a group of the form $P = 1 + \mathcal{J}$ where $\mathcal{J}$ is a finitely generated nilpotent R-algebra; here, $1 + \mathcal{J}$ is understood to be a subgroup of the unit group $\mathcal{A}^\times$ of the R-algebra $\mathcal{A} = R \bigoplus \mathcal{J}$. Following the terminology of [Isa95], we say that a subgroup Q of $P = 1 + \mathcal{J}$ is an *algebra subgroup* if there exists a subalgebra $\mathcal{L}$ of $\mathcal{J}$ such that $Q = 1 + \mathcal{L}$; similarly, we say that a subgroup N of $P = 1 + \mathcal{J}$ is an *ideal subgroup* if there exists a two-sided ideal $\mathcal{I}$ of $\mathcal{J}$ such that $N = 1 + \mathcal{I}$ (notice that an ideal subgroup is normal). Finally, we observe that, if R is endowed with a topology, then this topology induces naturally a topology on $\mathcal{J}$, and so it also induces a topology on the algebra group $P = 1 + \mathcal{J}$.

As a standard example, the unitriangular group over an arbitrary ring R is an algebra group over R; indeed, $U_n(R) = 1 + \mathcal{J}$ where $\mathcal{J}$ is the R-algebra consisting of all strictly uppertriangular matrices with coefficients in R. Notice that, in the case where $R = \mathbb{F}_q$ is a finite field, the commutator subgroup is also an algebra group over R, because it is equal to $1 + \mathcal{J}^2$; how-

ever, this is not always the case: for arbitrary algebra groups, the commutator subgroup may be properly contained in $1+\mathcal{J}^2$ (note that $1+\mathcal{J}/1+\mathcal{J}^2 \simeq 1+\mathcal{J}/\mathcal{J}^2$ is an abelian group). As an example, let D_{16} be the dihedral group of order 16, and consider the group algebra $\mathcal{A} = \mathbb{F}_2[D_{16}]$; if $\mathcal{J} = \mathcal{J}(\mathcal{A})$ is the Jacobson radical of $\mathcal{A}$, then $|[1+\mathcal{J}, 1+\mathcal{J}]| = 2^9$, but the subalgebra generated by the elements of $[1+\mathcal{J}, 1+\mathcal{J}] - 1$ has dimension 11 (see [Isa07, Chapter 2]). Given an arbitrary R-algebra $\mathcal{J}$, we can define the Lie bracket $[x, y] = xy - yx$ for $x, y \in \mathcal{J}$, which endows $\mathcal{J}$ with a (natural) structure of a Lie algebra. In particular, it is easy to see that $[\mathcal{J}, \mathcal{J}] \leq \mathcal{J}^2$. In the case where $\mathcal{J}$ is the niltriangular algebra, we have $1+[\mathcal{J}, \mathcal{J}] = [1+\mathcal{J}, 1+\mathcal{J}] = 1+\mathcal{J}^2$, so it is natural to ask if this happens in general; however, in [JZ04] A. Jaikin-Zapirain constructed an example of an algebra group (over a finite field) satisfying $1 + [\mathcal{J}, \mathcal{J}] \neq [1 + \mathcal{J}, 1 + \mathcal{J}]$.

One way to construct examples of algebra groups (over a commutative ring R) is to take a subset $\mathcal{P}$ of $\{(i, j) \mid 1 < i < j < n\}$ and to consider the R-linear span $\mathcal{J} = \langle e_{i,j} \mid (i, j) \in \mathcal{P}\rangle$ where $e_{i,j}$, for $1 \leq i < j \leq n$, denotes the usual $n \times n$ elementar matrix (with 1 in the (i, j)-th entry and 0 elsewhere). Note that, in order to $1+\mathcal{J}$ be an algebra group we must have $(i, k) \in \mathcal{P}$ whenever $(i, j), (j, k) \in \mathcal{P}$ (because $\mathcal{J}$ must be multiplicatively closed). For such a subset $\mathcal{P}$,

$$U_{\mathcal{P}}(R) = \{g \in U_n(R) \mid g_{i,j} = 0 \text{ if } i < j \text{ and } (i, j) \notin \mathcal{P}\}$$

is a subgroup of the unitriangular group $U_n(R)$, which is an algebra group over R and to which we refer as a *pattern group (over R)*.

Pattern groups behave very closely to the unitriangular groups; indeed, I.M. Isaacs proved the following result.

Proposition 1.1.1 ([Isa07, Corollary 2.2]). *Let $U_{\mathcal{P}}(q)$ be a pattern group (over $R = \mathbb{F}_q$) and let $\mathcal{J} = U_{\mathcal{P}}(q) - 1$ be the corresponding algebra. Then the commutator subgroup $[U_{\mathcal{P}}(q), U_{\mathcal{P}}(q)]$ is equal to $1 + \mathcal{J}^2$; furthermore, it is also a pattern group.*

Pattern groups provide us a good source of examples of algebra groups:

- The unitriangular group $U_n(q)$ is a pattern group with $\mathcal{P} = \{(i, j) \mid 1 \leq i < j \leq n\}$.

- If $\mathfrak{u}_n(q) = U_n(q) - 1$ is the niltriangular algebra over $\mathbb{F}_q$, then for any $1 \leq k < n$, the algebra group $1 + \mathfrak{u}_n(q)^k$ is a pattern group with

  $$\mathcal{P} = \{(i, j) \mid 1 \leq i < j \leq n,\ j - i \geq k\};$$

  further, if we choose $1 \leq r < n$, then there is also a pattern group associated with

  $$\mathcal{P} = \{(i, j) \mid 1 \leq i < j \leq n,\ j - i \geq k\} \setminus \{(i, j) \mid j - i \leq r\}.$$

- The direct product $U_{n_1}(q) \times U_{n_2}(q)$ of two unitriangular groups (over the same field) is isomorphic to the pattern group associated to

$$\mathcal{P} = \{(i,j) \mid 1 < i < j \leq n_1\} \cup \{(i,j) \mid n_1 < i < j \leq n_1 + n_2\}$$

(notice that a similar construction shows that any finite direct product of pattern groups is also a pattern group).

From the similarity with the unitriangular group, we might ask as well whether Higman's Conjecture is true in this situation. However, Z. Halasi and P. Pálfy found a pattern group for which the number of conjugacy classes is not a polynomial function in q with integer coefficients; see [HP11, Corollary 4.5]; nevertheless, they proved that Higman's conjecture is true for a small class of pattern groups.

Proposition 1.1.2. *Let $U_{\mathcal{P}}(q)$ be a pattern group which is a normal subgroup of $U_n(q)$ and with nilpotency class equal to two. Then the number of irreducible characters of $U_{\mathcal{P}}(q)$ is given by a polynomial function of q with integer coefficients, and in fact it is given by a polynomial function of $q-1$ with nonnegative integer coefficients.*

Now, we observe that an algebra group $P = 1 + \mathcal{J}$ over the finite field $\mathbb{F}_q$ is a p-group (where p is the characteristic of $\mathbb{F}_q$); indeed, the mapping $1 + x \mapsto x$ defines a bijection from P to $\mathcal{J}$ and $\mathcal{J}$ is a vector space over $\mathbb{F}_q$, and so $|P| = |\mathcal{J}|$ is a power of q (hence a power of p as well). Therefore, the degree of any irreducible character of P must be a power of p (see, for example, [Isa94, Theorem 3.12]). Looking at what happens in the case of the character theory of $\mathrm{GL}_n(q)$, where the degrees of the irreducible characters depend on q, we might ask if the same happens in the case of algebra groups. In fact,, I.M. Isaacs proved the following result.

Theorem 1.1.3 ([Isa95, Theorem A]). *If P is an algebra group over $\mathbb{F}_q$, then every irreducible character of P has q-power degree.*

Proof. (sketch) If P is abelian, then the theorem is trivially true (because all irreducible characters have degree one), so we assume that P is not abelian. By induction on the dimension of $\mathcal{J}$, suppose that the statement is true for all proper algebra groups $Q = 1 + \mathcal{L}$ with $\dim \mathcal{L} < \dim \mathcal{J}$. The proof of the theorem will rely on the following auxiliary result.

Lemma 1.1.4 ([Isa95, Lemma F]). *Fix a positive integer q, and let G be a finite group. Let M be a normal subgroup of G, and suppose that $\mathfrak{H}$ is a collection of subgroups of G satisfying:*

(1) The intersection of every two elements of $\mathfrak{H}$ is equal to M.

(2) $G = \bigcup_{H \in \mathfrak{H}} H$.

(3) $|G : H| = q$ *for every subgroup* $H \in \mathfrak{H}$.

(4) The group G/M *is abelian of order* q^2.

(5) The irreducible characters of M *and of the elements of* $\mathfrak{H}$ *all have degree a power of* q.

Then, every irreducible character of G *has degree a power of* q.

In order to prove the theorem, we need to show that such collection exists for every non-abelian algebra group $P = 1 + \mathcal{J}$. Firstly suppose that $\mathcal{J}^2$ has codimension one in $\mathcal{J}$, so that we can write $\mathcal{J} = \mathcal{J}^2 + \mathbb{F}_q u$ for some $u \in \mathcal{J} \setminus \mathcal{J}^2$. Let us consider the subalgebra $\langle u \rangle$ generated by u. Then, we also have $\mathcal{J} = \mathcal{J}^2 + \langle u \rangle$ and [Isa95, Lemma 3.1] implies that $\mathcal{J} = \langle u \rangle$, and thus $\mathcal{J}$ is abelian (which implies that P is abelian), a contradiction. Therefore, we may assume that $\mathcal{J}^2$ has codimension in $\mathcal{J}$ greater than or equal to 2. Let U be a vector subspace of $\mathcal{J}$ of codimension 2 containing $\mathcal{J}^2$; notice that every vector subspace V with $\mathcal{J}^2 \subseteq V$ is an ideal of $\mathcal{J}$, and hence $1 + V$ is an algebra group. Finally, let $\mathfrak{H}$ be the collection of all subgroups $1 + V$ where V is an hyperplane of $\mathcal{J}$ which contains U.

Given any two such hyperplanes V_1 and V_2, we have $(1 + V_1) \cap (1 + V_2) = 1 + U$, and so the collection $\mathfrak{H}$ satisfies the first condition. It is clear that the union of all subgroups of $\mathfrak{H}$ is equal to P, and hence we have that the second condition holds. By construction, we have $|P/(1+V)| = q$ for every hyperplane $V \subseteq \mathcal{J}$, and thus the fourth condition also holds. Finally, it is clear that $P/(1 + U) \simeq \mathcal{J}/U$ is an abelian group (because $\mathcal{J}^2 \leq U$) of order q^2 (by construction).

For the last condition, $1 + U$ and the subgroups in $\mathfrak{H}$ are all proper algebra subgroups of P, and so the induction hypothesis guarantees that they all have irreducible characters with q power degree. Therefore, P has a collection $\mathfrak{H}$ satisfying the conditions of the lemma above, and thus every irreducible character of P has q-power degree. $\square$

A generalisation of the theorem above was given by C. André and A.P. Nicolás.

Theorem 1.1.5 ([AN08, Theorem 2.3])**.** *Let* R *be a finite commutative ring, with residue field* $R/\mathcal{J}(R) \simeq \mathbb{F}_q$, *and let* $\mathcal{J}$ *be a finite-dimensional nilpotent* R*-algebra. Then, every irreducible character of* $P = 1 + \mathcal{J}$ *has* q*-power degree.*

The proof of this theorem uses a very similar construction as the one of Theorem 1.1.3 to find a collection of subgroups which satisfies the hypotheses of Lemma 1.1.4. Using the theorem above, André and Nicolás generalised the result to the case where R is the ring of integers of any finite extension of the p-adic field.

Theorem 1.1.6 ([AN08, Theorem 2.4]). *Let R be the ring of integers of a finite extension E of the p-adics, and suppose that $R/\mathcal{J}(R)$ is a finite ring with q elements. Let $\mathcal{J}_E$ be a finite-dimensional nilpotent E-algebra, and let $\mathcal{J}$ be a multiplicative closed R-module of $\mathcal{J}_E$. Then, every continuous irreducible representation of $P = 1 + \mathcal{J}$ has dimension a power of q.*

Proof. It is well-known that R is a discrete valuation ring (see [Ati18]), and hence there is an element $\varpi \in R$ such that $\mathcal{J}(R) = \varpi R$ is the unique maximal ideal of R. Let $m \in \mathbb{N}$ be arbitrary, and consider the algebra group $P_m = 1 + \varpi^m \mathcal{J}$. Since $\varpi^m \mathcal{J}$ is a (two-sided) ideal of $\mathcal{J}$, P_m is a normal subgroup of P, and so we can form the quotient P/P_m. On the other hand, $\mathcal{J}/\varpi^m \mathcal{J}$ is an R-algebra, and so we may consider the algebra group $P'_m = 1 + (\mathcal{J}/\varpi^m \mathcal{J})$; we note that P'_m is an algebra group over the quotient ring $R_m = R/\varpi^m R$,and that R_m is a local ring with residue field isomorphic to $\mathbb{F}_q$. By the previous theorem we conclude that all irreducible representations of P'_m have dimension equal to a power of q. As in the proof of [AN08, Theorem 2.4], we can see that P_m is isomorphic to P'_m, and so it follows that all irreducible representations of P_m have dimension equal to a power of q.

Now, let $\pi \colon P \to \mathrm{GL}(V)$ be an arbitrary irreducible continuous representation of P. Since $\{P_m \mid m \in \mathbb{N}\}$ is a basis of open neighbourhoods of the identity, there exists $m \in \mathbb{N}$ such that P_m lies in the kernel of π. Therefore, the representation π determines (in the natural way) an irreducible representation of P_m, and so V must have dimension equal to a power of q, as required. □

In the particular case where $P = U_n(q)$ is the unitriangular group over $\mathbb{F}_q$, the previous result was first conjectured by D. Thompson and was later proven by D. Kazhdan in the case where $\mathbb{F}_q$ has characteristic $p \geq n$; Kazhdan's result appears in [Sri06, Theorem 7.7(v)] (see also [Kaz77]). In fact, under the assumption $p \geq n$, D. Kazhdan proved that every irreducible character of $U_n(q)$ is induced from a linear character of some algebra subgroup of $U_n(q)$ (see also [Kaz77]); the proof is strongly based on the method of coadjoint orbits introduced by A. Kirillov (see [Kir62]). In [Isa95], I.M. Isaacs asked whether every irreducible character of an arbitrary algebra group over $\mathbb{F}_q$ is induced from a linear character of an algebra subgroup; notice that this result would imply Isaacs's theorem on character degrees. In [And], C. André proved that this is true in the particular case where $\mathcal{J}^p = 0$ where p is the characteristic of $\mathbb{F}_q$; later, in [Hal04], Z. Halasi gave a proof for the general case using Isaacs Theorem 1.1.3. This theorem was claimed to be proven by E.Gutkin in [Gut73]; however, in [Isa95], I.M. Isaacs showed with an example that Gutkin's argument was not correct.

Theorem 1.1.7 ([Hal04, Theorem 1.2]). *Let $P = 1 + \mathcal{J}$ be an algebra group over $\mathbb{F}_q$, and let χ be an irreducible character of P. Then, there exist an algebra subgroup over Q of P and a linear character of Q such that $\chi = \mathrm{Ind}_Q^P \lambda$.*

The detailed proof of this theorem can be found in [Hal04]; however, we shall give a brief description of it. We first recall the following.

Lemma 1.1.8 ([Hal04, Lemma 3.1]). *Let $P = 1 + \mathcal{J}$ be an algebra group over $\mathbb{F}_q$, and let χ be an irreducible character of P. Then, the following are equivalent.*

(1) There exist a proper algebra subgroup $Q \leq P$ and φ an irreducible character of Q such that $\chi = \operatorname{Ind}_Q^P \varphi$.

(2) The restriction of χ to $1 + \mathcal{J}^2$ is not irreducible.

Proof. Firstly, suppose that there exist a proper algebra subgroup $Q = 1 + \mathcal{L}$ of P, and an irreducible character φ of Q such that $\chi = \operatorname{Ind}_Q^P \varphi$. Let

$$K = Q(1 + \mathcal{J}^2) = 1 + \mathcal{L} + \mathcal{J}^2;$$

in particular, $1 + \mathcal{J}^2 \leq K$ and $K \neq P$ (if $K = P$, [Isa95, Lemma 3.1] implies that $Q = P$). Therefore, we have $\chi = (\operatorname{Ind}_Q^K \varphi)^P$. It follows that χ_K is not irreducible, and thus $\chi_{1+\mathcal{J}^2}$ is not irreducible as well.

Now, let us assume that $\chi_{1+\mathcal{J}^2}$ is not irreducible, and choose an irreducible constituent ψ of $\chi_{1+\mathcal{J}^2}$. Let $Q = 1 + \mathcal{L}$ be a maximal algebra subgroup such that ψ can be extended to Q, and note that $Q \neq P$ because $\chi_{1+\mathcal{J}^2}$ is not irreducible. Let ϕ be an extension of ψ to Q which is also an irreducible constituent of the restriction χ_Q. Then, for any $x \in \mathcal{J} \setminus \mathcal{L}$,

$$N_x = 1 + \mathbb{F}_q x + \mathcal{L}$$

is a algebra subgroup of P satisfying $|N_x : Q| = q$. Let ϑ be an irreducible character of N_x such that ϕ is a constituent of ϑ_Q. By Theorem 1.1.3, $\vartheta(1)$ and $\phi(1)$ are powers of q, and so either ϑ is an extension of ϕ (which cannot happen by the maximal choice of Q), or $\vartheta = \operatorname{Ind}_Q^{N_x} \phi$. Therefore, for all $x \in \mathcal{J}$, the inertia group $I_{N_x}(\phi)$ of ϕ in N_x is equal to Q, and so we have $I_P(\phi) = q$. By [Isa94, Problem 6.1], we conclude that $\operatorname{Ind}_Q^P \phi$ is irreducible, and hence $\chi = \operatorname{Ind}_Q^P \phi$, as required. □

Now, we are able to give the proof of Halasi's theorem.

Proof of Theorem 1.1.7. Let χ be a non-linear irreducible character of P. Then, the restriction of χ to $1 + \mathcal{J}^2$ is not irreducible (otherwise it would be P-invariant and so it would be linear (by [Hal04, Theorem 1.3])). By the previous lemma, it follows that there exist a proper algebra subgroup Q of P and an irreducible character φ of Q such that $\chi = \operatorname{Ind}_Q^P \varphi$. Therefore, by induction on $|P|$, we conclude that there exist an algebra subgroup Q' and a linear character λ of Q' such that $\varphi = \operatorname{Ind}_{Q'}^P \lambda$. The theorem follows (by transivity of the induction). □

Later, using methods similar to the above, C. André and A. Nicolás extended the theorem above.

Theorem 1.1.9 ([AN08, Theorem 2.6]). *Let R be a finite Galois ring such that $R/\mathcal{J}(R) \simeq \mathbb{F}_q$, let $\mathcal{J}$ be a finite nilpotent algebra over R, and let $P = 1 + \mathcal{J}$. Then, for every irreducible character χ of P, there exist a subalgebra $\mathcal{L}$ of $\mathcal{J}$ and a linear character λ of $Q = 1 + \mathcal{L}$ such that $\chi = \mathrm{Ind}_Q^P(\lambda)$.*

C. André and A. Nicolás in [AN08] also proved the theorem in the case where R is the ring of integers of a finite extension of the p-adic field.

Theorem 1.1.10 ([AN08, Theorem 2.4]). *Let R be the ring of integers of a finite extension E of the p-adic field, and suppose that $R/\mathcal{J}(R)$ is a finite ring with q elements. Let $\mathcal{J}_E$ be a finite-dimensional nilpotent algebra over E, let $\mathcal{J}$ be a multiplicative closed R-module, and let $\pi\colon P \to \mathrm{GL}(V)$ be an irreducible continuous representation of $P = 1 + \mathcal{J}$. Then, there exist a subalgebra $\mathcal{L}$ of $\mathcal{J}$ and a continuous linear character λ of $Q = 1 + \mathcal{L}$ such that $V = \mathrm{Ind}_Q^P(\mathbb{C}_\lambda)$ where $\mathbb{C}_\lambda$ denotes the (canonical) one-dimensional Q-module which affords the character λ.*

Proof. Let the notation be as in the proof of Theorem 1.1.6. Let $m \in \mathbb{N}$ be such that P_m lies in the kernel of π, so that π defines naturally an irreducible representation $\pi'\colon P/P_m \to \mathrm{GL}(V')$ of P/P_m. As in the proof of Theorem 1.1.6, P/P_m is an algebra group over the finite ring $R/\varpi^m R$, and thus the previous theorem guarantees that there exist a subalgebra $\mathcal{L}'$ of $\mathcal{J}/\mathcal{J}_m$ and a linear character λ' of $Q' = 1 + \mathcal{L}'$ such that $V' \simeq \mathrm{Ind}_{Q'}^{P'} \mathbb{C}_{\lambda'}$. If $\mathcal{L}$ denotes the inverse image of $\mathcal{L}'$ under the canonical epimorphism $\mathcal{J} \to \mathcal{J}/\mathcal{J}_m$, then λ' defines naturally a linear character of $Q = 1 + \mathcal{L}$, and so we conclude that $V = \mathrm{Ind}_Q^P \mathbb{C}_\lambda$. □

More recently, in [Boy11], M. Boyarchenko proved the following generalisation.

Theorem 1.1.11 ([Boy11, Theorem 1.3]). *Let $\Bbbk$ be a non-Archimedean local field, and let $\mathcal{J}$ be a nilpotent $\Bbbk$-algebra. Then, every irreducible representation of $1 + \mathcal{J}$ is admissible and unitarisable. Furthermore, for every irreducible smooth representation $\pi\colon 1 + \mathcal{J} \to \mathrm{GL}(V)$ of $1 + \mathcal{J}$, there exist a subalgebra $\mathcal{J}'$ of $\mathcal{J}$ and a smooth character λ of $1 + \mathcal{J}'$ such that*

$$V \simeq \mathrm{Ind}_{1+\mathcal{J}'}^{1+\mathcal{J}}(\mathbb{C}_\lambda) = \text{c-}\mathrm{Ind}_{1+\mathcal{J}'}^{1+\mathcal{J}}(\mathbb{C}_\lambda).$$

The proof given by M. Boyarchenko also applies to the case where $\Bbbk$ is a finite field, and it does not use [Isa95, Theorem A]. A detailed explanation of Boyarchenko's theorem and methods will be given in Chapter 2.

1.2 Irreducible characters of groups associated with split basic algebras

One way to see the unitriangular group $U_n(q)$ is as a p-Sylow subgroup of the general linear group $\mathrm{GL}_n(q)$. Of particular interest is its normalizer, that is, the Borel subgroup $B_n(q)$ of $\mathrm{GL}_n(q)$ consisting of all invertible uppertriangular matrices. Note that $B_n(q)$ is the semidirect product $B_n(q) = T_n(q) \ltimes U_n(q)$, where $T_n(q)$ denotes the maximal torus consisting of all invertible diagonal matrices, acting on the $U_n(q)$ by conjugation. Since $U_n(q)$ is a normal subgroup of $B_n(q)$, a good description of the irreducible characters of $U_n(q)$ might provide, via Clifford theory, also a good description of the irreducible characters of $B_n(q)$.

In [Sze96], B. Szegedy considers the more general situation of unit groups of the special family of split basic algebras (to which he refers as DN-algebras). Let $\Bbbk$ be a field, and let $\mathcal{A}$ be a finite-dimensional (associative) $\Bbbk$-algebra with identity. We say that $\mathcal{A}$ is a *split basic* $\Bbbk$-*algebra* if the set consisting of all nilpotent elements of $\mathcal{A}$ equals to the Jacobson radical $\mathcal{J}(\mathcal{A})$ of $\mathcal{A}$, and if the (semisimple) quotient $\mathcal{A}/\mathcal{J}(\mathcal{A})$ is isomorphic to a (finite) direct sum of copies of $\Bbbk$.

Every split basic $\Bbbk$-algebra $\mathcal{A}$ decomposes as a direct sum $\mathcal{A} = \mathcal{D} \bigoplus \mathcal{J}(\mathcal{A})$ where $\mathcal{D}$ is a subalgebra of $\mathcal{A}$ spanned as a vector space by a set of minimal orthogonal central idempotents of $\mathcal{A}$ (see for example [Hal06, Lemma 2.1]). We will refer to $\mathcal{D}$ as a diagonal algebra of $\mathcal{A}$, and denote by T the unit group $\mathcal{D}^\times$ of $\mathcal{D}$; notice also that the unit group $\mathcal{A}^\times$ decomposes as the semidirect product $\mathcal{A}^\times = T \ltimes (1 + \mathcal{J}(\mathcal{A}))$.

In particular, for every finite-dimensional nilpotent algebra $\mathcal{J}$ the $\mathbb{F}_q$-algebra $\mathcal{A} = \mathbb{F}_q \oplus \mathcal{J}$ is a split basic $\mathbb{F}_q$-algebra. The representation of the unit group $\mathcal{A}^\times = \mathbb{F}_q^\times \times (1 + \mathcal{J})$ is essentially the "same" as the representation of the algebra group $1+\mathcal{J}$, because every irreducible character of $\mathcal{A}^\times$ is the product of a linear character of $\mathbb{F}_q^\times$ and an irreducible character of $1 + \mathcal{J}$. In particular, the degree of every irreducible character of $\mathcal{A}^\times$ is a power of q, and one can easily conclude from Halasi's Theorem that every irreducible character is induced from a linear character of the unit group of a subalgebra of $\mathcal{A}$ (which is also a split basic $\mathbb{F}_q$-algebra). More generally, we may expect that the central results in the case of finite algebra groups may also hold in the case of unit groups of finite split basic algebras; in fact, in [Sze96], B. Szegedy proved the following extension of Isaacs's Theorem.

Theorem 1.2.1 ([Sze96, Theorems 3.1 and 3.2])**.** *Let $\mathcal{A}$ be a split basic $\mathbb{F}_q$-algebra, let $\mathcal{A}^\times$ denote the unit group of $\mathcal{A}$, and let χ be an irreducible character of $\mathcal{A}^\times$. Then, there exist a subgroup H of $\mathcal{A}^\times$ and a linear character λ of H such that $\chi = \mathrm{Ind}_H^{\mathcal{A}^\times}(\lambda)$. Furthermore, both the size of every conjugacy class and the degree of every irreducible characters are of the form: $q^r(q-1)^s$ for some non negative integers r and s.*

Although Szegedy's Theorem states that the unit group of a split basic $\mathbb{F}_q$-algebra is an M-group (that is, every irreducible characters is induced from linear character of some subgroup), it does not state that the inducing group may be chosen to be the unit group of a subalgebra. However, in [Hal06], Z. Halasi used a different argument to show that, in fact, the inducing subgroup can be chosen as the unit group of a subalgebra.

Theorem 1.2.2 ([Hal06, Theorem 1.3])**.** *Let $\mathcal{A}$ be a split basic $\mathbb{F}_q$-algebra, let $\mathcal{A}^\times$ be the unit group of $\mathcal{A}$, and let χ be an irreducible character of $\mathcal{A}^\times$. Then there exist a subalgebra $\mathcal{B}$ of $\mathcal{A}$ and a linear character λ of the unit group $\mathcal{B}^\times$ of $\mathcal{B}$ such that $\chi = \mathrm{Ind}_{\mathcal{B}^\times}^{\mathcal{A}^\times}(\lambda)$.*

Below, we sketch the proof of this theorem; firstly, we mention the following result (also due to Z. Halasi).

Theorem 1.2.3 ([Hal06, Theorem 3.5])**.** *Let $\mathcal{A}$ be a split basic $\mathbb{F}_q$-algebra, let $\mathcal{A}^\times$ be the unit group of $\mathcal{A}$, and let χ be an irreducible character of the algebra group $1+\mathcal{J}$ where $\mathcal{J} = \mathcal{J}(\mathcal{A})$ is the Jacobson radical of $\mathcal{A}$. Suppose that χ is invariant under the conjugation action of $\mathcal{A}^\times$. Then, there exist a proper (two-sided) ideal $\mathcal{L}$ of $\mathcal{A}$ and a linear character λ of the algebra subgroup $1+\mathcal{L}$ which is invariant under the action of the diagonal subgroup T of $\mathcal{A}^\times$ and such that $\chi = \mathrm{Ind}_{1+\mathcal{L}}^{1+\mathcal{J}}(\lambda)$.*

(Notice that, in the particular case where $\mathcal{A}$ is the split basic $\mathbb{F}_q$-algebra $\mathcal{A} = \mathbb{F}_q \otimes \mathcal{J}$ for some finite-dimensional nilpotent $\mathbb{F}_q$-algebra $\mathcal{J}$, the previous theorem reduces to the Halasi's Theorem 1.1.7).

Proof of Theorem 1.2.2 (sketch). Since every subalgebra of a split basic $\mathbb{F}_q$-algebras containing the identity is also a split basic $\mathbb{F}_q$-algebra (see for example [Sze96, Lemma 2.2], it is enough to prove that if χ is non-linear, then there exist a proper subalgebra $\mathcal{A}'$ of $\mathcal{A}$ and an irreducible character χ' of $(\mathcal{A}')^\times$ such that $\chi = \mathrm{Ind}_{(\mathcal{A}')^\times}^{\mathcal{A}^\times}(\chi')$.

We consider the semidirect decomposition $\mathcal{A}^\times = T \ltimes (1+\mathcal{J})$ and let ω be an irreducible component of the restriction $\chi_{1+\mathcal{J}}$. Then, by [Hal06, Lemma 3.1], the centraliser $C_{\mathcal{A}^\times}(\omega)$ is the unit group of the subalgebra $\mathcal{A}' = \mathcal{D}' + \mathcal{J}$ of $\mathcal{A}$. By Clifford's Theorem (see [Isa94, Theorem 6.11]), we conclude that χ is induced by an irreducible character of $C_{\mathcal{A}^\times}(\omega)$. If $\mathcal{D}' \neq \mathcal{D}$, then $\mathcal{A}' = \mathcal{D}' + \mathcal{J}$ is a proper subalgebra of $\mathcal{A}$.

Let us assume that $\mathcal{D}' = \mathcal{D}$, that is, ω is invariant under the action of T. Then, by [Isa94, Corollary 6.28], we can extend ω to a character of $\mathcal{A}^\times$ (because $|\mathcal{A}^\times : 1+\mathcal{J}|$ and $|1+\mathcal{J}|$ are coprime); by [Isa94, Corollary 6.17], it follows that χ is an extension of ω, and hence ω is not linear.

By Theorem 1.2.3 we know that $\omega = \mathrm{Ind}_{1+\mathcal{L}}^{1+\mathcal{J}}(\psi)$ for some ideal $\mathcal{L}$ of $\mathcal{J}$ and some T-invariant irreducible character of $1+\mathcal{L}$. Then, $\mathcal{B}' = \mathcal{D} + \mathcal{L}$ is an proper subalgebra of $\mathcal{A}$.

By [Isa94, Corollary 6.28], we may choose an extension ψ of ψ to the unit group of $\mathcal{B}$. Since $(\mathrm{Ind}_{\mathcal{B}^\times}^{\mathcal{A}^\times} \phi)_{1+\mathcal{J}} = \mathrm{Ind}_{1+\mathcal{L}}^{1+\mathcal{J}} \psi$ by [Isa94, Problem 5.2], we conclude that $\mathrm{Ind}_{\mathcal{B}^\times}^{\mathcal{A}^\times}(\phi) = \chi\mu$ for some linear character μ of T. Therefore, for $\omega' = \mu_{\mathcal{B}^\times}^{-1}\phi$ we see that $\chi = \mathrm{Ind}_{\mathcal{B}^\times}^{\mathcal{A}^\times} \omega'$, as required. $\square$

We see that Halasi's proof relies heavily on Clifford theory, and so on the fact that the given group is finite. In the Chapter 5, we shall give a different proof of the theorem above in the case of a split basic algebra over a non-Archimedean local field (such as a finite extension of the p-adic field, or the field of Laurent polynomials over a finite field) which also applies to the finite case.

1.3 The Kirillov method for algebra groups

In [Kir62], A. Kirillov described what is now known as the *Kirillov orbit method*. Kirillov used this method to study the unitary representations of connected and simply connected real nilpotent Lie groups. The method may also be applied to algebra groups over any field $\Bbbk$ provided that the characteristic of the field is zero or not smaller than the nilpotency class of the algebra associated with the given algebra group; either of these conditions is required in order to guarantee that the exponential map may be defined.

We can also apply the construction described in by the orbit method to a particular class of finite groups. In [Laz54], M. Lazard considered the category Nilp_n of all nilpotent groups with nilpotency class less than n and such that the k-power map (given by the mapping $g \mapsto g^k$) is invertible for all $k < n$, and the category $\mathfrak{nilp}_n$ of all $\mathbb{Z}[\frac{1}{n!}]$-Lie algebras with nilpotency class less than n. Then, with every Lie algebra $\mathfrak{g}$ in $\mathfrak{nilp}_c$, M. Lazard associated a nilpotent group, denoted $\exp(\mathfrak{g})$, which has the same underlying set $\mathfrak{g}$, and where the product of $x, y \in \exp\ \mathfrak{g}$ is given by the formula

$$x * y = \sum_{i \le n} CH_i(x, y)$$

where, for each $1 \le i < n$, $CH_i(x, y)$ is the homogeneous component of degree i of the usual Campbell-Hausdorff-Baker formula; by the way of example, we list below the first terms (here,$[., .]$ denotes the Lie bracket of $\mathfrak{g}$):

(1) $CH_1(x, y) = x + y$;

(2) $CH_2(x, y) = \frac{1}{2}[x, y]$;

(3) $CH_3(x, y) = \frac{1}{12}([x, [x, y]] - [y, [x, y]])$;

(4) $CH_4(x, y) = \frac{1}{24}[x, [y, [y, x]]]$.

Note the product is well defined because $CH_i(x, y)$ is a polynomial with coefficients in $\mathbb{Z}[\frac{1}{i!}]$, and the sum is finite because $\mathfrak{g}$ is a nilpotent algebra with nilpotency class less than n.

It can be proven that $\exp(\mathfrak{g})$ lies in Nilp_n, and that the mapping $\mathfrak{g} \mapsto \exp(\mathfrak{g})$ defines a functor $\exp\colon \mathfrak{nilp}_n \to \mathrm{Nilp}_n$. Indeed, M. Lazard in [Laz54] proved the following theorem.

Theorem 1.3.1 (Lazard)**.** *The functor* $\exp\colon \mathfrak{nilp}_n \to \mathrm{Nilp}_n$ *is an equivalence of categories; its quasi-inverse will be denoted by* $\log$.

In the case where $G = U_n(q)$ is the unitriangular group over a finite field of characteristic $p > n$, Lazard's construction gives $G = \exp(\mathfrak{g})$ where $\mathfrak{g} = \mathfrak{u}_n(q)$ is the niltriangular algebra over $\mathbb{F}_q$; in this particular situation, we may choose the usual exponential map given by

$$\exp(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^{n-1}}{(n-1)!}, \qquad x \in \mathfrak{u}_n(q).$$

Similarly, $1 + \mathcal{J} = \exp(\mathcal{J})$ for every finite-dimensional nilpotent algebra over a finite field whose characteristic p satisfies $\mathcal{J}^p = 0$.

Defining the map $\log\colon \exp(\mathfrak{g}) \to \mathfrak{g}$ to be the inverse of map $\exp$, we can define an conjugacy action on $\mathfrak{g}$ by

$$g \cdot x = \log\left(g \exp(x) g^{-1}\right), \qquad g \in \mathfrak{g},\ x \in \mathfrak{g};$$

this action defines a linear action of the Lie algebra $\mathfrak{g}$, and will be denoted by gxg^{-1} for simplicity.

The orbit method gives us a bijection between the set of all irreducible characters of a finite group G in Nilp_n, and the orbits of the coadjoint action of G on the Pontryagin dual $\mathfrak{g}^*$ of the additive group of the Lie algebra $\mathfrak{g} = \log\ G$; as usual the coadjoint G-action is defined by $(g\lambda)(x) = \lambda(gxg^{-1})$ for all $g \in G$, all $\lambda \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$. The following theorem of M. Lazard can be found in [Laz54].

Theorem 1.3.2. *Let G be a finite group in* Nilp_n*, and let* $\mathfrak{g} = \log\ G$ *be the corresponding Lie algebra in* $\mathfrak{nilp}_n$*. Then, G acts by conjugation on $\mathfrak{g}$ and by the coadjoint action on the linear dual $\mathfrak{g}^*$. If Ω is a G-orbit on $\mathfrak{g}^*$, then the function $\chi_\Omega : G \to \mathbb{C}$ given by*

$$\chi_\Omega(g) = \frac{1}{\sqrt{|\Omega|}} \sum_{\lambda \in \Omega} \lambda(\log(g)), \qquad g \in G,$$

is an irreducible character of G. Furthermore, every irreducible character of G is of this form, and two characters χ_Ω and $\chi_{\Omega'}$ associated with coadjoint G-orbits Ω and Ω', respectively, are equal if and only if $\Omega = \Omega'$. Therefore, there is a bijection between irreducible characters of G and G-orbits on $\mathfrak{g}^*$.

In [BS08], M. Boyarchenko and M. Sabitova extended the orbit method to the case where G is a profinite group (that is, a topological group that is isomorphic to the inverse limit of discrete finite groups; see Example 2.1.2 in chapter 2). Here, we denote by $\mathrm{Fun}(G)$ (resp., $(\mathrm{Fun}(\mathfrak{g}))^G$) the set consisting of all functions $G \to \mathbb{C}$ (resp., $\mathfrak{g} \to \mathbb{C}$); moreover, $\mathrm{Fun}(G)^G$ (resp., $\mathrm{Fun}(\mathfrak{g})$) are the subsets consisting of all functions which are invariant for the conjugacy action of G (resp., for the coadjoint action of G).

Theorem 1.3.3. *Let G be a profinite group, and suppose that there exist an abelian profinite group $\mathfrak{g}$ and a homeomorphism $\exp\colon \mathfrak{g} \to G$ such that:*

(1) For each $g \in G$, the mapping $x \mapsto \log(g \exp(x) g^{-1})$ defines a group automorphism (where $\log$ is the inverse of $\exp$).

(2) The pullback map $\exp^* : \mathrm{Fun}(G)^G \to \mathrm{Fun}(\mathfrak{g})^G$ (defined by $f \mapsto f \circ \exp$) commutes with the group convolution.

Then, every G-orbit on the Pontryagin dual $\mathfrak{g}^*$ of $\mathfrak{g}$ is finite. Furthermore, there is a bijection between the orbits on $\mathfrak{g}^*$ and the irreducible characters of G such that, for each G-orbit Ω, the irreducible character χ_Ω associated with Ω is given by the formula

$$\chi_\Omega(g) = \frac{1}{\sqrt{|\Omega|}} \sum_{\lambda \in \Omega} f \circ \lambda(\log(g)), \qquad g \in G.$$

The methods above can be applied to any algebra group over a finite field with sufficiently large characteristic, and also to any algebra group over the ring of integers of a finite extension of a p-adic field. In both of these cases, the theorems above show that the character degrees are equal to the square root of the size of the corresponding orbit on the dual of the associated nilpotent algebra. (Indeed, it can be proven that the size of a coadjoint orbit is a full square; see for example [CG04]). However, it is well-known that the orbit method does not work in the general case; see the papers [IK98] by I.M. Isaacs, and also [JZ04] by A. Jaikin-Zapiran.

Although the formulas above give the value of the irreducible characters, it is not obvious how we can obtain the corresponding irreducible representations. One way of doing this, was suggested by A.A. Kirillov; for a similar construction, see also [Dix77]. Let $\mathfrak{g}$ be an arbitrary nilpotent Lie algebra over a field $\Bbbk$, and let $f \in \mathfrak{g}^*$ where $\mathfrak{g}^*$ denotes the linear dual of $\mathfrak{g}$. By a *polarization* of $\mathfrak{g}$ at f we mean a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $f([\mathfrak{h}, \mathfrak{h}]) = 0$, and such that $\mathfrak{h}$ is maximal among all vector subspaces of $\mathfrak{g}$ satisfying this property. It can be proven that such Lie subalgebra always exist; see [Kir62] or [Dix77]. In fact, in [And], C. André observed that, if $\mathfrak{g}$ is an associative nilpotent algebra, then such a polarisation can be chosen to be a subalgebra of $\mathfrak{g}$.

To conclude this section, we note that, if $\mathfrak{g}$ is a finite-dimensional nilpotent Lie algebra over a finite field $\mathbb{F}_q$, $f \in \mathfrak{g}^*$ is a linear functional of $\mathfrak{g}$, and λ is a non trivial linear character of the additive group $\mathbb{F}_q^+$ of $\mathbb{F}_q$, then the composition $\lambda \circ f : \mathfrak{g} \to \mathbb{C}^\times$ is a linear character of the additive group $\mathfrak{g}^+$ of $\mathfrak{g}$. Moreover, if $\mathfrak{h}$ is a polarization of $\mathfrak{g}$ at f, then the formula

$$\lambda_f(h) = (\lambda \circ f \circ \log)(h) \qquad h \in \exp(\mathfrak{h}),$$

defines a linear character of the group $\exp(\mathfrak{h})$; in [Kaz77], D. Kazhdan adapted a result of A.A. Kirillov (see [Kir62]) to prove the following result.

Theorem 1.3.4. *Let G be a finite group in Nilp_n, and let $\mathfrak{g} = \log(G)$. Let $f \in \mathfrak{g}^*$, and let Ω be the coadjoint G-orbit which contains f. Then, if $\mathfrak{h}$ is a polarization of $\mathfrak{g}$ at f, the induced representation*

$$V_f = \mathrm{Ind}_{\exp(\mathfrak{h})}^G(\mathbb{C}_{\widetilde{f}})$$

is irreducible and affords the character χ_Ω of G.

Chapter 2

Smooth Representations of ℓ-groups

In this section, we shall review some basic theory of ℓ-groups and of their representations; our main reference is [BH06].

2.1 ℓ-groups

Roughly, we can separate the theory of locally compact topological groups into two different classes: the theory of locally compact totally disconnected groups on the one hand, and the theory of Lie groups on the other. Let G be a locally compact group, and let G° denote the connected component of the identity (hence, G° is the (unique) maximal connected subgroup of G); for basic notions of topological groups, we refer to [DE14]. Then, we have the short exact sequence:

$$1 \to G^\circ \to G \to G/G^\circ \to 1$$

Since G° is a closed subgroup of G and also a connected locally compact subgroup, the quotient G/G° is a Hausdorff, locally compact totally disconnected group. By the Gleason-Yamabe theorem (see [Gle51] and [Yam53]), we know that G° has a normal compact subgroup K such that the quotient G°/K is isomorphic to a Lie group.

In this thesis, we shall focus on the first class of groups, that is, topological groups which are Hausdorff, locally compact and totally disconnected. In the literature (see for example [BH06]), these are sometimes called *locally profinite groups* (see Example 2.1.2 bellow) or *l.c.t.d. groups*; however, we shall use the terminology introduced by M. Boyarchenko in [Boy11], and refer to them as *ℓ-groups*. Therefore, by an *ℓ-group* we mean a topological group G such that, for the underlying topology, G is an Hausdorff, locally compact and totally disconnected topological space. (We observe that every totally disconnected group is Hausdorff; indeed, for every point is closed.)

One major difference between ℓ-groups and Lie groups is that, for any Lie group, there is always an open neighbourhood of the identity containing only the trivial subgroup. By the way of example, if we consider the additive Lie group $\mathbb{R}^+$ of the real numbers, then the open interval $]-1, 1[$ does not contain any non-trivial subgroup. However, in the case of ℓ-groups, it occurs the opposite situation: any open neighbourhood of the identity always contains an open (hence, closed) compact subgroup. In fact, this property can be considered as an alternative definition of an ℓ-group (see [BH06, Section 1.1]).

Theorem 2.1.1. *If G is a Hausdorff topological group, then the following are equivalent:*

- *G is an ℓ-group.*
- *Every neighbourhood of the identity contains an open compact subgroup of G.*

Example 2.1.2. Let $(I, \leq)$ be a directed partially ordered set, and let $\{G_i\}_{i\in I}$ be a family of finite groups. Suppose that there exists a group homomorphism $f_{i,j} : G_j \to G_i$ whenever $i, j \in I$ are such that $i \leq j$, and that the following conditions are satisfied:

(1) $f_{i,i}$ is the identity map for all $i \in I$.

(2) $f_{i,k} = f_{i,j} \circ f_{j,k}$ for all $i, j, k \in I$ with $i \leq j \leq k$.

(Hence, the pair $\{\{G_i\}_{i\in I}, \{f_{i,j}\}_{i\leq j}\}$ is an *inverse system* of finite groups). Let $\varprojlim G_i$ be the subset of the direct product $\prod_{i\in I} G_i$ consisting of all sequences $(g_i)_{i\in I}$ satisfying $g_i = f_{i,j}(g_j)$ for all $i, j \in I$ with $i \leq j$; then, $\varprojlim G_i$ is in fact a subgroup of $\prod_{i\in I} G_i$. If we endow each G_i with the discrete topology and consider the product topology in $\prod_{i\in I} G_i$, then $\varprojlim G_i$ is a closed subset of $\prod_{i\in I} G_i$; inded, $\varprojlim G_i$ becomes a compact totally disconnected group.

In a more concrete example, if we fix a prime number p and take $I = \mathbb{N}$ (with the usual ordering), then we may set $G_i = \mathbb{Z}/p^i\mathbb{Z}$ for all $i \in \mathbb{N}$, and define the group homomorphism $f_{i,j} : \mathbb{Z}/p^j\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$, for all $i, j \in \mathbb{N}$ with $i \leq j$, by $f_{i,j}(n + p^j\mathbb{Z}) = n + p^i\mathbb{Z}$ for all $n \in \mathbb{Z}$; then,

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^i\mathbb{Z}$$

is (isomorphic to) the usual ring of the p-adic integers (notice that $\mathbb{Z}_p$ is indeed a ring).

Example 2.1.3. One important class of ℓ-groups are the matrix groups over p-adic fields (that is, finite extensions of the field $\mathbb{Q}_p$ of p-adic numbers). We recall briefly the definition of $\mathbb{Q}_p$, where p is a fixed prime number. For every non-zero rational number $x \in \mathbb{Q}^\times$, there exist unique integers $m, n, r \in \mathbb{Z}$, with m and n relatively prime and not divisible by p, such that $x = p^r\frac{m}{n}$ then, we define $|x|_p = p^{-r}$. If we set $|0|_p = 0$, then we obtain a real-valued function defined on $\mathbb{Q}$ to which we refer as the *p-adic absolute value* on $\mathbb{Q}$. This function satisfies the following properties for all $x, y \in \mathbb{Q}$:

(1) $|x|_p \geq 0$, and $|x|_p = 0$ if and only if $x = 0$,

(2) $|xy|_p = |x|_p|y|_p$,

(3) $|x + y|_p \leq (|x|_p + |y|_p)$;

therefore, the p-adic absolute value $|.|_p$ is a *non-Archimedean valuation*. The p-adic absolute value induces naturally a (metric) topology in $\mathbb{Q}$ with an open basis consisting of all open balls

$$B(x, \varepsilon) = \{y \in \mathbb{Q} \mid |x - y|_p < \varepsilon\}$$

where $x \in \mathbb{Q}$ and ε is any positive number. We define the field of p-adic numbers $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to the p-adic absolute value hence, by definition, $\mathbb{Q}_p$ is the smallest field containing $\mathbb{Q}$ which is complete with respect to a non-Archimedean valuation whose restriction to $\mathbb{Q}$ equals $|.|_p$ (since there is no danger of ambiguity, we will also denote by$|.|_p$ the valuation on $\mathbb{Q}_p$).

It can be proved that $\{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$ is a subring of $\mathbb{Q}_p$, and that it is isomorphic to $\mathbb{Z}_p$ (as defined above); hence, we will always identify $\mathbb{Z}_p$ with this subring of $\mathbb{Q}_p$, that is,

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}.$$

It is easy to see that $\mathbb{Z} \subseteq \mathbb{Z}_p$, and that $\mathbb{Z}_p$ is a discrete valuation ring with (unique) maximal ideal $p\mathbb{Z}_p$ and (finite) residue field isomorphic to $\mathbb{F}_p$. Moreover, $\mathbb{Q}_p$ is the quotient field of $\mathbb{Z}_p$; indeed, every element of $\mathbb{Q}_p$ can be written uniquely as a product $p^k u$ where $k \in \mathbb{Z}$ and $u \in \mathbb{Z}_p$ is a unit; the fractional ideals

$$p^k\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq p^{-k}\}, \qquad k \in \mathbb{Z},$$

are open subgroups of $\mathbb{Q}_p$, and form a basis of open neighbourhoods of 0 in $\mathbb{Q}_p$. In particular, it follows from Theorem 2.1.1 that $\mathbb{Q}_p$ is an ℓ-group (with respect to addition); notice also that $\mathbb{Q}_p$ is the union of all its compact open subgroups.

The previous examples can be generalised for an arbitrary *non-Archimedean local field*, that is, a field $\Bbbk$ which is endowed with a non-Archimedean valuation $\|\cdot\|$ such that $\Bbbk$ is complete and locally compact with respect to the metric topology induced by $\|\cdot\|$; thus, $\Bbbk$ is, either a finite extension of some p-adic field $\mathbb{Q}_p$ (if $\Bbbk$ has characteristic zero), or the field of formal Laurent series over some finite field $\mathbb{F}_q$ (if $\Bbbk$ has prime characteristic). (For detailed information on non-Archimedean local fields, we refer to [Cas86]; see also [Nar13].)

Example 2.1.4. Every non-Archimedean local field $\Bbbk$ is the quotient field of a discrete valuation ring $\mathfrak{o} = \mathfrak{o}_{\Bbbk}$. Let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$; hence, the residue class field $\mathfrak{o}/\mathfrak{p}$ is finite

(see [Cas86, Corollary on pg. 46]. If $q = |\mathfrak{o}/\mathfrak{p}|$ and $\varpi \in \mathfrak{o}$ is a prime element of $\Bbbk$ (that is, $\mathfrak{p} = \varpi\mathfrak{o}$, then every element of $\Bbbk$ can be written uniquely as a product $\varpi^k u$ where $k \in \mathbb{Z}$ and $u \in \mathfrak{u}$ is a unit. As in the case of $\mathbb{Q}_p$, the fractional ideals $\varpi^k\mathfrak{o}$, for $k \in \mathbb{Z}$, are open compact subgroups of $\Bbbk$, and form a basis of open neighbourhoods of 0 in $\Bbbk$. Therefore, Theorem 2.1.1 implies that $\Bbbk$ is an ℓ-group (with respect to addition); we also note that $\Bbbk$ is the union of all its compact open subgroups.

On the other hand, let us consider of k-vector space $\mathcal{M}_n(\Bbbk)$ consisting of all $n \times n$ matrices with coefficients in $\Bbbk$. If we endow $\mathcal{M}_n(\Bbbk)$ with the topology induced naturally by the topology of $\Bbbk$, then $\mathcal{M}_n(\Bbbk)$ is an ℓ-group (with respect to addition): the subsets $\varpi^k\mathcal{M}_n(\mathfrak{o})$, for $k \in \mathbb{Z}$, are open compact subgroups of $\mathcal{M}_n(\Bbbk)$, and form a basis of open neighbourhoods of $0 \in \mathcal{M}_n(\Bbbk)$. As before, $\mathcal{M}_n(\Bbbk)$ is the union of all its compact open subgroups.

Example 2.1.5. As in the previous example, let $\Bbbk$ be a non-Archimedean local field with ring of integers $\mathfrak{o} = \mathfrak{o}_\Bbbk$ and prime element $\varpi \in \mathfrak{o}$. Then, the multiplicative group $\Bbbk^\times$ is also an ℓ-group: the congruence unit groups $1 + \varpi^k\mathfrak{o}$, for $n \in \mathbb{N}$, are compact open subgroups, and form a basis of open neighbourhoods of $1 \in \Bbbk^\times$. However, in this situation, the union of the open compact subgroups is the (proper) subgroup $(1+\varpi\mathfrak{o}) \times \mu_\Bbbk$ where $\mu_\Bbbk$ denotes the group of roots of the unity in $\Bbbk$ with order not divisible by the characteristic of the residue field; note that $(1+\varpi\mathfrak{o}) \times \mu_\Bbbk$ is the unit group of $\mathfrak{o}$. Therefore, the multiplicative group $\Bbbk^\times$ has a maximal open compact subgroup, and is isomorphic to the direct product of this maximal compact subgroup and the additive group $\mathbb{Z}^+$ of the integers (equipped with the discrete topology).

More generally, the general linear group $\mathrm{GL}_n(\Bbbk)$ (consisting of all invertible matrices of $\mathcal{M}_n(\Bbbk)$) is an open subset of $\mathcal{M}_n(\Bbbk)$; since inversion of matrices is continuous (because it is given by a polynomial map), $\mathrm{GL}_n(\Bbbk)$ is a topological group. The subgroups $\mathrm{GL}_n(\mathfrak{o})$ and $1 + \varpi^k\mathcal{M}_n(\mathfrak{o})$, for $k \in \mathbb{N}$, are compact open, and form a basis of open neighbourhoods of $1 \in \mathrm{GL}_n(\Bbbk)$. Therefore, $\mathrm{GL}_n(\Bbbk)$ is an ℓ-group.

Other examples of ℓ-groups are:

- the (standard) Borel subgroup $B_n(\Bbbk)$ of $\mathrm{GL}_n(\Bbbk)$ consisting of all invertible uppertriangular matrices;
- the upper unitriangular subgroup $U_n(\Bbbk)$ of $\mathrm{GL}_n(\Bbbk)$ consisting of all unipotent matrices in $B_n(\Bbbk)$;
- the maximal torus $T_n(\Bbbk)$ consisting of all diagonal matrices in $B_n(\Bbbk)$.

We note that $U_n(\Bbbk)$ is a normal subgroup of $B_n(\Bbbk)$, and that $B_n(\Bbbk)$ is the semidirect product $B_(\Bbbk) = T_n(\Bbbk) \ltimes U_n(\Bbbk)$; moreover, $T_n(\Bbbk)$ is isomorphic to a direct product of n copies of the

multiplicative group $\mathbb{k}^\times$. Furthermore, we mention that the union of all open compact subgroups of $B_n(\mathbb{k})$ is the subgroup consisting of all uppertriangular matrices whose diagonal entries lie in $\mathfrak{o}$, and hence $B_n(\mathbb{k})$ is isomorphic to the semidirect product of an ℓ_c-group with a discrete group.

Following [Boy11], we say that an ℓ-group G is an *ℓ_c-group* if it is a filtered union of its compact open subgroups; that is, if every element $g \in G$ lies in some open compact subgroup, and if any two open compact subgroups are both contained in some (other) open compact subgroup. As a standard example, the unitriangular group $U_n(\mathbb{k})$ over a non-Archimedean local field $\mathbb{k}$ is an ℓ_c-group: if $\mathfrak{o}$ is the ring of integers of $\mathbb{k}$ and $\varpi \in \mathfrak{o}$ is the prime element of $\mathbb{k}$, then the subgroups

$$U^k = \{g \in U_n(\mathbb{k}) \mid g_{i,j} \in \varpi^{(j-i)k}\mathfrak{o}\}, \qquad k \in \mathbb{N},$$

form a basis of open compact subgroups satisfying $U^{k+1} \subsetneq U^k$ such that $U_n(\mathbb{k}) = \bigcup_{k \in \mathbb{N}} U^k$.

In this thesis, we will only consider topological groups which are second countable (that is, the underlying topology has a countable basis of open subsets). In what follows, we state some well-known theorems that will be used throughout the text. We start with Baire category theorem; see for example [Gem90, Proposition 12 in Chapter 10]. (We recall that a subset of a topological space is said to be *nowhere-dense* if its closure has empty interior.)

Theorem 2.1.6 (Baire Category Theorem). *Let X be a locally compact Hausdorff topological space. Then, X is not the countable union of nowhere-dense subsets of X.*

As a first elementary application of Baire category theorem, we show the following.

Proposition 2.1.7. *Let G and H be second countable ℓ-groups, and let $\phi\colon G \to H$ be a continuous epimorphism. Then, ϕ is an open map.*

Proof. Let U be an open compact subgroup of G. Since G is second countable, the quotient space G/U is countable, and thus there is a countable subset $X \subset G$ such that G is the disjoint union

$$G = \bigcup_{x \in X} xU.$$

Since ϕ is an epimorphism, we have

$$H = \bigcup_{x \in X} \phi(x)\phi(U).$$

Since U is compact, the cosets $\phi(x)\phi(U) = \phi(xU)$, for $x \in X$, are also compact, and so they are closed. By Baire Category Theorem, there must exist $x \in X$ such that $\phi(x)\phi(U)$ has

non-empty interior. Since left multiplication by $\phi(x)$ is an homeomorphism, we conclude that $\phi(x)\phi(U)$ has non-empty interior. Finally, if V is a non-empty open subset of $\phi(U)$, then

$$\phi(U) = \bigcup_{h\in\phi(U)} hV,$$

and so $\phi(U)$ is open in H. The result follows from the fact that the open compact subgroups of G form a basis of open neighbourhoods of the identity. □

From Proposition 2.1.7 it is straightforward to deduce that the usual isomorphism theorems are in fact homeomorphisms (for second countable ℓ-groups):

- If G and H are second countable ℓ-groups and $\phi : G \to H$ is a continuous epimorphism, then the canonical isomorphism $\widetilde{\phi} : G/\ker(\phi) \to H$ is an homeomorphism.
- If G is a second countable ℓ-group, H is a closed subgroup of G and N is a normal closed subgroup of G, then HN is a closed subgroup of G and the canonical isomorphism

  $$HN/H \simeq H/H \cap N$$

  is an homeomorphism.
- If G and H are second countable ℓ-groups, $\phi : G \to H$ is a continuous epimorphism and N is a closed normal subgroup of H, then the canonical isomorphisms

  $$G/\phi^{-1}(N) \simeq H/N \simeq (G/ker(\phi))/(\phi^{-1}(N)/ker(\phi))$$

  are homeomorphisms.

The following result will be fundamental for our work.

Theorem 2.1.8 (Orbit-Stabiliser Theorem)**.** *Let G be a second countable ℓ-group which acts continuously on (the left of) a locally compact Hausdorff topological space X. For any $x \in X$, let*

$$C_G(x) = \{g \in G \mid g \cdot x = x\}$$

be the stabiliser of x in G. Then, the mapping $g \mapsto g \cdot x$ induces naturally an homeomorphism $G/C_G(x) \to G \cdot x$ where $G \cdot x$ is the G-orbit of x.

Proof. It is enough to prove that the map is open. Let U be an open subset of G, and let K be an open compact subgroup of G which is contained in U (K exists because G is an ℓ-group).

Since G is second countable, there exists a countable subset Y of G such that

$$G = \bigcup_{y \in Y} yK.$$

Then,

$$G \cdot x = \bigcup_{y \in Y} (yK) \cdot x,$$

where the sets $(yK) \cdot x$, for $y \in Y$, are all homeomorphic to $K \cdot x$ and $K \cdot x$ is a compact (hence, a closed) subset of X. By the Baire Category Theorem, we conclude that $K \cdot x$ has non-empty interior, and so $K \cdot x$ must be open because, if $V \subset K \cdot x$ is open, then

$$K \cdot x = \bigcup_{k \in K} k \cdot V.$$

Therefore, $U \cdot x$ is open, as we wanted. $\square$

If X is a topological space and $Y \subseteq X$ is a subset, we say that Y is *locally closed* if it is the intersection of a closed subset and an open subset (equivalently, if Y is open in its closure). Locally closed subsets are of particular importance. From the Orbit-Stabiliser Theorem, we deduce the following result.

Proposition 2.1.9. *Let G be a second countable ℓ-group which acts continuously on a locally compact Hausdorff topological space X. Then, for any $x \in X$, the orbit $G \cdot x$ is a locally closed subset of X.*

Proof. By the Orbit-Stabiliser Theorem, we have an homeomorphism $G/C_G(x) \to G \cdot x$. Therefore, since G is locally compact, the quotient space $G/C_G(x)$ is also locally compact (because $C_G(x)$ is closed), and so the orbit $G \cdot x$ is also locally compact. It follows that $G \cdot x$ is open in its closure, and so it is a locally closed subset of X. $\square$

2.2 Smooth representations

Roughly, the main purpose of (group) representation theory is to understand the ways in which a group G can act linearly on vector spaces (subject to various appropriate hypotheses). An *(abstract) representation* of an arbitrary group G is a pair (π, V) where V is a vector space (over some field) and $\pi : G \to \mathrm{GL}(V)$ is a group homomorphism where $\mathrm{GL}(V)$ denotes the group of all linear automorphisms of V; we do not require V to be finite-dimensional, but we will always assume that V is a complex vector space. For simplicity, we will refer to V as a

(left) G-module with (linear) G-action defined by

$$g \cdot v = \pi(g)v, \qquad g \in G,\ v \in V.$$

In general, for a topological group G, it is natural to consider representations which preserve the topological nature of G; in our situation this can be achieved by requiring that the given representation is smooth. A G-module V is said to be *smooth* if the group stabiliser

$$C_G(v) = \{g \in G \mid gv = v\}$$

is open in G for all $v \in V$; in some literature, smooth representations are also referred to as "algebraic representations" (see [BZ76]). The following proposition clarifies the term "smooth"; see [BH06, Section 2.1].

Proposition 2.2.1. *Let G be an ℓ-group, and let V be an G-module. Then the following are equivalent:*

(1) V is a smooth G-module.

(2) For all $v \in V$, there exists an open compact subgroup K of G such that $kv = v$ for all $k \in K$ (that is, such that v is fixed by K).

(3) If V is endowed with the discrete topology, then the action map $\alpha\colon G \times V \to V$ (given by the mapping $(g, v) \mapsto g \cdot v$) is continuous.

Proof. From the definition, we know that the module V is smooth if and only if, for all $v \in V$, the group stabiliser $C_G(v) = \{g \in G \mid gv = v\}$ is open in G. On the other hand, since G is an ℓ-group, a subgroup is open if and only if it contains an open compact subgroup. This proves that (1) and (2) are equivalent.

Next, suppose that V is a smooth G-module, and choose $v \in V$. Then,

$$\alpha^{-1}(v) = \{(g, w) \in G \times V \mid g \cdot w = v\},$$

and note that $C_G(v) \times \{v\} \subseteq \alpha^{-1}(v)$. Since V is smooth and discrete, we conclude that $C_G(v) \times v$ is open in $G \times V$. If $(g, w) \in \alpha^{-1}(v)$, then $(g, w) \in gC_G(v) \times \{v\} \subseteq \alpha^{-1}(v)$, and thus $\alpha^{-1}(v)$ is the union of open sets (note that $gC_G(v)$ is also open). It follows that $\alpha^{-1}(v)$ is open, and hence

$$\alpha^{-1}(U) = \bigcup_{v \in U} \alpha^{-1}(v)$$

is open for all (open) subset U of V. Therefore, the action map is continuous.

Finally, suppose that α is continuous, and let $v \in V$. Since $\{v\}$ is open in V, $\alpha^{-1}(v)$ is open in $G \times V$, and hence $\alpha^{-1}(v) \cap (G \times \{v\})$ is open in $G \times \{v\}$. Since G is homeomorphic to $G \times \{v\}$ and

$$\alpha^{-1}(v) \cap (G \times \{v\}) = C_G(v) \times \{v\},$$

we conclude that $C_G(v)$ is open, and this implies that the G-module V is smooth. □

As for abstract representations, a *homomorphism* between two smooth G-modules V and W of an ℓ-group, is a linear map $\phi\colon V \to W$ such that

$$\phi(gv) = g\phi(v), \qquad g \in G,\ v \in V;$$

we will denote by $\mathrm{Rep}(G)$ the category of smooth representations of G.

Given an ℓ-group G, a G-module V of G may obviously fail to be smooth; however, V always contains a submodule which is smooth; the notion of G-submodule is the usual one: a G-module V' is a *submodule* of V if V' is a G-invariant subspace of V (that is, $gV' \subseteq V'$ for all $g \in G$). Let

$$V^\infty = \{v \in V \mid C_G(v) \text{ is open}\}$$

be the vector subspace of V consisting of all *smooth vectors*; note that V^∞ is invariant for the action of G: if $v \in V^\infty$, then $C_G(v)$ is open, and so $C_G(gv) = gC_G(v)g^{-1}$ is also open, which means that $gv \in V^\infty$ for all $g \in G$. Therefore, we have a smooth G-submodule V^∞ of V. We also note that, if $\phi\colon V \to W$ is a homomorphism of abstract representations, then ϕ sends smooth vectors of V to smooth vectors of W (because $C_G(v) \subseteq C_G(\phi(v))$, and so $C_G(\phi(v))$ is open). We refer to V^∞ as the *smoothing* of V; notice that the mapping $V \mapsto V^\infty$ defines a functor from the category of (abstract) G-modules to the category of smooth G-modules.

Proposition 2.2.2 ([BH06, Exercises (1)-(3), Section 2.3])**.** *Let G be an ℓ-group, let $\mathrm{Rep}(G)$ denote the category of smooth G-modules, and let $\mathrm{ARep}(G)$ denote the category of (abstract) G-modules. Then, the smoothing functor $V \mapsto V^\infty$ is additive and left exact; moreover, it is right adjoint to the inclusion functor $F : \mathrm{Rep}(G) \to \mathrm{ARep}(G)$.*

Proof. The proof is straightforward. □

Example 2.2.3. Let $f\colon \mathbb{Q}_p \hookrightarrow \mathbb{C}$ be an arbitrary injective map, and let V be the complex vector space linearly spanned by the set $\{x \cdot f \mid x \in \mathbb{Q}_p\}$ where

$$(x \cdot f)(y) = f(x + y), \qquad x, y \in \mathbb{Q}_p.$$

If we define $\pi\colon \mathbb{Q}_p \to \mathrm{GL}(V)$ by

$$\pi(x)(y \cdot f) = (x+y) \cdot f, \qquad x, y \in \mathbb{Q}_p,$$

then V becomes a $\mathbb{Q}_p$-module which is not smooth; indeed, the stabiliser $C_{\mathbb{Q}_p}(f)$ is the trivial subgroup 0 of $\mathbb{Q}_p$ (hence, it is not open). Furthermore, one can see that $V^\infty = 0$, and so

$$\mathrm{Hom}_{\mathrm{ARep}(\mathbb{Q}_p)}(W, V) = \mathrm{Hom}_{\mathrm{ARep}(\mathbb{Q}_p)}(W, V^\infty) = 0$$

for every smooth $\mathbb{Q}_p$-module W (by the right adjuntion of the Proposition 2.2.2).

If V is a smooth G-module of an ℓ-group G, then every every G-submodule V' of V is also smooth; on the other hand, the quotient vector space V/V' defines naturally a G-module V/V' of G which is also smooth. We say that a smooth G-module V of an ℓ- group G is *irreducible* if $V \neq 0$ and if V and 0 are the only G-invariant subspaces of V. Note that if V is any irreducible G-module and $v \in V \setminus 0$ is fixed, then V is linearly spanned by the set $\{gv \mid g \in X\}$ where X is a complete set of representatives of left cosets of $C_G(v)$ in G. Since $C_G(v)$ is open, if G is second countable, then X is a countable set, and so we conclude that the dimension of any irreducible representation of a second countable ℓ-group is countable.

In the case of finite groups, it is well-known that every representation (over the complex field) has an irreducible sub-representation, and that (by Maschke's Theorem) every representation decomposes as a direct sum of irreducible representations (that is, every representation is semisimple). However, in the case of smooth representation of ℓ-groups this is not always the case.

Example 2.2.4. Let $\pi\colon \mathbb{Q}_p^\times \to \mathrm{GL}(\mathbb{C}^2)$ be the map defined by

$$\pi(a) = \begin{bmatrix} 1 & \log(|a|_p) \\ 0 & 1 \end{bmatrix}, \qquad a \in \mathbb{Q}_p^\times.$$

Then, $\mathbb{C}^2$ with the action defined by π becomes a smooth $\mathbb{Q}_p^\times$-module, and

$$W = \{(x, 0) \in \mathbb{C}^2 \mid x \in \mathbb{C}\}$$

is a $\pi(\mathbb{Q}_p^\times)$-invariant vector subspace of $\mathbb{C}^2$. Suppose that W has a complement W'. Then W' must be spanned by a vector (x, y) with $y \neq 0$, and we have

$$a \cdot (x, y) = (x + \log(|a|_p y), y), \qquad a \in \mathbb{Q}_p^\times.$$

Since W' has to be $\mathbb{Q}_p^\times$-invariant, there must exist a map $\lambda\colon \mathbb{Q}_p^\times \to \mathbb{C}$ such that

$$a \cdot (x,y) = \lambda(a)(x,y) = (x + \log(|a|_p y), y), \qquad a \in \mathbb{Q}_p^\times.$$

Since $y \neq 0$, it follows that $\lambda(a) = 1$ and $\log(|a|_p) = 1$ for all $a \in \mathbb{Q}_p^\times$, a contradiction. Therefore, the smooth $\mathbb{Q}_p^\times$-module $\mathbb{C}^2$ is not semisimple.

The lack of irreducible subrepresentations forces to use subquotients.

Proposition 2.2.5. *Let G be an ℓ-group, and let V be a smooth G-module. Then,*

(1) There exist G-submodules W and W' of V with $W \subseteq W'$ such that the quotient W'/W is irreducible.

(2) If V is finitely generated, then there exists a G-submodule W of V such that the quotient V/W is irreducible.

Proof. Note that (1) follows from (2) applied to the submodule of V generated by a non-zero vector $v \in V$.

For (2), we consider the set $\mathcal{P}$ consisting of all G-submodules of V ordered by inclusion. Let $\{W_i \mid i \in I\}$ be a totally ordered set of $\mathcal{P}$; we claim that the union $\bigcup_{i\in I} W_i$ lies in $\mathcal{P}$. Suppose that $V = \bigcup_{i\in I} W_i$ and let $v_1, ..., v_n$ be the generators of V. Since $\{W_i \mid i \in I\}$ is totally ordered, there exists $i \in I$ such that $v_1, ..., v_n \in W_i$. Therefore, we must have $W_i = V$, a contradiction. By Zorn's lemma, it follows that there exists a maximal G-submodule W of V, and hence the quotient V/W is irreducible. □

In some cases smooth representations decompose as a direct sum of its irreducible subrepresentations; the proof is straightforward.

Proposition 2.2.6 ([BH06, Proposition 2.2, pg. 14])**.** *Let V be a smooth G-module of an ℓ-group G Then the following are equivalent:*

(1) V decomposes as a direct sum of irreducible G-submodules.

(2) V decomposes as the sum of all its irreducible submodules G-submodules.

(3) Every G-submodule of V has a G-invariant complement.

Example 2.2.7. Let G be a compact ℓ-group (hence, a profinite group), and let V be an irreducible smooth G-module. Let $v \in V$ be an arbitrary non-zero vector, and let X be a complete set of representatives of the quotient space $G/C_G(v)$. Since G is compact and $C_G(v)$ is open, the set X must be finite, and thus V is finitely generated by the set $\{g \cdot v : g \in X\}$ (because V is irreducible). Therefore, every irreducible smooth G-module of a compact ℓ-group is finite-dimensional.

The representation theory of compact groups is very similar to the representation theory of finite groups. In fact, every smooth representation of a compact group is semisimple; the following proposition also shows that, although the smooth modules of an ℓ-group are not in general semisimple, they can be restricted to the open compact subgroups in order to obtain semisimplicity.

Proposition 2.2.8 ([BH06, Lemma 2.2, pg. 14]). *Let K be a compact ℓ-group, and let V be a smooth K-module. Then V is semisimple.*

Proof. Let $\widetilde{V}$ be the sum of all irreducible K-invariant subspaces of V, suppose that $\widetilde{V} \neq V$, and let $v \in V \setminus \widetilde{V}$. Let W be the K-invariant subspace of V generated by v. Then, as in the previous example, we conclude that W is finite-dimensional, and so it decomposes as a direct sum of irreducible K-invariant subspaces. It follows that $W \subseteq \widetilde{V}$, a contradiction (because $v \in W$). □

The simplest case of modules of any group G are the modules of dimension one; note that such a representation is essentially a homomorphisms $\lambda \colon G \to \mathbb{C}^\times$, and so a one-dimensional module is smooth if and only if the kernel of λ is open. We also have the following characterisation of one-dimensional smooth modules.

Proposition 2.2.9 ([BH06, Proposition 1.6, pg. 10]). *Let G be a ℓ-group and let $\mathbb{C}_\lambda$ denote the canonical one-dimensional module of G which is associated with a homomorphism $\lambda \colon G \to \mathbb{C}^\times$. Then, the following are equivalent:*

(1) λ is continuous.

(2) The G-module $\mathbb{C}_\lambda$ is smooth.

Furthermore, if λ satisfies these conditions and G is the union of its compact open subgroups (in particular, if G is an ℓ_c-group), then the image of λ is contained in the unit circle of $\mathbb{C}$.

Proof. Suppose that λ is continuous, and let $V \subseteq \mathbb{C}^\times$ be an open neighbourhood of $1 \in \mathbb{C}^\times$ which does not contain any non-trivial subgroup of $\mathbb{C}^\times$, Then $\lambda^{-1}(V)$ open in G, and so it contains an open compact subgroup K of G. Since $\lambda(K) \subseteq V$ is a subgroup of $\mathbb{C}^\times$, we must have $\lambda(K) = 1$. Since $K \subseteq \ker(\lambda)$, we conclude that $\ker(\lambda)$ is open (hence, $\mathbb{C}_\lambda$ is smooth).

Conversely, suppose that $\ker(\lambda)$ is open, and let $V \subseteq \mathbb{C}^\times$ be an open subset. Since the inverse image $\lambda^{-1}(V)$ is a union of left cosets of $\ker(\lambda)$ in G, it is an open subset of G, and this proves that λ is continuous.

For the last assertion, note that the unit circle of $\mathbb{C}$ is the unique maximal compact subgroup of $\mathbb{C}^\times$. If K is a compact subgroup of G, then $\lambda(K)$ is a compact subgroup of $\mathbb{C}^\times$, and so it

must be contained in the unit circle. Since G is the union of its open compact subgroups, we conclude that the image of λ is contained in the unit circle, as required. $\square$

If G is an ℓ- group and $\lambda\colon G \to \mathbb{C}^\times$ is a continuous group homomorphism, then we will refer to λ as a *smooth character* of G.

Example 2.2.10. If $\mathbb{k}$ is a non-Archimedean local field, then its norm $\|\cdot\|$ clearly defines a smooth character of the multiplicative group $\mathbb{k}^\times$ whose kernel equals the congruence unit group $1+\mathfrak{p}$ where $\mathfrak{p}$ is the (unique) maximal ideal of the ring of integers of $\mathbb{k}$; note that $1+\mathfrak{p}$ is the maximal open compact subgroup of $\mathbb{k}$, and that $\|\cdot\|$ takes values outside the unit circle.

On the other hand, if λ is an arbitrary smooth character of $\mathbb{k}^\times$, then the composite $\lambda\circ \det\colon \mathrm{GL}_n(\mathbb{k}) \to C^\times$ is a smooth character of the ℓ-group $\mathrm{GL}_n(\mathbb{k})$ which may or may not take values outside the unit circle (depending on λ). Indeed, it can be proved that every finite-dimensional irreducible smooth $\mathrm{GL}_n(\mathbb{k})$-module is one-dimensional and corresponds to a smooth character $\lambda\circ\det$ for some smooth character λ of $\mathbb{k}^\times$.

The additive group $\mathbb{k}^+$ of a non-Archimedean local field is an ℓ_c-group, and so the image of every smooth character of $\mathbb{k}^+$ lies in the unit circle of $\mathbb{C}$. By the way of example, if $\|\cdot\|$ denotes the norm of $\mathbb{k}$, then the mapping $x \mapsto e^{2\pi i\|\cdot\|}$ defines a smooth character of $\mathbb{k}^+$. In fact, the character group of $\mathbb{k}^+$ (that is, the group consisting of all smooth characters of $\mathbb{k}^+$ equipped with the pointwise multiplication) has a very nice structure.

Proposition 2.2.11 ([BH06, Proposition 1.7, pg. 11]). *Let $\mathbb{k}$ be a non-Archimedean local field, let $\mathbb{k}^\circ$ denote the the character group of $\mathbb{k}^+$, and let $\lambda\in\mathbb{k}^\circ$ be a fixed non-trivial smooth character of $\mathbb{k}^+$. For every $y\in\mathbb{k}$, let $\lambda_y\colon \mathbb{k}\to\mathbb{C}$ be the map defined by*

$$\lambda_y(x) = \lambda(yx), \qquad x\in\mathbb{k}.$$

Then, the mapping $y\mapsto\lambda_y$ defines a group isomorphism between $\mathbb{k}^+$ and $\mathbb{k}^\circ$.

Proof. It is clear that the mapping $y\mapsto\lambda_y$ defines an injective group homomorphism from $\mathbb{k}^+$ to $\mathbb{k}^\circ$. In order to prove that it is also surjective, let $\mathfrak{o}=\mathfrak{o}_{\mathbb{k}}$ be the ring of integers of $\mathbb{k}$, let $\mathfrak{p}$ be the unique maximal ideal of $\mathfrak{o}$, and let ϖ be the prime element of $\mathbb{k}$ (hence, $\mathfrak{p}=\varpi\mathfrak{o}$). Let $\mu\in\mathbb{k}^\circ$ be arbitrary, and let $l\in\mathbb{Z}$ be the smallest nonnegative integer such that $\mathfrak{p}^l\subseteq\ker(\mu)$, and similarly let $d\in\mathbb{Z}$ be the smallest nonnegative integer such that $\mathfrak{p}^d\subseteq\ker(\lambda)$. For every $u\in\mathfrak{o}^\times$, the smooth character $\lambda_{\varpi^{d-l}u}$ is equal to μ on $\mathfrak{p}^l$ (because both characters equal to one on $\mathfrak{p}^l$). Moreover, that, for every $u,u'\in\mathfrak{o}^\times$, the smooth characters $\lambda_{\varpi^{d-l}u}$ and $\lambda_{\varpi^{d-l}u'}$ are equal on $\mathfrak{p}^{l-1}$ if and only if $u\equiv u' (\operatorname{mod}\mathfrak{p})$. If $q=|\mathfrak{o}/\mathfrak{p}|=|\mathfrak{p}^{l-1}/\mathfrak{p}^l|$, then the subgroup $\mathfrak{p}^{l-1}$ has $q-1$ non trivial smooth characters which are trivial on $\mathfrak{p}^l$. As u ranges through $\mathfrak{o}^\times/(1+\mathfrak{p})$ the $q-1$

smooth characters $\lambda_{\varpi^{d-l}u}$ are distinct and non-trivial on $\mathfrak{p}^{l-1}$, and are trivial on $\mathfrak{p}^l$. Therefore, there exists $u_1 \in \mathfrak{o}^\times$ such that $\lambda_{\varpi^{d-l}u_1}$ equals μ on $\mathfrak{p}^{l-1}$.

We can iterate this process, and find a sequence of elements $u_n \in \mathfrak{o}^\times$ such that $\lambda_{\varpi^{d-l}u_n}$ is equal to μ on $\mathfrak{p}^{l-n}$; furthermore, we have $u_{n+1} \equiv u_n (\mathrm{mod}\, \mathfrak{p}^n)$. Therefore, $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and so it converges to some $u \in \mathfrak{o}^\times$. It follows that $\mu\lambda_{\varpi^{d-l}u}$. □

The previous proposition is classical and means that every non-Archimedean local field is *self-dual*; it is well-known that finite fields are also self-dual (see [LN97]). The following result is an easy generalisation of the previous proposition.

Proposition 2.2.12. *Let V be a finite-dimensional vector space over a non-Archimedean local field $\Bbbk$, let V° denote the character group of the additive group V^+, and let $\lambda \in \Bbbk^\circ$ be a fixed non-trivial smooth character of $\Bbbk^+$. Moreover, let V^* denote the dual vector space of V and, for every $f \in V^*$, let $\lambda_f \colon V \to \mathbb{C}^\times$ be the map defined by*

$$\lambda_f(v) = \lambda(f(v)), \qquad v \in V.$$

Then, the mapping $f \mapsto \lambda_f$ defines a group isomorphism between V^* and V°.

Proof. Straightforward. □

Example 2.2.13. Let $\Bbbk$ be a non-Archimedean local field, and consider the unitriangular group $U_n(\Bbbk)$ (which we already know to be an ℓ_c-group). Let $1 \leq i < n$, let $\alpha \in k$, and define the map $\xi_{i,\alpha} : U_n(\Bbbk) \to \mathbb{C}^\times$ by

$$\xi_{i,\alpha}(x) = \lambda(\alpha x_{i,i+1}), \qquad x \in U_n(\Bbbk),\ x = (x_{i,j});$$

as above, $\lambda \in \Bbbk^\circ$ is a fixed non-trivial smooth character of the additive group $\Bbbk^+$. Then, $\xi_{i,\alpha}$ is a smooth character of $U_n(\Bbbk)$ which is nontrivial if and only if $\alpha \neq 0$. Furthermore, since the commutator subgroup $U_n(\Bbbk)' = [U_n(\Bbbk), U_n(\Bbbk)]$ consists of all $x \in U_n(\Bbbk)$ satisfying $x_{i,i+1} = 0$ for all $1 \leq i < n$ (hence, $U_n(\Bbbk)'$ is closed), and since the smooth characters of $U_n(\Bbbk)$ are in one-to-one correspondence with the smooth characters of the quotient group $U_n(\Bbbk)/U_n(\Bbbk)'$, we conclude that every non-trivial smooth character λ of $U_n(\Bbbk)$ decomposes as a product

$$\lambda = \prod_{i\in I} \xi_{i,\alpha_i}$$

for some non-empty subset $I \subseteq \{1, ..., n\}$ and some non-zero elements $\alpha_i \in \Bbbk^\times$, $i \in I$. (Note that $U_n(\Bbbk)/U_n(\Bbbk)' \simeq \Bbbk^+ \times ...\Bbbk^+$ ($n-1$ copies)).

A classical result in the theory of representations of finite groups is Schur's Lemma, in the case of smooth modules of ℓ-groups we have the following version of this classical result.

Lemma 2.2.14 (Schur's Lemma; [BH06, Section 2.6, pg. 21]). *Let G be an ℓ-group, and let V and W be irreducible G-modules. Then $\mathrm{Hom}_{\mathrm{Rep}(G)}(V, W)$ is zero if V is not isomorphic to W, and $\mathrm{Hom}_{\mathrm{Rep}(G)}(V, W) \simeq \mathbb{C}$ if V is isomorphic to W.*

Proof. Let $f : V \to W$ be a homomorphism of smooth representations, and note that the image $\mathrm{im}(f)$ is a G-submodule of W, whereas the kernel $\ker(f)$ is a G-submodule of V. Therefore, if $\ker(f) = 0$, then $\mathrm{im}(f)$ must be equal to W, and so f is an isomorphism. On the other hand, if $\ker(f) = V$, then f must be the zero map. In particular, it follows that

$$\mathrm{End}_{\mathrm{Rep}(G)}(V) = \mathrm{Hom}_{\mathrm{Rep}(G)}(V, V)$$

is a division algebra.

Now, let us fix $v \in V \setminus 0$; since V is irreducible, we must have $V = Gv$, and so every $f \in \mathrm{End}_{\mathrm{Rep}(G)}(V)$ is uniquely determined by the image $f(v) \in W$. Since W has countable dimension (because the quotient of G by any open compact is countable), we conclude that $\mathrm{End}_{\mathrm{Rep}(G)}(V)$ also has countable dimension. Let $f \in \mathrm{End}_{\mathrm{Rep}(G)}(V)$, and suppose that f is not scalar. Consider the subfield $\mathbb{C}(f)$ of $\mathrm{End}_{\mathrm{Rep}(G)}(V)$, and note that $\mathbb{C}(f)$ is a proper extension of $\mathbb{C}$ (which implies that f is transcendental over $\mathbb{C}$). Therefore, the subset $\{(f - a) - 1 \mid a \in \mathbb{C}\}$ is linearly independent over $\mathbb{C}$, and so the dimension of $\mathbb{C}(f)$ is uncountable, a contradiction. It follows that $\mathrm{End}_{\mathrm{Rep}(G)}(V) \simeq \mathbb{C}$ as required. $\square$

A consequence of the Schur's lemma is that the center of any ℓ-group acts by scalar multiplication on any irreducible smooth module; in particular, irreducible smooth modules of abelian ℓ-groups are all of dimension one. We also recall that, in the case of finite groups, if $\dim \mathrm{End}_{\mathrm{Rep}(G)}(V) = 1$, then the given module V is irreducible. However, this is not the case in our more general situation of ℓ-groups. For example, if V is the smooth module of the group of invertible triangular matrices over $\mathbb{Q}_p$ smoothly induced (for the definition see Section 2.3 below) from the trivial module of the subgroup of the invertible diagonal matrices, then $\mathrm{End}_{\mathrm{Rep}(G)}(V) = 1$, but V is not irreducible (see [BH06, Chapter 9.10] for a proof).

As we mentioned before, irreducible smooth modules of an ℓ-group are frequently, either of dimension one, or of infinite dimension. However, in some important situations, we are able to reduce to finite-dimensional representations. If G is an ℓ-group, then a smooth module V is said to be *admissible* if, for every open compact subgroup K of G, the vector subspace

$$V^K = \{v \in V \mid kv = v \; \forall k \in K\}$$

is finite-dimensional. We recall that a module V of an ℓ-group is smooth if and only if every $v \in V$ is fixed by some open compact subgroup of G, that is, every $v \in V$ lies in V^K for some open compact subgroup K of G; therefore,

$$V = \bigcup_K V^K$$

where the union is over all open compact subgroups of G; consequently, if V is admissible, then V is a union of finite-dimensional modules for open compact subgroups.

Proposition 2.2.15 ([BH06, Corollary 1, pg. 16]). *Let G be an ℓ-group. Then,*

$$0 \to U \to V \to W \to 0$$

is an exact sequence of smooth G-modules if and only if the (naturally induced) sequence

$$0 \to U^K \to V^K \to W^K \to 0$$

is exact for all open compact subgroup K of G.

Proof. Straightforward. □

It is well-known that, if G is any group, then the dual vector space V^* of any (abstract) G-module V is also a G-module, via the linear G-action defined by

$$(g \cdot f)(v) = f(vg), \qquad g \in G,\ f \in V^*,\ v \in V.$$

However, in the setting of smooth representations of ℓ-groups, one must be more careful, since the dual G-module V^* of a smooth G-module V is not necessarily smooth. For this reason, it is convenient to consider the *smooth dual* $V^\vee$ of a smooth G-module V which is defined as the (smooth) G-submodule of V^* consisting of all *smooth linear forms* (that is, linear forms $f \in V^*$ for which the centraliser $C_G(f)$ is an open subgroup of G); see Proposition 2.2.2).

Proposition 2.2.16. *Let G be a ℓ-group, let V be a smooth G-module, and let K be an open compact subgroup of G. Then:*

(1) The restriction mapping $f \mapsto f_{V^K}$ defines a linear isomorphism $(V^\vee)^K \simeq (V^K)^*$ (where $(V^K)^*$ denotes the dual vector space of V^K).

(2) The correspondence $V \mapsto V^\vee$ defines naturally an exact contravariant functor $\mathrm{Rep}(G) \to \mathrm{Rep}(G)$.

Proof. Firstly, note that V decomposes as the direct sum of K-submodules

$$V = V^K \oplus V(K)$$

where $V(K)$ is the vector subspace of V linearly spanned by the set $\{v - kv \mid v \in V, k \in K\}$. Due to this decomposition, every $f \in (V^K)^*$ can be linearly extended by zero to a linear function $\overline{f} \in V^*$; indeed, it is easy to see that $\overline{f} \in (V^\vee)^K$. On the other hand, every element of $(V^\vee)^K$ is of this form (because it is equals zero on $V(K)$), and this proves the first assertion.

Now, let $\phi\colon V \to W$ be a homomorphism of smooth G-modules, and define $\phi^*\colon W^* \to V^*$ by

$$\phi^*(f)(v) = f(\phi(v)), \qquad f \in W^*,\ v \in V.$$

It is to see that ϕ^* is a homomorphism of G-modules and that ϕ^* maps smooth vectors to smooth vectors. Therefore, ϕ^* defines (by restriction) a homomorphism of smooth G-modules $\phi^\vee\colon W^\vee \to V^\vee$. On the other hand, let $U \to V \to W$ be an exact sequence of smooth G-modules, and consider the sequence of G-modules $W^\vee \to V^\vee \to U^\vee$. If K is an arbitrary open compact subgroup of G, then the sequence $(W^\vee)^K \to (V^\vee)^K \to (U^\vee)^K$ is isomorphic to the sequence $(W^K)^* \to (V^K)^* \to (W^K)^*$, and thus it is exact (because $U^K \to V^K \to W^K$ is exact (by Proposition 2.2.15)). The result follows (again by Proposition 2.2.15). ☐

Every vector space V may be naturally embedded in its bi-dual $(V^*)^*$ via the mapping $v \mapsto v^*$ where

$$v^*(f) = f(v), \qquad v \in V,\ f \in V^*.$$

It is well-known that, if V is finite-dimensional, then this map is in fact an isomorphism; a similar statement holds for admissible smooth representations.

Proposition 2.2.17 ([BH06, Proposition 2.9, pg. 24; Proposition 2.10, pg. 25])**.** *Let G be a ℓ-group, and V be a smooth G-module. Then, the following are equivalent.*

(1) V is admissible.

(2) $V^\vee$ is admissible.

(3) The natural embedding $V \hookrightarrow (V^\vee)^\vee$ defines an isomorphism $V \simeq (V^\vee)^\vee$.

Furthermore, if V is admissible, then V is irreducible if and only if $V^\vee$ is irreducible.

Proof. Note that, for every open compact subgroup K, the mapping $v \mapsto v^*$ (as defined above) restricts to a linear injective map

$$V^K \hookrightarrow ((V^K)^*)^* \simeq ((V^K)^*)^*.$$

Since V (resp., $(V^\vee)^\vee$) is the union of V^K (resp., $((V^\vee)^\vee)^K$) where K runs over all open compact subgroups of G, the given map is surjective if and only if, for every open compact subgroup K, the map $V^K \hookrightarrow ((V^K)^*)^*$ is surjective, which happens if and only if V^K has finite dimension. By the definition, we conclude that the embedding $V \hookrightarrow (V^\vee)^\vee$ is surjective if and only if V is admissible.

Since $(V^\vee)^K \simeq (V^K)^*$ for every open compact subgroup K of G, it is clear that V is admissible if and only if $V^\vee$ is admissible.

For the last assertion, it is enough to observe that, if V is admissible, then the isomorphism $V \simeq (V^\vee)^\vee$ implies that the mapping $W \mapsto W^\vee$ between that G-submodules of V and the G-submodules of $V^\vee$. □

Proposition 2.2.17 asserts that the mapping $V \mapsto V^\vee$ defines a contravariant functor on the category of admissible representations, and that it is its own quasi-inverse. We also observe that, in the case of an arbitrary smooth representation (not necessarily admissible), the irreducibility of the smooth dual implies the irreducibility of the original representation.

Another classic way of constructing new smooth representations is to consider the tensor product of two representations: if G is ℓ-group and V_1 and V_2 are smooth G-modules, then the (internal) tensor product $V_1 \otimes V_2$ becomes a G-module for the action defined (on the pure tensors) by

$$g \cdot (v_1 \otimes v_2) = (g \cdot v_1) \otimes (g \cdot v_2), \qquad g \in G,\ v_1 \in V_1,\ v_2 \in V_2.$$

We note that $V_1 \otimes V_2$ is indeed a smooth G-module because

$$C_G(v_1) \cap C_G(v_2) \leq C_G(v_1 \otimes v_2), \qquad v_1 \in V_1,\ v_2 \in V_2.$$

In particular, if G is an ℓ-group V is a smooth G-module and $\lambda\colon G \to \mathbb{C}^\times$ is a smooth character, then the tensor product $V \otimes \mathbb{C}_\lambda$ identifies naturally with the vector space V with a new (smooth) G-action given by

$$g * v = \lambda(g)(g \cdot v), \qquad g \in G,\ v \in V;$$

we denote this (new) smooth G-module by $V \otimes \lambda$, and notice that V is irreducible if and only if $V \otimes \lambda$ is irreducible.

On the other hand, let H and K be ℓ-groups, and equip direct product $G = H \times K$ with the product topology; hence, G is an ℓ-group. Let V_1 be a smooth H-module, and let V_2 be a smooth K-module. Then, we define the (external) tensor product $V_1 \boxtimes V_2$ to be the G-module

whose underlying vector space is $V_1 \otimes V_2$ and where the G-action is defined by

$$(h,k)\cdot(v_1\otimes v_2)=(h\cdot v_1)\otimes(k\cdot v_2), \qquad h\in H,\ k\in K,\ v_1\in V_1,\ v_2\in V_2;$$

since

$$C_H(v_1)\cap C_K(v_2)\leq C_G(v_1\otimes v_2), \qquad v_1\in V_1,\ v_2\in V_2,$$

it is clear that $V_1 \boxtimes V_2$ is a smooth G-module. We note that, for an arbitrary ℓ-group G, the tensor product $V_1 \otimes V_2$ of smooth G-modules V_1 and V_2, may be identified with restriction of $V_1 \boxtimes V_2$ to the diagonal subgroup

$$\Delta(G)=\{(g,g)\mid g\in G\}$$

of $G\times G$ (which can be naturally identified with G).

The tensor product provides a good description of the smooth representation of the direct product of ℓ-groups.

Proposition 2.2.18 ([Fla79, Theorem 1]). *Let H and K be ℓ-groups, let V_1 be a smooth H-module, and let V_2 be a smooth K-module. If V_1 and V_2 are irreducible and admissible, then the smooth $(H\times K)$-module $V_1\boxtimes V_2$ is also irreducible and admissible; furthermore, every irreducible admissible smooth $(H\times K)$-module is of this form.*

Proof. Omitted. □

2.3 Induction Functors

In this section, we relate the smooth modules of an ℓ-group G with smooth modules of a closed subgroup H of G (which is also an ℓ-group). Firstly, we observe that every G-module V is also an H-module (by restriction of the G-action) which we will denote by $\mathrm{Res}_H^G(V)$, or simply by V_H; it is clear that V_H is smooth whenever V is smooth (because $C_H(v)=H\cap C_G(v)$ for all $v\in V$).

Let H be a closed subgroup of an ℓ-group G, and let W be an arbitrary H-module. Then, W is a module for the group algebra $\mathbb{C}[H]$, and thus we may form the tensor product $\mathbb{C}[G]\otimes_{\mathbb{C}[H]}W$ where H acts on the group algebra $\mathbb{C}[G]$ via right multiplication (so that $\mathbb{C}[G]$ becomes a right $\mathbb{C}[H]$-module). Then, $\mathbb{C}[G]\otimes_{\mathbb{C}[H]}W$ becomes a left $\mathbb{C}[G]$-module (with respect to the G-action given by left multiplication on the first factor), and hence it gives rise to an abstract representation of G. It is well-known that the G-module $\mathbb{C}[G]\otimes_{\mathbb{C}[H]}W$ is isomorphic to the

G-module $\text{a-Ind}_H^G(W)$ consisting of all functions $f : G \to W$ which satisfy the condition

$$f(hg) = hf(g), \qquad h \in H,\ g \in G,$$

endowed with the action defined by

$$(gf)(x) = f(xg), \qquad g, x \in G,\ f \in \text{a-Ind}_G^H(W).$$

We refer to $\text{a-Ind}_H^G(W)$ as the G-module (abstractly) induced by W.

Assume now that W is a smooth H-module. The induced G-module $\text{a-Ind}_H^G(W)$ is not necessarily smooth; however, we can always consider the smooth G-submodule $\text{a-Ind}_H^G(W)^\infty$ which we will denote by $\text{Ind}_H^G(W)$ and refer to as the smooth G-module *smoothly induced* by W; hence

$$\text{Ind}_H^G(W) = \text{a-Ind}_H^G(W)^\infty.$$

Looking at the definition of a smooth vector, we see that $f \in \text{a-Ind}_H^G(W)$ is smooth if and only if there exists an open compact subgroup K (depending on f) such that

$$f(gk) = f(g), \qquad g \in G,\ k \in K.$$

If $\phi : W_1 \to W_2$ is a homomorphism of H-modules, then the mapping $f \mapsto \phi \circ f$ defines a homomorphism of G-modules

$$\text{Ind}_H^G(\phi) : \text{Ind}_H^G(W_1) \to \text{Ind}_H^G(W_2),$$

and thus we obtain a functor $\text{Ind}_H^G : \text{Rep}(H) \to \text{Rep}(G)$.

Proposition 2.3.1 ([BH06, Proposition 2.4, pg. 18])**.** *Let G be an ℓ-group, and let H a closed subgroup of G. Then, the functor $\text{Ind}_H^G : \text{Rep}(H) \to \text{Rep}(G)$ is additive and exact.*

Proof. Straighforward. $\square$

As above, let H be a closed subgroup of an ℓ-group G, and let W be a smooth H-module. Then, we have a canonical H-morphism $\alpha_W : \text{Ind}_H^G(W) \to W$ given by the mapping $f \mapsto f(1)$, and thus W is isomorphic to a quotient of $\text{Ind}_H^G(W)$. On the other hand, we also have a version of Frobenius reciprocity.

Proposition 2.3.2 (Frobenius Reciprocity; [BH06, 25])**.** *Let H be a closed subgroup of an ℓ-group, let W be a smooth H-module, and let V be a smooth G-module. Then, the mapping*

$\phi \mapsto \alpha_W \circ \phi$ defines a linear isomorphism

$$\operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) \simeq \operatorname{Hom}_H(\operatorname{Res}_H^G V, W)$$

which is functorial in both V and W.

Proof. We only prove surjectivity; the remaining are straightforward. Let $f : \operatorname{Res}_H^G(V) \to W$ be an homomorphism of H-modules, and define $f_* : V \to \operatorname{Ind}_W^G(W)$ by

$$f_*(v)(g) = f(gv), \qquad v \in V,\ g \in G.$$

Then, f_* is an homomorphism of smooth G-modules, and we have

$$(\alpha_W \circ f_*)(v) = \alpha_W(f_*(v)) = f_*(v)(1) = f(v), \qquad v \in V,$$

as we wanted. □

Frobenius reciprocity asserts that, the induction functor is right adjoint to the restriction functor; however, in most of the cases, it may fail to be left adjoint. Another type of induction which we will frequently use is the *compact induction* (or *induction with compact supports*). Let H be a closed subgroup of an ℓ-group, let W be a smooth H-module, and define

$$\text{c-}\operatorname{Ind}_H^G(W) = \{f \in \operatorname{Ind}_H^G(W) \mid f \text{ has compact support modulo } H\}$$

here, a function $f \in \operatorname{Ind}_H^G(W)$ has compact support modulo H if $\operatorname{supp}(f) \subseteq HC$ for some compact subset C of G. Since $\operatorname{Ind}_H^G(W)$ is a smooth G-module (and $\text{c-}\operatorname{Ind}_H^G(W)$ is a submodule of $\operatorname{Ind}_H^G(W)$), $\text{c-}\operatorname{Ind}_H^G(W)$ is also a smooth G-module, to which we refer as the smooth G-module *compactly induced* (or simply *c-induced*) by W. Note that, if the quotient space G/H is compact, then we clearly have $\text{c-}\operatorname{Ind}_H^G(W) = \operatorname{Ind}_H^G(W)$. As in the case of smooth induction, we obtain a functor $\text{c-}\operatorname{Ind}_H^G : \operatorname{Rep}(H) \to \operatorname{Rep}(G)$ (in fact, $\text{c-}\operatorname{Ind}_H^G$ is a subfunctor of Ind_H^G), and we also have the following (the proof is just a repetition of the proof of the similar result for Ind_H^G).

Proposition 2.3.3. *Let G be an ℓ-group, and let H be a closed subgroup of G. Then, the functor $\text{c-}\operatorname{Ind}_H^G : \operatorname{Rep}(H) \to \operatorname{Rep}(G)$ is additive and exact.*

The functor $\text{c-}\operatorname{Ind}_H^G$ is specially well-behaved when the subgroup H is open in G.

Proposition 2.3.4 ([BH06, Lemma 2.5, pg. 19])**.** *Let H be an open subgroup of an ℓ-group G, and let W be a smooth H-module. For every $w \in W$, let $f_w : G \to W$ be the function*

defined by $f_w(h) = hw$ *for all* $h \in H$*, and* $f_w(g) = 0$ *for all* $g \in G \setminus H$*. Then, the mapping* $w \mapsto f_w$ *defines an injective homomorphism* $\alpha^c_W : W \to \text{c-Ind}_H^G(W)$ *of* H*-modules whose image is the vector subspace of* $\text{c-Ind}_H^G(W)$ *consisting of all functions with support contained in* H*. Moreover, if* $\mathcal{B}$ *is a basis for* W *and* $X \subseteq G$ *is a complete set of representatives of* G/H*, then the set* $\{gf_w \mid w \in \mathcal{B}, g \in x\}$ *is a basis for* $\text{c-Ind}_H^G(W)$*.*

Proof. Straightforward. □

As in the case of the smooth induction, the following version of Frobenius reciprocity holds for c-induction.

Proposition 2.3.5 ([BH06, Proposition 2.5, pg.20])**.** *Let* H *be an open subgroup of an* ℓ*-group* G*, let* W *be a smooth* H*-module, and let* V *be a smooth* G*-module. Then, the mapping* $\phi \mapsto \phi \circ \alpha^c$ *defines a linear isomorphism*

$$\mathrm{Hom}_G(\text{c-Ind}_H^G W, V) \simeq \mathrm{Hom}_H(W, \mathrm{Res}_H^G V)$$

which is functorial in both V *and* W*.*

Proof. Straightforward. □

In contrast to what happens in the case of smooth induction, the functor c-Ind_H^G is left adjoint to the restriction functor Res_H^G (provided that the subgroup H is open in G). This is another aspect where the theory of representations of finite groups (for example) differs from the representation theory of ℓ-groups: in the finite group case, the induction functor is left and right adjoint to the restriction functor, but in the setting of smooth representations that is not always true. However, if the subgroup H is open and the quotient G/H is compact (which is equivalent to G/H being finite), then $\text{c-Ind}_H^G = \mathrm{Ind}_H^G$ and the induction functor is, in fact, left and right adjoint to the restriction functor. In particular, we conclude the following:

Proposition 2.3.6 ([BH06, Lemma 2.7, pg. 21])**.** *Let* H *be an open subgroup of an* ℓ*-group* G*, and suppose that the quotient* G/H *is finite. Then, the following holds.*

(1) If V *is a smooth* G*-module, then* V *is semisimple as a smooth representation of* G *if and only if it is semisimple as a smooth representation of* H*.*

(2) If W *is a semisimple smooth* H*-module, then the smoothly induced* G*-module* $\mathrm{Ind}_H^G(W)$ *(which is equal to the smooth c-induced* G*-module) is semisimple.*

Proof. Omitted. □

An useful lemma to deal with compact induction in the general situation is the following result.

Lemma 2.3.7. *Let H be a closed subgroup of an ℓ-group G, and let W be a smooth H-module. Let K be an open compact subgroup of G, and let Γ be a complete set of representatives of the double coset space $H\backslash G/K$. Then the mapping $f \mapsto (f(g))_{g\in\Gamma}$ defines an isomorphism of (smooth) G-modules*

$$(\text{c-Ind}_H^G W)^K \simeq \bigoplus_{g\in\Gamma} W^{H\cap gKg^{-1}}$$

Proof. Straightforward. (We note that, for every open compact subgroup K of G, a basis for $(\text{c-Ind}_H^G W)^K$ is described in [BH06, pg. 32].) □

An immediate consequence of this lemma is the following.

Lemma 2.3.8. *Let H be a closed subgroup of a ℓ-group G. If W is a non-zero smooth H-module, then $\text{c-Ind}_H^G(W)$ is non-zero; in particular, $\text{Ind}_H^G(W)$ is also non-zero.*

Proof. Obvious. □

Another consequence is the following.

Proposition 2.3.9. *Let H be a closed subgroup of a ℓ-group G, and suppose that the quotient space G/H is compact. Then, the functor c-Ind_H^G (which equals Ind_H^G) takes admissible smooth H-modules to admissible smooth G-modules.*

Proof. For any compact open subgroup K of G, the number of double cosets in $H\backslash G/K$ is finite, and so from the direct sum decomposition above we deduce that, if W is admissible, then $W^{H\cap gKg^{-1}}$ for all $g \in G$ is finite-dimensional, and thus $(\text{c-Ind}_H^G W)^K$ is finite-dimensional. □

We can generalise to ℓ-groups the following well-known theorem of Mackey.

Lemma 2.3.10 (Mackey Lemma)**.** *Let G be an ℓ-group, let $H \leq G$ be a closed subgroup, let V be a smooth G-module, and let W be a smooth H-module. Then, there is an isomorphism of G-modules*

$$V \otimes \text{c-Ind}_H^G(W) \simeq \text{c-Ind}_H^G(V_H \otimes W).$$

Proof. Firstly, note that $V \otimes \text{c-Ind}_H^G W$ is linearly spanned by the set $\{v \otimes f \mid v \in V, f \in \text{c-Ind}_H^G(W)\}$, and thus we may define a map $\psi\colon V \otimes \text{c-Ind}_H^G(W) \to \text{c-Ind}_H^G(V_H \otimes W)$ by

$$\psi(v \otimes f)(g) = (g \cdot v) \otimes f(g), \qquad v \in V,\ f \in \text{c-Ind}_H^G(W),\ g \in G$$

(this map is indeed well-defined). By restricting to open compact subgroups of G, it is not hard to check that ψ is in fact an isomorphism. □

2.4 Jacquet Functor

Let G be an ℓ-group, and let V be a smooth G-module. As before, we may consider the G-invariant submodule

$$V^G = \{v \in V \mid g.v = v,\ \forall g \in G\}$$

of V. By definition, V^G is the largest vector subspace of V where G acts trivially; however, the notion of G-coinvariant quotient will be more useful to us. Let

$$V(G) = \langle g \cdot v - v \mid g \in G, v \in V \rangle_{\mathbb{C}}$$

and define the *vector space of G-coinvariants* to be the quotient

$$V_G = V/V(G);$$

hence, V_G is the largest quotient of V where G acts trivially.

Proposition 2.4.1. *Let G be an ℓ-group Then the mapping $V \mapsto V_G$ defines a right exact functor $J_G\colon \operatorname{Rep}(G) \to \operatorname{Vec}(\mathbb{C})$ from the category $\operatorname{Rep}(G)$ to the category $\operatorname{Vec}(\mathbb{C})$ of complex vector spaces.*

Proof. Let V and W be smooth G-modules, and let $f\colon V \to W$ a G-homomorphism. Then,

$$f(g \cdot v - v) = g \cdot f(v) - f(v), \qquad g \in G,\ v \in V,$$

and $f(V(G)) \subset W(G)$. Therefore, f induces a linear map $f_G\colon V_G \to W_G$, and so J_G is in fact a functor from $\operatorname{Rep}(G)$ to $\operatorname{Vec}(\mathbb{C})$. The remaining assertions are straightforward. $\square$

We refer to the functor $J_G\colon \operatorname{Rep}(G) \to \operatorname{Vec}(\mathbb{C})$ as the *Jacquet functor*, and to the vector space $J_G(V) = V_G$ as the *Jacquet module* associated with V. Of particular interest is the case where G is an ℓ_c-group.

Proposition 2.4.2. *Let G be an ℓ_c-group, and let V be a smooth G-module. Then,*

$$V = V(G) \oplus V^G;$$

in particular, there is an isomorphism $V^G \simeq V/V(G) = V_G$ of smooth G-modules.

Proof. Firstly, suppose that G is a a compact group. If this is the case, then V is semisimple (see Proposition 2.2.8) and, in particular, the submodule V^G has a complement in V. Since $V(G)$ is zero under the natural projection $\pi\colon V \to V^G$, we see $V(G)$ is contained in any complement

of V^G. Now, if W is any non-trivial irreducible submodule of V, then $W(G) = W$, and so $W \subseteq W(G) \subseteq V(G)$. It follows that $V(G)$ is the sum of all irreducible non-trivial submodules of V, and thus we must have $V = V^G \oplus V(G)$.

In the general case, since G is the union of all its open compact subgroups, we have

$$V^G = \bigcap_K V^K$$

where the intersection runs over all open compact subgroups K of G. On the other hand, since G is an ℓ_c-group, it is also clear that

$$V(G) = \bigcup_K V(K)$$

where the union runs over all the open compact subgroups K of G. Since $V = V^K \oplus V(K)$ (as a K-module), we easily conclude that $V = V(G) \oplus V^G$, as required. □

The following result is as easy consequence of the previous proposition (and of Proposition 2.4.1).

Proposition 2.4.3. *If G is an ℓ_c-group, then the Jacquet functor $J_G : \mathrm{Rep}(G) \to \mathrm{Vec}(\mathbb{C})$ is exact.*

Proof. By identifying V^G with V_G, it is easy to see that the functor is left exact, and the proposition 2.4.1 give us that it is also right exact. □

It follows from Example 2.2.4 that an irreducible smooth module of an ℓ-group may not have irreducible quotients; however, in the case of an ℓ_c-group G, the exactness of the Jacquet functor implies that, for a certain class of ℓ_c-groups, every smooth G-module has an irreducible quotient. Firstly, we prove the following auxiliary result.

Proposition 2.4.4. *Let G be an ℓ_c-group, and let V be a smooth G-module. Then, the smooth dual $V^\vee$ is an injective object in the category $\mathrm{Rep}(G)$.*

Proof. For every smooth G-module W, we have the following chain of natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Rep}(G)}(V, W^\vee) \to \mathrm{Hom}_{\mathrm{Rep}(G)}(W \otimes_{\mathbb{C}} V, \mathbb{C}) \to \mathrm{Hom}_{\mathrm{Vec}(\mathbb{C})}((W \otimes_{\mathbb{C}} V)_G, \mathbb{C}).$$

Since $(W \otimes_{\mathbb{C}} V)_G = J_G(W \otimes_{\mathbb{C}} V)$ (and since G is an ℓ_c-group), the previous proposition implies that $\mathrm{Hom}_{\mathrm{Vec}(\mathbb{C})}(J_G(- \otimes V), \mathbb{C})$ is the composition of three exact functors, and thus $\mathrm{Hom}_{\mathrm{Rep}(G)}(-, V^\vee)$ is an exact functor. The result follows. □

As a consequence, we deduce the following:

Proposition 2.4.5. *Let G be an ℓ_c-group and let V be an admissible G-module. Then, V is an injective object in the category $\mathrm{Rep}(G)$ of smooth G-modules.*

Proof. Since V is admissible, we have that an isomorphism of G-modules $(V^\vee)^\vee \simeq V$ (by Proposition 2.2.17), and so V is injective (by the previous proposition). □

A special class of groups is the class of groups for which all irreducible representations are admissible. In particular, abelian groups have this property (because every irreducible representation of an abelian group is one-dimensional). Other examples include the finite groups and, more generally, the compact groups (by the Peter-Weyl theorem). We also mention the result of M. Boyarchenko (see [Boy11]) which asserts that every irreducible smooth representation of an algebra group over a non-Archimedean local field has this property.

Proposition 2.4.6. *Let G be a ℓ_c-group such that every irreducible representation is admissibl., Then, every smooth representation has an irreducible quotient.*

Proof. Note that, since every irreducible G-module is admissible, every irreducible smooth G-module is injective (by the previous proposition). On the other hand, every smooth G-module V has a finitely generated submodule W (just take the G-submodule generated by any non-zero element of V), and thus W has an irreducible quotient (by Proposition 2.2.5). Therefore, every smooth G-module has an irreducible quotient. □

We can also twist a smooth G-module by a given smooth character of an ℓ-group G. More precisely, let G be an ℓ-group, let V be a smooth G-module, and let $\lambda\colon G \to \mathbb{C}^\times$ be a smooth character of G. Then, we may define a new G-action on V by the rule:

$$g \circ v = \lambda(g)(g \cdot v), \qquad g \in G,\ v \in V.$$

If we denote by $\widetilde{V}$ the resulting smooth G-module, then the Jacquet functor gives the smooth G-module $\widetilde{V}_G = J_G(\widetilde{V})$ which by the definition is the the maximal quotient of V where G acts via the smooth character $\lambda^{-1}\colon G \to \mathbb{C}$ (recall that $\lambda^{-1}(g) = \lambda(g)^{-1}$ for all $g \in G$).

A more constructive way to define this twist is as follows. Let G, V and λ be as above, and consider the vector subspace

$$V(\lambda) = \langle gv - \lambda(g)v \mid g \in G\ v \in V \rangle_{\mathbb{C}}$$

of V. Then, we define

$$V_\lambda = V/V(\lambda);$$

notice that this quotient is exactly the maximal quotient where G acts via the smooth character λ (hence, in the above notation, we have $\widetilde{V}_G = V_{\lambda^{-1}}$).

Proposition 2.4.7 ([BZ76, Proposition 2.35])**.** *Let G be an ℓ-group and $\lambda : G \to \mathbb{C}^\times$ a smooth character of G. Then, the mapping $V \mapsto V_\lambda$ defines a right exact functor $J_G^\lambda \colon \operatorname{Rep}(G) \to \operatorname{Vec}(\mathbb{C})$.*

Proof. It is enough to consider the smooth G-module $V \otimes \mathbb{C}_{\lambda^{-1}}$, and to apply the Jacquet functor to this G-module. $\square$

In the notation as above, we refer to the J_G^λ the *twisted Jacquet functor* associated with λ; notice that the twisted Jacquet functor associated with the trivial character of G is just the usual Jacquet functor. As before, for ℓ_c-groups, the twisted Jacquet functors are exact.

Proposition 2.4.8 ([BZ76, Proposition 2.35])**.** *Let G be an ℓ_c-group, and let $\lambda \colon G \to \mathbb{C}^\times$ be a smooth character of G. Then, the twisted Jacquet functor J_G^λ is exact.*

The twisted Jacquet functors (and hence, also the usual Jacquet functors) are particularly useful in the following situation. Let G be an ℓ-group, and let V be a smooth G-module. Moreover, let H be a closed normal subgroup of G, and let $\lambda : H \to \mathbb{C}$ be a smooth character of H. If

$$C_G(\lambda) = \{g \in G \mid \lambda(ghg^{-1}) = \lambda(h) \text{ for all } h \in H\}$$

is the centralizer of λ in G, then

$$z(gv - \lambda(g)v) = (zgz^{-1})(zv) - \lambda(zgz^{-1})(zv) \in V(\lambda), \quad z \in C_G(\lambda),\ g \in G,\ v \in V,$$

and so $V(\lambda)$ is a $C_G(\lambda)$-submodule of $\operatorname{Res}^G_{C_G(\lambda)}(V)$. It follows that $V_\lambda = V/V(\lambda)$ is a (smooth) $C_G(\lambda)$-module, and thus the twisted Jacquet functor J_G^λ may be seen as a functor from the category $\operatorname{Rep}(G)$ to the category $\operatorname{Rep}(C_G(\lambda))$.

For every $g \in G$, $h \in H$ and $v \in V$, we have

$$g.(hv - \lambda(h)v) = (ghg^{-1})(gv) - \lambda^{g^{-1}}(ghg^{-1})(gv)$$

and thus $gV(\lambda) = V(\lambda^{g^{-1}})$; here, for every $g \in G$, we define the map $\lambda^g : H \to \mathbb{C}^\times$ by $\lambda^g(h) = \lambda(ghg^{-1})$ for all $g \in H$ (it is obvious that λ^g is a smooth character of H). In particular, we conclude that $V_\lambda \neq 0$ if and only if $V_\mu \neq 0$ for all $\mu \in \lambda^G$ where

$$\lambda^G = \{\lambda^g \mid g \in G\}$$

is the G-orbit which contains λ. Furthermore, we have a canonical map

$$\epsilon : V \to \prod_{\mu \in \lambda^G} V_\mu$$

with

$$\ker(\epsilon) = \bigcap_{\mu \in \lambda^G} V_\mu.$$

Since, G permutes the subspaces V_μ, for $\mu \in \lambda^G$, we see $\ker(\epsilon)$ is a G-submodule of V. In particular, if V is irreducible, then either $\ker(\epsilon) = 0$, or $\ker(\epsilon) = V$; moreover, $\ker(\epsilon) = V$ if and only if $V_\lambda = 0$ (by definition). Therefore, if $V_\lambda \neq 0$, then ϵ is injective, and so $\mathrm{Res}_H^G(V)$ is isomorphic to a submodule of the product $\prod_{\mu \in \lambda^G} V_\mu$ (notice that ϵ is indeed a homomorphism of H-modules).

Finally, let $\lambda' \colon H \to \mathbb{C}^\times$ be a smooth character of H, and suppose that $\lambda' \notin \lambda^G$. Then, $(V_\mu)_{\lambda'} = 0$ for all $\mu \in \lambda^G$ (by the definition), and thus $V_\mu = V_\mu(\lambda')$ for all $\mu \in \lambda^G$. It follows that

$$V_\mu \subseteq (\prod_{\mu' \in \lambda^G} V_{\mu'}(\lambda'), \qquad \mu \in \lambda^G,$$

and so

$$\prod_{\mu \in \lambda^G} V_\mu \subseteq (\prod_{\mu \in \lambda^G} V_\mu)(\lambda').$$

Therefore, if $(\prod_{\mu \in \lambda^G} V_\mu)_{\lambda'} = 0$, then $V_{\lambda'} = 0$. This completes the proof of the following result.

Proposition 2.4.9. *Let G be an ℓ-group, let V be an irreducible smooth G-module, and let H be a closed normal subgroup of G. Then, there exists a smooth character $\lambda \colon H \to \mathbb{C}^\times$ such that $V_\lambda \neq 0$; furthermore, for every smooth character $\mu \colon H \to \mathbb{C}^\times$ of H, we have $V_\mu \neq 0$ if and only if $\mu \in \lambda^G$.*

2.5 Haar Measure

We start this section by briefly recalling some concepts of measure theory which we will be using in later sections; we use [Rud06, Chapters 1 and 2] as our main reference.

Let X be an arbitrary topological space, and let $\mathcal{B}$ be the σ-algebra on X which is generated by the open sets of X (that is, $\mathcal{B}$ is the smallest σ-algebra containing all the open sets); we recall that a *σ-algebra* on X is a family of subsets of X which contains X and is closed under relative complements and at most countable unions. The σ-algebra $\mathcal{B}$ is called the *Borel σ-algebra*, and we refer to the elements of the σ-algebra $\mathcal{B}$ as the Borel subsets of X. On the other hand, by a

measure on X (with σ-algebra $\mathcal{B}$) we mean a function $\mu\colon \mathcal{B} \to \mathbb{R}^+ \cup \{\infty\}$ such that, if $\{S_n\}_{n\in N}$ is a countable collection of pairwise disjoint Borel subsets of X, then

$$\mu\Big(\bigcup_{n\in\mathbb{N}} S_n\Big) = \sum_{n\in\mathbb{N}} \mu(S_n)$$

A measure μ on X is said to be *regular* if:

(1) If $S \in \mathcal{B}$, then $\mu(S) = \inf\{\mu(U) \mid S \subseteq U,\ U \text{ is open}\}$;

(2) If $U \subseteq X$ is open, then $\mu(U) = \sup\{\mu(K) \mid K \subseteq U,\ K \text{ is compact}\}$.

If G is an ℓ-group and $g \in G$, then we define the left translation $L_g : G \to G$ and the right translation $R_g : G \to G$ by

$$L_g(g') = gg' \qquad \text{and} \qquad R_g(g') = g'g, \qquad g' \in G;$$

notice that both L_g and R_g are homeomorphisms which preserve the Borel subsets of G. If μ is a measure on G, we say that μ is *left invariant* (resp., *right invariant*) if, for every $g \in G$ and every Borel subset S of G, we have $\mu(gS) = \mu(L_g(S)) = \mu(S)$ (resp., $\mu(Sg) = \mu(R_g(S)) = \mu(S)$); furthermore, we say that μ is a *left Haar measure* (resp., a *right Haar measure*) if μ is regular and left (resp., right) invariant. The following theorem asserts that there always exists an Haar measure on G.

Theorem 2.5.1 ([DE14, Theorem 1.3.4]). *Let G be a locally compact group. Then, there exists a non-zero left (resp., right) Haar measure on G. Moreover, the Haar measure on G is uniquely determined up to a positive multiple.*

If G is an ℓ-group and μ is an (left) Haar measure on G, then with every $g \in G$ we can associate a new Haar measure μ_g on G by setting $\mu_g(S) = \mu(Sg)$ for all Borel subset S of G. Therefore, by the unicity of the Haar measure, for every $g \in G$, there exists a positive real number $\delta_G(g)$ such that $\mu_g = \delta_G(g)\mu$, and thus we get a map $\delta_G\colon G \to \mathbb{R}^+$ to which we refer as the *modular character* of G; notice that the modular function δ_G is independent of the initial choice of the Haar measure. We say that the ℓ-group G is *unimodular* if $\delta_G(g) = 1$ for all $g \in G$; we observe that, if this is the case, then every left Haar measure is also a right Haar measure (that is, the left Haar measure is also right invariant).

Theorem 2.5.2 ([DE14, Theorem 1.3.1]). *Let G be an ℓ-group. Then, the modular function $\delta_G\colon G \to \mathbb{R}^+$ is a smooth character of G. Furthermore, if G is abelian or compact, then G is unimodular; in particular, every ℓ_c-group is unimodular.*

The modular character allows us to relate compact induction and smooth induction. In order to describe this relation, we first recall the notion of left (and right) Haar integral on an arbitrary ℓ-group G. Let $C_c^\infty(G)$ be the vector space consisting of all functions $f : G \to \mathbb{C}$ which are locally constant and have compact support. Then, the group G acts on the left of $C_c^\infty(G)$ by left translations: if $g \in G$ and $f \in C_c^\infty(G)$, then $L_g f \in C_c^\infty(G)$ is defined by

$$(L_g f)(x) = f(g^{-1}x), \qquad x \in G;$$

similarly, G acts on the right of $C_c^\infty(G)$ by right translations: if $g \in G$ and $f \in C_c^\infty(G)$, then $R_g f \in C_c^\infty(G)$ is defined by

$$(R_g f)(x) = f(xg), \qquad x \in G.$$

By a *left Haar integral* on G we mean a non-zero linear functional $I\colon C_c^\infty(G) \to \mathbb{C}$ such that

(1) $I(L_g f) = I(f)$ for all $g \in G$ and all $f \in C_c^\infty(G)$;

(2) $I(f) \in \mathbb{R}_0^+$ for all $f \in C_c^\infty(G)$ such that $f \geq 0$.

We define a *right Haar integral* on G in a similar way (using right translation instead of left translation). It can be proved that every ℓ-group G admits a left Haar integral (and also a right Haar integral); see [BH06, Proposition 3.1] and its corollary. There is a close connection between left (resp., right) Haar integrals on G and left (resp., right) Haar measures G: indeed, given any left (resp., right) Haar integral I, we may consider the characteristic function $\mathbb{I}_S : G \to \mathbb{C}$ of any subset $S \subseteq G$, and define the left (resp., right) Haar measure μ on G by the rule $\mu(K) = I(\mathbb{I}_K)$ for all non-empty compact open subset $K \subseteq G$. As usual, we express the relation between the Haar integral and the Haar measure by the notation

$$I(f) = \int_G f(g) d\mu(g), \qquad f \in C_c^\infty(G);$$

for simplicity, we sometime write $I(f) = \int_G f d\mu$. The usual properties of integration hold in this context: by way of example, we may define

$$\int_K f(g) d\mu(k) = \int_G \mathbb{I}_K(g) f(g) d\mu(g), \qquad f \in C_c^\infty(G).$$

We observe that, if the group G is unimodular, then every left Haar integral on G is also a right Haar integral on G (and vice-versa).

Now, let G be an ℓ-group, let H be a closed subgroup of G, and let $\lambda : H \to \mathbb{C}^\times$ be a smooth character of H. We consider the vector space $C_c^\infty(G/H, \lambda)$ consisting of all functions

$f : G \to \mathbb{C}^\times$ which are smooth under the left translation by elements of G, compactly supported modulo H, and satisfy

$$f(gh) = \lambda(h)f(g), \qquad g \in G,\ h \in H.$$

Then, $C_c^\infty(G/H, \lambda)$ becomes a smooth G-module (under left translation), and in fact there exists an isomorphism of (smooth) G-modules

$$C_c^\infty(G/H, \lambda) \simeq \text{c-Ind}_H^G(\lambda).$$

In this situation, it can be proved (see [BH06, Proposition 3.4]) that there exists a non-zero linear functional $I_\lambda \colon C_c^\infty(G/H, \lambda) \to \mathbb{C}$ such that

$$I_\lambda(L_g f) = I_\lambda(f), \qquad g \in G,\ f \in C_c^\infty(G/H, \lambda)$$

if and only if $\lambda = (\delta_G)_H \delta_H^{-1}$ where δ_G and δ_H are the modular characters of G and H, respectively. If these conditions are satisfied then, similarly to the above, we may define the "integral"

$$I_\lambda(f) = \int_{G/H} f(g) d\dot{\mu}, \qquad f \in C_c^\infty(G/H, \lambda)$$

where $\dot{\mu}$ is a positive semi-invariant measure on G/H; for details see [BH06, Section 3.4].

Theorem 2.5.3 (Duality Theorem; [BH06, Theorem 3.5, pg. 32])**.** *Let H be a closed subgroup of a ℓ-group G, and let W be a smooth H-module. Then, there exists a linear isomorphism*

$$\text{c-Ind}_H^G(W)^\vee \simeq \text{Ind}_H^G(\delta_{G/H} \otimes W^\vee)$$

where $\delta_{G/H} = (\delta_G)_H \delta_H^{-1}$. In particular, if G is unimodular, then

$$\text{c-Ind}_H^G(W)^\vee \simeq \text{Ind}_H^G(W^\vee)$$

Proof. By the definition, $\delta_{G/H} \otimes W^\vee$ is the H-module whose underlying vector space is $W^\vee$ with (twisted) H-action defined by

$$h \cdot \check{w} = \delta_{G/H}(h)(h\check{w}), \qquad h \in H,\ \check{w} \in W^\vee.$$

For every $\Psi \in \text{Ind}_H^G(\delta_{G/H} \otimes W^\vee)$ and every $\phi \in \text{c-Ind}_H^G(W)$, the mapping $g \mapsto \Psi(g)(\phi(g))$

defines a function in $C_c^\infty(G/H, \delta_{G/H})$, and thus we may define

$$\Phi(\psi) = \int_{G/H} \Psi(g)(\phi(g)) d\dot{\mu}(g)$$

where $\dot{\mu}$ is as above. The mapping $\Psi \mapsto (\psi \mapsto \Phi(\psi))$ defines the desired isomorphism $\text{c-Ind}_H^G(W)^\vee \to \text{Ind}_H^G(\delta_{G/H} \otimes W^\vee)$ (for details see [BH06, Theorem 3.5]) $\square$

Now, let V be a complex vector space, and let us denote the space of locally constant, compactly supported functions $f : G \to V$ by $C_c^\infty(G; V)$; in particular, this space is isomorphic to the tensor product $C_c^\infty(G) \otimes_{\mathbb{C}} V$ (via the natural isomorphism which maps a pure tensor $f \otimes v$ to the function defined by the mapping $g \mapsto f(g)v$). With the identification above, we may define an integral on $C_c^\infty(G; V)$, that is, there exists an unique linear map $I_V \colon C_c^\infty(G; V) \to V$ such that

$$I_V(f \otimes v) = \int_G f(g) d\mu(g) v, \qquad f \in C_c^\infty(G),\ v \in V.$$

This integral benefits of the same properties of the Haar integral, and we shall denote it as

$$I_V(\psi) = \int_G \psi(g) d\mu(g), \qquad \psi \in C_c^\infty(G, V).$$

2.6 Hecke Algebra

With any group G, we can associate the complex group algebra $\mathbb{C}[G]$, that is, the associative algebra over $\mathbb{C}$ consisting of finite linear combinations of the elements of G endowed with the convolution product. It is well-known that, if the group G is finite, the representations of G are in one-to-one correspondence with the representations of $\mathbb{C}[G]$. However, if the group is not finite, this relation is lost, and so we must define (if possible) a different algebra to play the role of the group algebra.

Let G be an ℓ-group which we will assume to be unimodular, and let μ be a Haar measure on G; we note that all the stated results are valid (with minor changes) in the case where G is not unimodular, but the proofs become more intricate. As before, we consider the vector space $C_c^\infty(G)$ and, for every $f_1, f_2 \in C_c^\infty(G)$, we define the convolution product $f_1 * f_2 \in C_c^\infty(G)$ by:

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g) d\mu(h), \qquad g \in G$$

It is straightforward to check that $C_c^\infty(G)$ becomes an associative algebra not necessarily with an identity; indeed, it has an identity if and only if the group G is compact. Henceforth, we shall use the (standard) notation $\mathcal{H}(G)$ for this algebra, and refer to it as the *Hecke algebra* of

G.

The Hecke algebra $\mathcal{H}(G)$ has many idempotent elements: for every open compact subgroup K of G, we define the function

$$e_K = \mu(K)^{-1}\mathbb{I}_K \in \mathcal{H}(G)$$

where $\mathbb{I}_K$ is the characteristic function of K; hence, for every $g \in G$, we have

$$e_K(g) = \begin{cases} \mu(K)^{-1}, & \text{if } g \in K \\ 0, & \text{otherwise} \end{cases}$$

The following holds for every open compact subgroup K of G:

(1) e_K is an idempotent (that is, $e_K = e_K * e_K$).

(2) A function $f \in \mathcal{H}(G)$ is K-invariant (that is, $f(kg) = f(g)$ for all $k \in K$ and all $g \in G$) if and only if $e_K * f = f$.

(3) The set $e_K * \mathcal{H}(G) * e_K$ is a subalgebra of $\mathcal{H}(G)$ with identity e_K.

Notice that the subalgebra $e_K * \mathcal{H} * e_K$, for an open compact subgroup of G, consists of all functions $f \in \mathcal{H}(G)$ which satisfy

$$f(k_1 g k_2) = f(g), \qquad k_1, k_2 \in K,\ g \in G;$$

we shall denote it by $\mathcal{H}(G, K)$. We note also that, since every function $f \in \mathcal{H}(G)$ is locally constant and has compact support, we should have $f \in \mathcal{H}(G, K)$ for some open compact subgroup K of G, and so

$$\mathcal{H}(G) = \bigcup_K \mathcal{H}(G, K)$$

where the union is over all open compact subgroups K of G.

Next, we consider $\mathcal{H}(G)$-modules; as above, G denotes an arbitrary ℓ-group. Let V be an $\mathcal{H}(G)$-module, and let us denote by $f * v$ the action of $f \in \mathcal{H}(G)$ on $v \in V$. As for G-modules, we shall not consider arbitrary $\mathcal{H}(G)$-modules; we shall concern ourselves only with those which are smooth: an $\mathcal{H}(G)$-module V is said to be *smooth* if $\mathcal{H}(G) * V = V$. Note that an $\mathcal{H}(G)$-module V is smooth if and only if, for all $v \in V$, there exists an open compact subgroup K such that $e_K * v = v$. We shall denote by $\mathrm{Mod}(\mathcal{H}(G))$ the category of smooth $\mathcal{H}(G)$-modules where the morphisms are homomorphisms of $\mathcal{H}(G)$-modules: if V_1 and V_2 are

$\mathcal{H}(G)$-modules, then a linear map $\psi : V_1 \to V_2$ is called an *homomorphism of* $\mathcal{H}(G)$*-modules* (or simply an $\mathcal{H}(G)$*-homomorphism*) if

$$\psi(f * v) = f * \psi(v), \qquad f \in \mathcal{H}(G),\ v \in V_1.$$

Now, let V be a smooth G-module, and let $f \in \mathcal{H}(G)$ and $v \in V$ be arbitrary. Then, we may choose an open compact subgroup K of G which fixes both f and v, and define

$$f * v = \sum_{x \in X} f(x)(xv)$$

where X is a complete set of representatives of the coset space G/K; notice that this sum is finite because $\mathrm{supp}(f)$ is compact. Then, we can prove that the mapping $(f, v) \mapsto f * v$ endows V with a structure of smooth $\mathcal{H}(G)$-module; indeed, this procedure establishes an equivalence between the categories $\mathrm{Rep}(G)$ and $\mathrm{Mod}(\mathcal{H}(G))$. (For more details, we refer to [BH06, Section 4.2].)

If V is a smooth $\mathcal{H}(G)$-module, then the G-action on V is defined as follows: for every $v \in V$, we choose a compact open subgroup K of G such that $e_K * m = m$ and define

$$g \cdot v = \mu(K)^{-1}(\mathbb{I}_{gK} * v) = e_{gK} * v, \qquad g \in G.$$

With this action, V becomes a smooth G-module.

We conclude this section with some observations concerning the behaviour of a smooth $\mathcal{H}(G)$-module (hence, also a smooth G-module) under the decomposition $\mathcal{H}(G) = \bigcup_K \mathcal{H}(G, K)$ where K runs over all open compact subgroups of G; as above, details can be found in [BH06, Section 4.2]. Let V be a smooth G-module, and let K be an open compact subgroup of G. Then, the mapping $f \mapsto e_K * f$ defines a projection $V \to V_K$; in particular, the vector space V_K is an $\mathcal{H}(G, K)$-module. On the other hand, the mapping $V \mapsto V_K$ defines a bijection between the isomorphism classes of irreducible smooth G-modules V satisfying $V_K \neq 0$ and the isomorphism classes of simple $\mathcal{H}(G, K)$-modules. Furthermore, if V is a non-zero smooth G-module, then V is irreducible if and only if, for every open compact subgroup K of G, either $V_K = 0$, or V_K is a simple $\mathcal{H}(G, K)$-module.

Chapter 3

Rodier Theory

Clifford theory (and also Mackey theory) is one of the most important tools in the study of representations of finite groups, since it allows to "reduce" to the representation theory of a normal subgroup. However, in general, Clifford theory cannot be applied in the context of infinite groups (in particular, of ℓ-groups); one of the main obstructions is that a smooth representation of an ℓ-group does not necessarily have irreducible subrepresentations. An attempt to replace Clifford theory (and Mackey theory) was developed by F. Rodier in the paper [Rod77] in the case where G is an ℓ-group and N is an abelian normal closed ℓ_c-subgroup of G. In this chapter, we briefly describe Rodier theory. Since it is fundamental for the work developed in this thesis, we will include some of the most relevant proofs; a more detailed explanation can be found in [Rod77].

3.1 ℓ-spaces and ℓ-sheaves

Although we are mainly concerned with ℓ-groups, in this section we consider an arbitrary *ℓ-space* X, that is, a topological space X whose underlying topology is locally compact, totally disconnected, and Hausdorff; in particular, the set $\mathcal{T}_c(X)$ consisting of all open compact subsets of X is a base for the topology of X (that is, for every $x \in X$ and every open subset U of X with $x \in U$, there exists a compact open subset $U_0 \in \mathcal{T}_c(X)$ such that $x \in U_0 \subseteq U$). For every open subset U of X, we denote by $\mathcal{C}_X^\infty(U)$ denote the (complex) vector space consisting of all locally constant functions $f : U \to \mathbb{C}$. Then, the mapping $U \mapsto \mathcal{C}_X^\infty(U)$ defines a sheaf of commutative $\mathbb{C}$-algebras (with respect to the pointwise multiplication of functions) on X, and thus the pair $(X, \mathcal{C}_X^\infty)$ is a ringed space; for the basic notions and properties of sheaf theory, we refer to [God58] (for a slightly more general approach see the unpublished notes [RS, Section II.2], where the authors consider, for any base $\mathcal{T}_0$ for the topology of X, the notion of a (pre)sheaf over X with base $\mathcal{T}_0$).

On the other hand, we denote by $C_c^\infty(X)$ the vector space consisting of all complex-valued functions defined on X which are locally constant and have compact support; in particular, $C_c^\infty(X)$ is a vector subspace of $C^\infty(X) = \mathcal{C}_X^\infty(X)$. It is obvious that $C_c^\infty(X)$ is a subalgebra of $\mathcal{C}^\infty(X)$, and that

$$C_c^\infty(X) = \bigcup_{U \in \mathcal{T}_c(X)} \mathcal{C}_X^\infty(U)$$

where, for every $U \in \mathcal{T}_c(X)$, we consider the natural embedding of $\mathcal{C}_X^\infty(U)$ as a subalgebra of $C_c^\infty(X)$. (Note that, in the case where $X = G$ is an ℓ-group, $C_c^\infty(G)$ is also the underlying vector space of the Hecke algebra $\mathcal{H}(G)$ of G; however, we will use different notations to distinguish the different multiplications on $C_c^\infty(G)$ and on $\mathcal{H}(G)$.)

The $\mathbb{C}$-algebra $C_c^\infty(X)$ does not necessarily have an identity; in fact, it has an identity if and only the toplogical space X is compact (because, if this is the case, then the characteristic function $\mathbb{I}_X$ of X lies in $C_c^\infty(X)$). However, $C_c^\infty(X)$ has a large family of idempotents: for every open compact subset $U \in \mathcal{T}_c(X)$, the characteristic function $\mathbb{I}_U$ of U lies in $C_c^\infty(X)$ and is clearly an idempotent.

In what follows, we describe the connection between the category $\mathrm{Mod}(C_c^\infty(X))$ *of smooth* $C_c^\infty(X)$*-modules* and the category $\mathrm{ShMod}(\mathcal{C}_X^\infty)$ of sheaves of $\mathcal{C}_X^\infty$ -modules on G. On the one hand, a $C_c^\infty(X)$-module M is said to be *smooth* if $C_c^\infty(X) \cdot M = M$; notice that, if X is compact, then every $C_c^\infty(X)$-module is smooth (because $C_c^\infty(X)$ has identity $\mathbb{I}_X$). On the other hand, by a *sheaf of* $\mathcal{C}_X^\infty$ *-modules* on X we mean a sheaf (of complex vector spaces) M on X such that, for every open subset U of X, the vector space $M(U)$ is a $\mathcal{C}_X^\infty(U)$-module; notice that, if $U \subseteq V$ are open subsets of X, then the restriction map $\mathcal{C}_X^\infty(V) \to \mathcal{C}_X^\infty(U)$ is a ring homomorphism, and thus every $\mathcal{C}_X^\infty(U)$-module also has a natural structure of $\mathcal{C}_X^\infty(V)$-module (so that the restriction map $M(V) \to M(U)$ becomes a homomorphism of $\mathcal{C}_X^\infty(V)$-modules).

For a proof of the following result, we refer [RS, Section II.2, pg.52]. We should mention that the usual notion of sheaf M on X requires that the mapping $U \mapsto M(U)$ is defined for all open subsets U of X; however, as explained in [RS, pgs. 53-54], we may use the slightly more general requirement that that mapping is defined only for the open subsets in an arbitrarily given base $\mathcal{T}_0$ for the topology of X (in particular, for the open compact subsets in $\mathcal{T}_c(X)$).

Proposition 3.1.1. *Let X be an ℓ-space, and let $\mathcal{M}$ a sheaf of complex vector spaces on X. Then, $\mathcal{M}$ is naturally a sheaf of $\mathcal{C}_X^\infty$-modules. In particular, we obtain an equivalence between the category* $\mathrm{ShVec}(X)$ *of sheaves of vector spaces on X and the category* $\mathrm{ShMod}(\mathcal{C}_X^\infty)$ *of sheaves of $\mathcal{C}_X^\infty$ -modules.*

Proof. For every $x \in X$, we define the *stalk* $\mathcal{M}_x$ a x to be the injective limit

$$\mathcal{M}_x = \varinjlim_{U \in \mathcal{V}(x)} \mathcal{M}(U)$$

where $\mathcal{V}(x)$ denotes the set consisting of all open neighbourhoods of x, and let

$$\widehat{\mathcal{M}} = \coprod_{x \in G} \mathcal{M}_x$$

be the *étalé space* associated with $\mathcal{M}$ (hence, $\widehat{\mathcal{M}}$ is the disjoint union of the stalks $\mathcal{M}_x$ for $x \in X$). As it is well-known, $\widehat{M}$ may be equipped with a topology with respect to which there exists a local homeomorphism $\pi \colon \widehat{\mathcal{M}} \to X$ such that

$$\pi(\mathcal{M}_x) = x, \qquad x \in X.$$

For every open subset U of X, a *section of* $\widehat{\mathcal{M}}$ *over* U is (by definition) a continuous map $s \colon U \to \widehat{\mathcal{M}}$ such that $\pi \circ s = \mathrm{id}_U$; equivalently, a section $s \colon U \to \widehat{\mathcal{M}}$ is a map such that

$$s(x) \in \mathcal{M}_x, \qquad x \in U,$$

and there exists $m \in \mathcal{M}(U)$ with

$$s(x) = \rho_{U,x}(m)$$

where $\rho_{U,x} : \mathcal{M}(U) \to \mathcal{M}_x$ is the natural projection. Then, the sheaf $\mathcal{M}$ may be naturally identified (up to isomorphism) with the sheaf $\mathcal{M}^+$ on X where, for every open subset U of X, the vector space consists of all sections $s : U \to \widehat{\mathcal{M}}$; for a detailed proof see [RS, Proposition II.2.1,pg. 51] (and also the remark on pg. 53). (This identification will be done throughout the thesis.)

Now, suppose that $U \in \mathcal{T}_c(X)$ is an arbitrary open subset of X, let $s : U \to \widehat{\mathcal{M}}$ be a section over U, and let $f \in \mathcal{C}^\infty_X(U)$. Then the mapping $x \mapsto f(x) \cdot s(x)$ defines a section $f \cdot s \colon U \to \widehat{\mathcal{M}}$ (because $f \in \mathcal{C}^\infty_X(U)$ is locally constant) and thus $f \cdot s \in \mathcal{M}(U)$. This shows that $\mathcal{M}(U)$ is a $\mathcal{C}^\infty_X(U)$-module for all open subset U of X, and thus $\mathcal{M}$ is a sheaf of $\mathcal{C}^\infty_X$-modules, as we required.

For the last assertion, it is enough to observe that, for every open set U of X, the mapping $\alpha \mapsto \alpha \mathbb{I}_U$ defines a ring homomorphism $\mathbb{C} \to \mathcal{C}^\infty_X(U)$ (hence the field $\mathbb{C}$ may be naturally embedded as a subring of $\mathcal{C}^\infty_X(U)$) and thus $\mathcal{C}^\infty_X(U)$ has a natural structure of $\mathbb{C}$-algebra. Therefore, the notions of a sheaf of a complex vector spaces on X and of a sheaf of $\mathcal{C}^\infty_X$-modules are equivalent (which implies that the categories $\mathrm{ShVec}(X)$ and $\mathrm{ShMod}(\mathcal{C}^\infty_X)$ are equivalent). $\square$

We next observe that there exists an equivalence between the category $\mathrm{ShMod}(\mathcal{C}_G^\infty)$ and the category $\mathrm{Mod}(C_c^\infty(X))$ of smooth $C_c^\infty(X)$-modules. Let $\mathcal{M}$ be a sheaf of $\mathcal{C}_X^\infty$-modules which, as above, we identify with the sheaf $\mathcal{M}^+$ of sections of the étalé space $\widehat{\mathcal{M}}$. Then, we define $\mathcal{M}_c(G)$ to be the vector space consisting of all compactly supported sections of $\widehat{\mathcal{M}}$; hence,

$$\mathcal{M}_c(X) = \bigcup_{U \in \mathcal{T}_c(X)} \mathcal{M}^+(U)$$

(see [RS, Proposition II.2.5, pg. 56]). There is a natural action of the $C_c^\infty(X)$ on the left of $\mathcal{M}_c(X)$ which endows $\mathcal{M}_c(X)$ with a structure of $C_c^\infty(X)$-module. On the other hand, if $s \in \mathcal{M}_c(X)$ is arbitrary, and if $U \in \mathcal{T}_c(X)$ is such that $\mathrm{supp}(s) \subseteq U$, then $\mathbb{I}_U \cdot s = s$ which implies that $\mathcal{M}_c(X)$ is indeed a smooth $C_c^\infty(X)$-module. We have the following result.

Theorem 3.1.2 ([RS, Proposition II.2.5, pg. 57])**.** *Let X be an ℓ-space, and let $\mathcal{M}$ be a sheaf of $\mathcal{C}_X^\infty$-modules on G. Then, $\mathcal{M}_c(X)$ is a smooth $C_c^\infty(X)$-module, and the mapping $\mathcal{M} \mapsto \mathcal{M}_c(X)$ defines an equivalence between the category $\mathrm{ShMod}(\mathcal{C}_X^\infty)$ of sheaves of $\mathcal{C}_X^\infty$ -modules over X and the category $\mathrm{Mod}(C_c^\infty(X))$ of smooth $C_c^\infty(X)$-modules.*

Proof (sketch). The quasi-inverse functor is described as follows. For every smooth $C_c^\infty(X)$-module M and every open compact subset $U \in \mathcal{T}_c(X)$, we define

$$\mathcal{M}(U) = \mathbb{I}_U \cdot M = \{m \in M \mid \mathbb{I}_U \cdot m = m\}$$

Then, it can be proved that the mapping $U \mapsto \mathcal{M}(U)$ extends to every open subset of X, and that we obtain a sheaf $\mathcal{M}$ of $\mathcal{C}_X^\infty$-modules over X (for details, see [RS, Proposition II.2.5]). On the other hand, we see that

$$\mathcal{M}_c(X) = \bigcup_{U \in \mathcal{T}_c(X)} \mathcal{M}(U) = \bigcup_{U \in \mathcal{T}_c(X)} \mathbb{I}_U \cdot M = M$$

(because M is smooth), and the theorem follows. $\square$

In the notation of the previous proof, for every $x \in X$, the stalk $\mathcal{M}_x$ may be identified with the quotient vector space $M/M(x)$ where

$$M(g) = \{m \in M \mid f \cdot m = 0 \text{ for some } f \in C_c^\infty(G) \text{ satisfying } f(g) \neq 0\};$$

for a proof see [RS, pg. 58-59]. We can give a different (but useful) description of the vector subspace $M(x)$ in the case where $X = G$ is an ℓ-group. Let $m \in M(g)$ be arbitrary, and let $f \in C_c^\infty(G)$ be such that $f(g) \neq 0$ and $f \cdot m = 0$. Let K be an open compact subgroup of

G such that f is constant in gK (such a open compact subgroup exists because f is locally constant and the open compact subgroups of G form basis of open neighbourhoods of 1). Then,

$$\mathbb{I}_{gK} = f(g)^{-1}\mathbb{I}_{gK}f,$$

and so

$$\mathbb{I}_{gK} \cdot m = (f(g)^{-1}\mathbb{I}_{gK}f) \cdot m = (f(g)^{-1}\mathbb{I}_{gK})(f \cdot m) = 0.$$

It follows that

$$M(g) = \{m \in M \mid \mathbb{I}_{gK} \cdot m = 0 \text{ for some open compact subgroup } K \text{ of } G\}.$$

In virtue of [BZ76, Proposition 1.14], the previous theorem motivates the definition of an ℓ-*sheaf* given in [BZ76] (which can be considered as an alternative definition of a sheaf of $\mathcal{C}_X^\infty$-modules). If X is an ℓ-space, then a pair $(X, \mathcal{F})$ is said to be an ℓ-*sheaf* on X if with every $x \in X$ there is an associated complex vector space $\mathcal{F}_x$ such $\mathcal{F}$ is a family of functions

$$s\colon X \to \widehat{\mathcal{F}} = \coprod_{x\in X} \mathcal{F}_x$$

satisfying

$$s(x) \in \mathcal{F}_x, \qquad x \in X$$

(to which we refer as the *(global) sections* of $\widehat{\mathcal{F}}$) such that the following conditions are satisfied:

(1) $\mathcal{F}$ is invariant under addition and under multiplication by functions on $C^\infty(X)$.

(2) If $f : X \to \widehat{\mathcal{F}}$ is a section such that $f_U = s_U$ for some open subset U of X with $x \in U$, then $f \in \mathcal{F}$.

(3) If $s \in \mathcal{F}$ and $x \in X$ is such that $s(x) = 0$, then $s_U = 0$ for some open subset U of X with $x \in U$.

(4) For every $x \in X$ and every $v \in \mathcal{F}_x$, there exists a section $s \in \mathcal{F}$ such that $s(x) = v$.

As above, we denote by $\mathcal{F}_c(X)$ the vector space consisting of all compactly supported sections $s \in \mathcal{F}$; as above, we see that $\mathcal{F}_c(X)$ is a smooth $C_c^\infty(X)$-module.

Using this terminology, the previous theorem can be restated as follows.

Theorem 3.1.3 ([BZ76, Proposition 1.14]). *Let X be an ℓ-space, and let M be a smooth $C_c^\infty(X)$-module. Then, there exists a unique (up to isomorphism) ℓ-sheaf $\mathcal{M}$ on X such that M is isomorphic as a $C_c^\infty(X)$-module to $\mathcal{M}_c(X)$.*

Here, by an *isomorphism* $\gamma^*\mathcal{F} \to \mathcal{F}'$ between two given ℓ-sheaves $\mathcal{F}$ and $\mathcal{F}'$ on ℓ-space X and X', respectively, we mean a pair (γ, γ_c) where $\gamma : X \to X'$ is a homeomorphism of ℓ-spaces and $\gamma_c : \mathcal{F}_c(X) \to \mathcal{F}'_c(X')$ is a linear isomorphism satisfying

$$\gamma_c(f \cdot s) = (f \circ \gamma^{-1}) \cdot \gamma_c(s), \qquad f \in C_c^\infty(X),\ s \in \mathcal{F}_c(X);$$

notice that the mapping $f \mapsto f \circ \gamma^{-1}$ defines an algebra isomorphism $C_c^\infty(X) \to C_c^\infty(X')$. We also observe that, for every $x \in X$, the isomorphism $\gamma^* : \mathcal{F} \to \mathcal{F}'$ induces a linear isomorphism of stalks $(\gamma^*)_x : \mathcal{F}_x \to \mathcal{F}'_{\gamma(x)}$. A particular case occurs when an arbitrary ℓ-group G acts continuously on an ℓ-space (hence, every $g \in G$ defines a homeomorphism $g : X \to X$ via the mapping $x \to g \cdot x$); then, by an *action of* G *on an* ℓ*-sheaf* $\mathcal{F}$ on X we mean a family of isomorphisms $g^* : \mathcal{F} \to \mathcal{F}$, for $g \in G$, such that

$$(gh)^* = g^* \circ h^*, \qquad g, h \in G$$

and $1^* : \mathcal{F} \to \mathcal{F}$ is such that $1_c = \mathrm{id}_{\mathcal{F}_c(X)}$ is the identity map of $\mathcal{F}_c(X)$.

We also observe that every homeomorphism $\gamma : X \to X'$ of ℓ-spaces X and X' induces an isomorphism $C_c^\infty(X) \to C_c^\infty(X')$ of $\mathbb{C}$-algebras (via the mapping $f \mapsto f \circ \gamma^{-1}$), and thus every sheaf $\mathcal{F}'$ on X' determines a sheaf on X, which we will denote by $\gamma^*\mathcal{F}$, where the sections are naturally given by composing with γ^{-1} (on the right). Then, we obtain an isomorphism of sheaves $\gamma^*\mathcal{F}' \simeq \mathcal{F}'$ where

$$(\gamma_c\mathcal{F}')_c(X) = \{s \circ \gamma \mid s \in \mathcal{F}'_c(X')\}$$

and where

$$(\gamma^*\mathcal{F}')_x \simeq \mathcal{F}'_{\gamma(x)}, \qquad x \in X.$$

3.2 Sheaves on the Pontryagin Dual of an abelian group

Let A be an abelian ℓ_c-group, and let A° denote the Pontryagin dual of A; we recall that, by Propositon 2.2.9, A° consists of all smooth characters $\lambda : A \to \mathbb{C}^\times$. It is well-known that A° has a structure of abelian group with respect to the pointwise multiplication of characters. We endow A° with the compact-open topology, that is, the smallest topology for which the sets

$$V(K, U) = \{\lambda \in A^\circ \mid \lambda(K) \subseteq U\}$$

are open for all compact subset $K \subseteq A$ and all open subset $U \subseteq \mathbb{C}^{\times}$. Then, we have the following result.

Proposition 3.2.1. *If A is an abelian ℓ_c-group, then A° is also an ℓ_c-group.*

Proof. If K is an open compact subgroup of A, then the annihilator $K^{\perp} = \{\lambda \in \widehat{K}\}$ of K in A° is an open compact subgroup of A°. Moreover, the set consisting of all subgroups of this form constitute a basis of neighbourhoods of the identity in A°, and its union equals A° (because, for every $\lambda \in A^{\circ}$, $\ker(\lambda)$ is open and thus there exists an open compact subgroup K of A with $K \subseteq \ker(\lambda)$). □

Now, let μ be a fixed Haar measure on A, and consider the Hecke algebra $\mathcal{H}(A)$ of A; recall that the underlying vector space of $\mathcal{H}(A)$ is $C_c^{\infty}(A)$. We already mentioned (see Section 2.6) that the category of smooth A-modules is equivalent to the category of smooth $\mathcal{H}(A)$-modules. On the other hand, let us consider the *Fourier transform* (on A°) which associates with every $f \in \mathcal{H}(A)$ the function $\widehat{f} : A^{\circ} \to \mathbb{C}$ defined by

$$\widehat{f}(\lambda) = \int_A f(a)\overline{\lambda(a)}d\mu(a), \qquad \lambda \in A^{\circ}.$$

Notice that, for every $\lambda \in A^{\circ}$, the mapping $x \mapsto f(x)\overline{\lambda}(x)$ defines a function in $C_c^{\infty}(A)$ (and hence the integral is well-defined). It is not hard to check that the Fourier transform $f \mapsto \widehat{f}$ defines an algebra isomorphism between $\mathcal{H}(A)$ (with the convolution product) and $C_c^{\infty}(A^{\circ})$ (with the pointwise product); the inverse homomorphism is given by the Fourier transform (on A): for every $f \in \mathcal{H}(A^{\circ})$, we define $\widehat{f} : A \to \mathbb{C}$ by

$$\widehat{f}(a) = \int_{A^{\circ}} f(\theta)\overline{\theta(a)}d\mu^{\circ}(\theta), \qquad a \in A$$

where μ° is a (fixed) Haar measure on A°. It follows that every smooth $\mathcal{H}(A)$-module (and hence every smooth A-module) has a structure of a smooth $C_c^{\infty}(A^{\circ})$-module (and vice-versa), and thus the results of the previous section imply that there is a one-to-one correspondence between smooth A-modules and sheaves of vector spaces on the ℓ_c-group A°.

For every smooth A-module V, let $\mathcal{V}$ denote the ℓ-sheaf on A° which is associated to V; we recall from Theorem 3.1.3 that V is isomorphic as a $C_c^{\infty}(A^{\circ})$-module to $\mathcal{V}_c(A^{\circ})$. Our next aim is to relate the stalks of $\mathcal{V}$ with the twisted Jacquet modules defined in Section 2.4. Let $\lambda \in A^{\circ}$ be arbitrary, and recall from the previous section (see also the proof of [BZ76, Proposition 1.14]) that the stalk $\mathcal{V}_{\lambda}$ is defined as que quotient space $V/V(\lambda)$ where

$$V(\lambda) = \{v \in V \mid \mathbb{I}_{\lambda K^{\perp}} \cdot v = 0 \text{ for some open compact subgroup } K \text{ of } A\}.$$

hence,

$$\mathcal{V}_c(A^\circ) = \coprod_{\theta \in A^\circ} V/V(\theta).$$

We claim that the quotient $V/V(\lambda)$ is in fact equal to the twisted Jacquet module $V_\lambda = J_\lambda^A(V)$ associated with λ; therefore, by the definition, it is enough to show that

$$V(\lambda) = \langle av - \lambda(a)v \mid a \in, v \in V\rangle_{\mathbb{C}}$$

Proposition 3.2.2. *Let the notation be as above, and let $\lambda \in A^\circ$ be arbitrary. Then, $V(\lambda)$ consists of all vectors $v \in V$ for which there exists an open compact subgroup K of A such that*

$$\int_K \overline{\lambda(a)}(a \cdot v) d\mu(a) = 0.$$

Furthermore, the stalk $\mathcal{V}_\lambda \simeq V/V(\lambda)$ is isomorphic to the twisted Jacquet module $V_\lambda = J_\lambda^A(V)$ associated with λ.

Proof. Let K be an arbitrary open compact subgroup of A. For every $v \in V$ we evaluate

$$\begin{aligned}
\mathbb{I}_{\lambda K^\perp} \cdot v = \widehat{\mathbb{I}_{\lambda K^\perp}} * v &= \int_A \widehat{\mathbb{I}_{\lambda K^\perp}}(a)(a \cdot v) d\mu(a) \\
&= \int_A \Big(\int_{A^\circ} \mathbb{I}_{\lambda K^\perp}(\theta)\overline{\theta(a)} d\hat{\mu}(\theta)\Big)(a \cdot v)\mu(a) \\
&= \int_A \Big(\int_{K^\perp} \overline{(\lambda\theta)(a)} d\hat{\mu}(\theta)\Big)(a \cdot v)\mu(a) \\
&= \int_A \overline{\lambda(a)}\Big(\int_{K^\perp} \overline{\theta(a)} d\hat{\mu}(\theta)\Big)(a \cdot v)\mu(a)
\end{aligned}$$

For every $a \in K$, we clearly have

$$\int_{K^\perp} \overline{\theta(a)} d\hat{\mu}(\theta) = \widehat{\mu}(K^\perp).$$

On the other hand, let $a \notin K$ be arbitrary. Since A is an ℓ_c-group, there exists an open compact subgroup $K_a \leq A$ containing both a and K. Since K is compact, the quotient group K_a/K is finite; moreover, every smooth character $\theta \in K^\perp$ determines (uniquely) a character $\widetilde{\theta}$ of K_a/K. In fact, we deduce that

$$\int_{K^\perp} \overline{\theta(a)} d\hat{\mu}(\theta) = \widehat{\mu}(K^\perp) \sum_{\widetilde{\theta} \in \widehat{K_a/K}} \widetilde{\theta}(aK) = 0$$

(because the sum $\sum_{\widetilde{\theta} \in \widehat{K_a/K}} \widetilde{\theta}$ equals the regular character of the finite group K_a/K). This

completes the proof of the first part of the proposition.

For the last assertion, it is enough to repeat the proof of [BH06, Lemma 8.1] (with minor obvious modifications). □

The previous proposition suggests to define the *spectrum* $\mathrm{Spec}(V)$ of a given smooth A-module V as the support $\mathrm{supp}(\mathcal{V})$ of the sheaf $\mathcal{V}$ on A° associated with V, that is,

$$\mathrm{Spec}(V) = \{\lambda \in A^\circ \mid \mathcal{V}_\lambda \neq 0\} = \{\lambda \in A^\circ \mid V_\lambda \neq 0\}$$

3.3 Rodier Theory for abelian groups

In this section, we fix a second countable ℓ-group G, and assume that G possesses a normal abelian closed ℓ_c-subgroup A. Let V be an arbitrary smooth G-module, and consider the restricted A-module $V_N = \mathrm{Res}_A^G(V)$ (which is also smooth), and the twisted Jacquet modules

$$J_\lambda^A(V_A) = V_\lambda, \qquad \lambda \in A^\circ.$$

In particular, we may consider the spectrum $\mathrm{Spec}(V_A)$, which we will denote by $\mathrm{Spec}_A(V)$ and refer to as the *spectrum of* V *with respect to* A; by the definition, we have

$$\mathrm{Spec}_A(V) = \{\lambda \in A^\circ \mid V_\lambda \neq 0\}$$

The group G acts naturally on A° via

$$\lambda^g(a) = \lambda(gag^{-1}), \qquad \lambda \in A^\circ,\ g \in G,\ a \in A,$$

and it is not hard to check that $\mathrm{Spec}_A(V)$ is invariant under the action of G. Furthermore, we have the following elementary result.

Lemma 3.3.1. *Let $\lambda \in A^\circ$, and let*

$$C_G(\lambda) = \{g \in G \mid \lambda^g = \lambda\}$$

denote the centralizer of λ. Then, $V(\lambda)$ is a $C_G(\lambda)$-submodule of V, and thus $V_\lambda = V/V(\lambda)$ is a smooth $C_G(\lambda)$-module. Moreover, we have

$$a \cdot v = \lambda(a)v, \qquad a \in A,\ v \in V_\lambda.$$

Proof. It is enough to observe that

$$z(av - \lambda(a)v) = (zaz^{-1})(zv) - \lambda(zaz^{-1})(zv)$$

for all $z \in C_G(\lambda)$, all $a \in A$ and all $v \in V$. $\square$

One of the main goals of this section is to prove that following result.

Theorem 3.3.2 ([Rod77, Théorème 3]). *Let G be a second countable ℓ-group, let A be a normal abelian closed ℓ_c-subgroup of G, and let $\lambda \in A^\circ$. Then, the twisted Jacquet functor J^A_λ, defines an equivalence between the category of the smooth G-modules V satisfying $\mathrm{Spec}_A(V) \subseteq \lambda^G$ and the category of smooth $C_G(\lambda)$-modules W where A operates via the character λ (that is, such that $aw = \lambda(a)w$ for all $a \in A$ and all $w \in W$). Furthermore, the inverse functor is given by the compact induction functor* $\text{c-Ind}^G_{C_G(\lambda)}$.

As a first step towards the proof of this theorem, let H be an arbitrary closed subgroup of G, and consider the $\mathbb{C}$-algebra $C^\infty_c(H \setminus G)$; notice that $H \setminus G$ is an ℓ-space with respect to the quotient topology. On the other hand, let W be an arbitrary smooth H-module and consider the smooth G-module $\text{c-Ind}^G_H(W)$ which is compactly induced by W. For every $f \in C^\infty_c(H \setminus G)$ and every $\phi \in \text{c-Ind}^G_H(W)$, we define the function $f \cdot \phi : G \to W$ by

$$(f \cdot \phi)(g) = f(Hg)\phi(g), \qquad g \in G;$$

it is easy to see that this function lies in $\text{c-Ind}^G_H(W)$, and that the mapping $(f, \phi) \mapsto f \cdot \phi$ endows $\text{c-Ind}^G_H(W)$ with a structure of smooth $C^\infty_c(H \setminus G)$-module.

By Theorem 3.1.3, we know that there exists a unique (up to isomorphism) ℓ-sheaf $\mathcal{F}$ on $H \setminus G$ such that $\mathcal{F}_c(H \setminus G) \simeq \text{c-Ind}^G_H(W)$ as a $C^\infty_c(H \setminus G)$-module. The vector space $\mathcal{F}_c(H \setminus G)$ becomes a smooth G-module with respect to the G-action defined by

$$(g \cdot s)(Hg') = s(Hg'g), \qquad g, g' \in G, s \in \mathcal{F}_c(H \setminus G),$$

and there is an isomorphism of G-modules

$$\mathcal{F}_c(H \setminus G) \simeq \text{c-Ind}^G_H(W)$$

which can be uniquely chosen so that the stalk $\mathcal{F}_{\bar{1}}$ of $\mathcal{F}$ at $\bar{1} = H \in H \setminus G$ becomes isomorphic to W as an H-module. This is exactly the assertion of [BZ76, Proposition 2.23(a)], and the proof is straigthforward; we note that, for every $g \in G$, we have an isomorphism of sheaves

$g^*\mathcal{F} \simeq \mathcal{F}$ which induces an isomorphism of stalks $\mathcal{F}_{Hg} \simeq (g^*\mathcal{F})_{\bar{1}}$, and thus (if necessary) we may replace $\mathcal{F}$ by $g^*\mathcal{F}$ in order to get the H-isomorphism $(g^*\mathcal{F})_{\bar{1}} \simeq W$.

This result may be applied to our particular situation where A is a normal abelian closed ℓ_c-subgroup of G and V is any smooth G-module. Let $\lambda \in A^\circ$ be such that $V_\lambda \neq 0$ (that is, $\lambda \in \mathrm{Spec}_A(V)$), and consider the centraliser $C_G(\lambda)$ of λ in G. By Lemma 3.3.1, we know that V_λ is a smooth $C_G(\lambda)$-module, and thus there is a sheaf $\mathcal{F}^\lambda$ on $C_G(\lambda)\backslash G$ (which is unique up to isomorphism) such that $\mathcal{F}_c(C_G(\lambda)\backslash G)$ is a smooth G-module and there are isomorphisms

$$\mathcal{F}_c(C_G(\lambda)\backslash G) \simeq \text{c-Ind}_{C_G(\lambda)}^G(V_\lambda)$$

of G-modules, and $(\mathcal{F}^\lambda)_{\bar{1}} \simeq V_\lambda$ of $C_G(\lambda)$-modules (as usual, for any $g \in G$, we use the bar notation $\bar{g} = C_G(\lambda)g \in C_G(\lambda)\backslash G$).

Let us consider the G-orbit

$$\lambda^G = \{\lambda^g \mid g \in G\} \subseteq A^\circ$$

which contains λ, and recall from Proposition 2.1.9 that λ^G is a locally closed subset of A°. Since G is second countable, the mapping $g \mapsto \lambda^g$ defines a homeomorphism

$$\varphi\colon C_G(\lambda)\backslash G \to \lambda^G$$

(by the Orbit-Stabiliser Theorem 2.1.8), and thus we may construct the ℓ-sheaf $\psi^*\mathcal{F}^\lambda$ on λ^G where

$$\psi = \phi^{-1}\colon \lambda^G \to C_G(\lambda)\backslash G$$

is the inverse of φ. We note that, for every $g \in G$, there are linear isomorphisms

$$(\psi^*\mathcal{F}^\lambda)_{\lambda^g} \simeq (\mathcal{F}^\lambda)_{\psi(\lambda^g)} = (\mathcal{F}^\lambda)_{\bar{g}} \simeq (g^*\mathcal{F}^\lambda)_{\bar{1}}$$

where we consider the natural action of G on the right of $C_G(\lambda)\backslash G$ given by right multiplication; in particular, we see that $(\psi^*\mathcal{F}^\lambda)_\lambda \simeq (\mathcal{F}^\lambda)_{\bar{1}} \simeq V_\lambda$. Moreover, we recall that

$$(\psi^*\mathcal{F}^\lambda)_c(\lambda^G) = \{s \circ \psi \mid s \in \mathcal{F}_c^\lambda(C_G(\lambda)\backslash G)\},$$

so that the linear isomorphism

$$\psi_c : (\psi^*\mathcal{F}^\lambda)_c(\lambda^G) \to \mathcal{F}_c^\lambda(C_G(\lambda)\backslash G)$$

is given by

$$\psi_c(s) = s \circ \varphi^{-1} = s \circ \psi, \qquad s \in (\psi^* \mathcal{F}^\lambda)_c(\lambda^G).$$

On the other hand, let $\mathcal{V}$ denote the ℓ-sheaf on A° which is associated with the smooth A-module $V_A = \mathrm{Res}_A^G(V)$, and consider the restriction $\mathcal{V}_{\lambda^G}$ of $\mathcal{V}$ to λ^G; by definition (see for example [BZ76, 1.16]), $\mathcal{V}_{\lambda^G}$ is an ℓ-sheaf on λ^G with stalks equal to V_θ for $\theta \in \lambda^G$, and where a section $s \in \mathcal{V}_{\lambda^G}$ is given by the restriction of some section in $\mathcal{V}$ to a neighbourhood of each $\theta \in \lambda^G$. Then, the mapping $s \mapsto s \circ \psi$ clearly defines an isomorphism of ℓ-sheaves between $\mathcal{F}^\lambda$ and $\mathcal{V}_{\lambda^G}$, and thus we get an isomorphism of ℓ-sheaves

$$\psi^* \mathcal{F}^\lambda \simeq \mathcal{V}_{\lambda^G};$$

indeed, we see that

$$\mathcal{V}_{\lambda^G}(\lambda^G) = \{s \circ \psi \mid s \in \mathcal{F}_c^\lambda(C_G(\lambda)\backslash G)\} = (\psi^* \mathcal{F}^\lambda)_c(C_G(\lambda)\backslash G)$$

Finally, we note that the natural action of G on the right of A° (given by the mapping $(\theta, g) \mapsto \theta^g$) induces an action of G on the left of $\mathcal{V}_{\lambda^G}(\lambda^G)$ which satisfies

$$(g \cdot s)(\theta) = s(\theta^g), \qquad g \in G,\ s \in \mathcal{V}_{\lambda^G}(\lambda^G);$$

in particular, we see that $\mathcal{V}_{\lambda^G}(\lambda^G)$ has a structure of smooth G-module. Since

$$(g \cdot s) \circ \psi = g \cdot (s \circ \psi), \qquad g \in G,\ s \in \mathcal{F}_c^\lambda(C_g(\lambda)\backslash G),$$

we conclude that there are isomorphisms of G-modules

$$(\mathcal{V}_{\lambda^G})_c(\lambda^G) \simeq \mathcal{F}_c^\lambda(C_G(\lambda)\backslash G) \simeq \text{c-Ind}_{C_G(\lambda)}^G(V_\lambda)$$

and this implies the following result.

Proposition 3.3.3. *Let G be a second countabe ℓ-group, and let A be a normal abelian closed ℓ_c-subgroup of G. Let $\lambda \in A^\circ$ be arbitrary, and let V be a smooth G-module such that $\mathrm{Spec}_A(V) \subseteq \lambda^G$. Then, there is an isomorphism of G-modules*

$$V \simeq \text{c-Ind}_{C_G(\lambda)}^G(V_\lambda).$$

Proof. It is enough to recall that

$$\mathrm{Spec}_A(V) = \{\theta \in A^\circ \mid V_\theta \neq 0\},$$

and thus the hypothesis $\mathrm{Spec}_A(V) \subseteq \lambda^G$ implies that there is an obvious linear isomorphism

$$\mathcal{V}_c(A^\circ) \simeq (\mathcal{V}_{\lambda^G})_c(\lambda^G).$$

The result follows because $V \simeq \mathcal{V}_c(A^\circ)$. □

Proof of Theorem 3.3.2. One the one hand, if V is a smooth G-module with $\mathrm{Spec}_A(V) \subseteq G$, then V_λ is a smooth $C_G(\lambda)$-module satisfying

$$a \cdot v = \lambda(a)v, \qquad a \in A,\ v \in V_\lambda$$

(by Proposition 3.3.1), whereas

$$V \simeq \text{c-Ind}_{C_G(\lambda)}^G(V_\lambda)$$

(by the previous proposition).

On the other hand, let W be a smooth $C_G(\lambda)$-module satisfying

$$a \cdot w = \lambda(a)w, \qquad a \in A,\ w \in W,$$

and consider the smooth G-module

$$V = \text{c-Ind}_{C_G(\lambda)}^G(W).$$

Then, as described above, [BZ76, Proposition 2.23(a)] implies that, up to isomorphism, there exists a unique ℓ-sheaf $\mathcal{F}^\lambda$ on $C_G(\lambda)\backslash G$ such that

$$V \simeq \mathcal{F}^\lambda_c(C_G(\lambda)\backslash G)$$

as G-modules, and

$$W \simeq (\mathcal{F}^\lambda)_{\bar{1}}$$

as $C_G(\lambda)$-modules.

Let $\mathcal{V}$ denote the ℓ-sheaf on A° which is associated with the smooth A-module $V_A = \mathrm{Res}^G_A(V)$, and consider the restriction $\mathcal{V}_G$ of $\mathcal{V}$ to λ^G. Then, we may repeat the argument above to conclude that there is an isomorphism of ℓ-sheaves between $\psi^*\mathcal{F}^\lambda$ and $\mathcal{V}_{\lambda^G}$ where $\psi\colon \lambda^G \to C_G(\lambda)\backslash G$ is the natural homeomorphism. In particular, we conclude that there is an isomorphism of ℓ-sheaves between $\mathcal{F}^\lambda$ and $\mathcal{V}_{\lambda^G}$ which induces an isomorphism of G-modules

$$\mathcal{F}^\lambda_c(C_G(\lambda)\backslash G) \simeq (\mathcal{V}_{\lambda^G})_c(\lambda^G),$$

and hence we get an isomorphism of G-modules

$$V \simeq \text{c-Ind}_{C_G(\lambda)}^G(W);$$

furthermore, we get isomorphisms of $C_G(\lambda)$-modules

$$\mathcal{V}_\lambda \simeq (\mathcal{V}_{\lambda^G})_\lambda \simeq (\psi^*\mathcal{F}^\lambda)_\lambda \simeq (\mathcal{F}^\lambda)_{\bar{1}} \simeq W.$$

Finally, we know from Proposition 3.2.2 that

$$\mathcal{V}_\lambda \simeq V_\lambda = J_\lambda^A(V),$$

and thus it is enough to show that $\mathrm{Spec}_A(V) \subseteq \lambda^G$. To see this, let $\theta \in A^\circ$ and $f \in \text{c-Ind}_{C_G(\lambda)}^G(W)$ be arbitrary; for simplicity, we will assume (without loss of generality) that $V = \text{c-Ind}_{C_G(\lambda)}^G(V_\lambda)$. Then for every open compact subgroup K of A, the integral

$$f^{(K)} = \int_K \overline{\theta(a)}(a \cdot f)d\mu(a)$$

is the element of $\text{c-Ind}_{C_G(\lambda)}^G(W)$ given by the rule

$$f^{(K)(g)} = \int_K \overline{\theta(a)}(a \cdot f)(g)d\mu(a) = f(g)\int_K \overline{\theta(a)}\lambda^g(a)d\mu(a), \qquad g \in G,$$

notice that

$$(a \cdot f)(g) = f(ga) = f(gag^{-1}g) = \lambda(gag^{-1})f(g) = \lambda^g(a)f(g), \quad a \in A,\ g \in G.$$

By the orthogonality (which hold because K is compact and every smooth character $\mu \in A^\circ$ has open kernel), we conclude that

$$\int_K \overline{\theta(a)}\lambda^g(a)d\mu(a) \neq 0$$

if and only if

$$\theta_K \in (\lambda^G)_K = \{(\lambda^g)_K \mid g \in G\}.$$

Therefore, if $\theta \notin \lambda^G$, then there exists an open compact subgroup K of A such that $\theta \notin (\lambda^G)_K$ (because A is an ℓ_c-group, and hence the union of all its open compact subgroups). For this

open compact subgroup K, we see that

$$f^{(K)}(g) = 0, \qquad g \in G,$$

and thus

$$\int_K \overline{\theta(a)}(a \cdot f) d\mu(a) = 0$$

It now follows from Proposition 3.2.2 that $f \in V(\theta)$ for all $\theta \in A^\circ \setminus \lambda^G$, and this implies that $V = V(\theta)$ (and hence $V_\theta = 0$) for all $\theta \in A^\circ \setminus \lambda^G$. $\square$

As a consequence of Theorem 3.3.2, we easily deduce that, if V is a smooth G-module satisfying $\mathrm{Spec}_A(V) \subseteq \lambda^G$ for some $\lambda \in A^\circ$, then V is irreducible if and only if the (smooth) $C_G(\lambda)$-module V_λ is irreducible; indeed, the equivalence of categories implies that, for every smooth G-modules V and V' with $\mathrm{Spec}_A(V), \mathrm{Spec}_A(V') \subseteq \lambda^G$, there is a linear isomorphism

$$\mathrm{Hom}_{\mathrm{Rep}(G)}(V, V') \simeq \mathrm{Hom}_{\mathrm{Rep}(C_G(\lambda))}(V_\lambda, V'_\lambda).$$

More generally, we have the following result.

Proposition 3.3.4. *Let G a second countable ℓ-group, and A be a normal abelian closed ℓ_c-subgroup of G. Then, a smooth G-module V is irreducible if and only if there exists $\lambda \in A^\circ$ such that*

(1) $\mathrm{Spec}_A(V) \subseteq \lambda^G$,

(2) V_λ *is an irreducible* $C_G(\lambda)$*-module.*

Proof. Let V be an arbitrary smooth G-module, and let $\lambda \in \mathrm{Spec}_A(V)$. By the observation above, it is enough to prove that, if $\mathrm{Spec}_A(V) \subsetneq \lambda^G$, then V is reducible; hence, we assume that $\mathrm{Spec}_A(V) \setminus \lambda^G$ is non-empty, and let $\theta \in \mathrm{Spec}_A(V) \setminus \lambda^G$. Let λ^G denotes the closure of λ^G, and note that, since λ^G is locally closed (by Proposition 2.1.9), $\overline{\lambda^G} \setminus \lambda^G$ is a closed subset of A°. On the one hand, if $\theta \in \overline{\lambda^G} \setminus \lambda^G$, then we set $Z = \overline{\lambda^G} \setminus \lambda^G$; we note that Z is G-invariant. On the other hand, if $\theta \notin \overline{\lambda^G} \setminus \lambda^G$, we set $Z = \overline{\theta^G}$, so that Z is also a closed G-invariant subset of A°. In any case, we consider the open subset $Z^c = A^\circ \setminus Z$ of A°, and observe that Z^c is also G-invariant.

Now, let $\mathcal{V}$ be the ℓ-sheaf on A° which is associated with V, and consider the ℓ-sheaves $\mathcal{V}' = \mathcal{V}_{Z^c}$ and $\mathcal{V}'' = \mathcal{V}_Z$ which are obtained by restriction to Z^c and to Z, respectively. As a consequence of [God58, Theorem 2.9.3] (see also [BZ76, 1.16 and 1.8]), we conclude that there is an exact sequence

$$0 \to \mathcal{V}'_c(Z^c) \to \mathcal{V}_c(A^\circ) \to \mathcal{V}''_c(Z) \to 0$$

of G-modules; in fact, since Z^c and Z are G-invariant, both $\mathcal{V}'_c(Z^c)$ and $\mathcal{V}''_c(Z)$ are G-modules (and both maps are G-homomorphisms). Finally, notice that, for every $\mu \in A^\circ$, there are linear isomorphisms

$$\begin{cases} \mathcal{V}'_\mu \simeq \mathcal{V}_\mu \simeq V_\mu, & \text{if } \mu \in Z^c, \\ \mathcal{V}''_\mu \simeq \mathcal{V}_\mu \simeq V_\mu, & \text{if } \mu \in Z. \end{cases}$$

It follows that both $\mathcal{V}'_c(Z^c)$ and $\mathcal{V}''_c(Z)$ are non-zero, and this implies that $V \simeq \mathcal{V}_c(A^\circ)$ is reducible, as required. □

The case where the G-orbits on A° are closed is of particular interest because, if this is the case, then the functor of compact induction preserves admissibility.

Theorem 3.3.5. *Let G a second countable ℓ-group, and let A be a normal abelian ℓ_c-subgroup of G. Let V be a smooth G-module such that $\operatorname{Spec}_A(V) \subseteq \lambda^G$, and suppose that the G-orbit λ^G is closed. Then, if the smooth $C_G(\lambda)$-module V_λ is admissible, then V is also admissible.*

Proof. In virtue of Proposition 3.3.3, we may assume that

$$V = \text{c-Ind}_{C_G(\lambda)}^G(V_\lambda);$$

indeed, for every $\theta \in \operatorname{Spec}_A(V)$, there is an isomorphism of G-modules $V \simeq \text{c-Ind}_{C_G(\theta)}^G(V_\theta)$. Suppose that V_λ is admissible, let K be an arbitrary open compact subgroup of G, and consider the vector space V^K consisting of all K-invariant functions $f \in V$; our aim is to prove that V^K is finite-dimensional. To see this, let $\Omega \subseteq G$ be a complete set of representatives of the double classes $C_G(\lambda)gK$ for $g \in G$; Then, for every $f \in V$, the support $\operatorname{supp}(f)$ of f is a union of these double classes (because $\operatorname{supp}(f)$ is compact and K is open). Let $g \in \Omega$, and consider the subgroup $C_{gKg^{-1}}(\lambda) = C_g(\lambda) \cap gKg^{-1}$ of $C_G(\lambda)$; notice that this subgroup is open and compact (because K is open and compact). If $f \in V^K$, then

$$h \cdot f(g) = f(gh) = f(gg^{-1}hg) = ((g^{-1}hg) \cdot f)(g) = f(g), \qquad h \in C_{gKg^{-1}}(\lambda)$$

and thus the vector $f(g) \in V_\lambda$ is fixed by $C_{gKg^{-1}}(\lambda)$ (that is, $f(g) \in (V_\lambda)^{C_{gKg^{-1}}(\lambda)}$). Since V_λ is admissible, we know that $(V_\lambda)^{C_{gKg^{-1}}(\lambda)}$ is finite-dimensional, and thus $(V_\lambda)^{C_{gKg^{-1}}(\lambda)}$ has a finite basis which we denote by $\mathcal{B}_g$. For each $v \in \mathcal{B}_g$, there exists a unique K-invariant function $f_{g,v} \in V^K$ which is supported on $C_G(\lambda)gK$ and such that $f_{g,v}(g) = v$. Then, the set

$$\{f_{g,v} \mid g \in \Omega, v \in \mathcal{B}_g\}$$

is a basis for V^K (see [BH06, Lemma 3.5.1]), and thus we must prove that this set is finite.

For every $f \in V^K$ and all $a \in K \cap A$, we deduce that

$$f(g) = (a \cdot f)(g) = f(ga) = \lambda(gag^{-1})f(g) = \lambda^g(a)f(g)$$

and thus

$$\mathrm{supp}(f) \subseteq \{g \in G \mid \lambda^g(K \cap A) = 1\} = \{g \in G \mid \lambda^g \in (K \cap A)^\perp\}.$$

It follows that, for every $g \in \Omega$, we have

$$C_G(\lambda)gK \subseteq \mathrm{supp}(f) \iff (\lambda^g)^K \subseteq \lambda^G \cap (K \cap A)^\perp;$$

we note that $\lambda^G \cap (K \cap A)^\perp$ is obviously a K-invariant subset of A° (because $K \cap A$ is a normal subgroup of K), and hence it is a disjoint union of K-orbits. Since $K \cap A$ is an open compact subgroup of A, $(K \cap A)^\perp$ is also an open compact subgroup of A°; moreover, since λ^G is a closed subset of A°, we conclude that $\lambda^G \cap (K \cap A)^\perp$ is an open compact subset of λ^G. Finally, since K is open, the K-orbits on A° are open subsets (because they are the image of open subsets of the quotient space $C_G(\lambda)\backslash G$ under the homeomorphism $C_G(\lambda)\backslash G \to \lambda^G$), and thus it follows that are only a finite number of K-orbits lying on the intersection $\lambda^G \cap (K \cap A)^\perp$. This clearly implies that the above basis of V^K is finite, and hence V^K is finite-dimensional, as required.

□

3.4 Rodier Theory for general groups

In this section, we generalise the results of the previous section to the situation where G is a second countable ℓ-group and N is an arbitrary normal closed ℓ_c-subgroup of G (not necessarily abelian). As before, given any smooth G-module, we define the spectrum of V with respect to N to be the subset

$$\mathrm{Spec}_N(V) = \{\lambda \in N^\circ \mid V_\lambda \neq 0\}$$

where N° is the Pontryagin dual of N (that is, the set consisting of all smooth characters of N).

We start with an elementary lemma.

Lemma 3.4.1. *Let G be a second countable ℓ-group, let N be a normal closed ℓ_c-subgroup of G, and let V be a smooth G-module. Then,* $\mathrm{Spec}_N(V)$ *is G-invariant (with respect to the*

natural action of G on the right of N° given by the conjugacy action), and

$$V_0 = \bigcap_{\lambda \in \mathrm{Spec}_N(V)} V(\lambda)$$

is a G-submodule of V. In particular, if V is irreducible and $\mathrm{Spec}_N(V)$ is non-empty, then the closure $\overline{[N,N]}$ in G of the commutator subgroup $[N,N]$ of N acts trivially in V (so that V becomes naturally a smooth $G/\overline{[N,N]}$-module).

Proof. Firstly, we observe that

$$V(\lambda^g) = g^{-1}V(\lambda), \qquad \lambda \in N^\circ,$$

and thus $\mathrm{Spec}_N(V)$ is G-invariant. Moreover, G permutes the set $\{V_\lambda \mid \lambda \in \mathrm{Spec}_N(V)\}$ which implies that V_0 is G-invariant. Now, suppose that V is irreducible and that $\mathrm{Spec}_N(V) \neq \emptyset$, and note that V_0 is the kernel of the natural map $\pi : V \to \coprod_{\lambda \in \mathrm{Spec}_N(V)} V_\lambda$ given by

$$\pi(v) = (v + V(\lambda))_{\lambda \in \mathrm{Spec}_N(V)}, \qquad v \in V.$$

Since $V_\lambda = V/V(\lambda) \neq 0$ for some $\lambda \in N^\circ$, we conclude that $V_0 \neq V$, and thus $V_0 = 0$ and the map π is injective. Since

$$n \cdot v = \lambda(n)v, \qquad n \in N,\ v \in V_\lambda,$$

it follows that $\overline{[N,N]}$ acts trivially on V_λ for all $\lambda \in \mathrm{Spec}_N(V)$, and so it acts trivially on $\coprod_{\lambda \in \mathrm{Spec}_N(V)} V_\lambda$. Since π is injective, we see that $\overline{[N,N]}$ acts trivially in V, as required. □

Most of the results of the previous section hold in the present situation; the main idea is to reduce a particular result to the pair $(G/\overline{[N,N]}), N/\overline{[N,N]}$ (note that $N/\overline{[N,N]}$ is a abelian normal closed ℓ_c-subgroup of $G/\overline{[N,N]}$). In particular, we deduce the following.

Theorem 3.4.2. *Let G be a second countable ℓ-group, let N be a normal ℓ_c-subgroup of G, and let $\lambda \in N^\circ$. Then, the twisted Jacquet functor $V \mapsto V_\lambda$ defines an equivalence between the category of the smooth G-modules V satisfying $\mathrm{Spec}_A(V) \subseteq \lambda^G$ and the category of smooth $C_G(\lambda)$-modules W where N operates via the character λ (that is, such that $nw = \lambda(n)w$ for all $n \in N$ and all $w \in W$). Furthermore, the inverse functor is given by the compact induction functor $W \mapsto \text{c-Ind}_{C_G(\lambda)}^G(W)$.*

As a consequence of Proposition 3.3.3 and 3.3.4 (which are also used in the proof of the previous theorem), we also deduce the following.

Proposition 3.4.3. *Let G be a second countable ℓ-group, let N be a normal closed ℓ_c-subgroup of G, and let V be an irreducible smooth G-module such that $\operatorname{Spec}_N(V) \neq \emptyset$. Then, $\operatorname{Spec}_N(V) \subseteq \lambda^G$ for some $\lambda \in N^\circ$, V_λ is an irreducible $C_G(\lambda)$-module, and there is an isomorphism of G-modules $V \simeq \operatorname{c-Ind}_{C_G(\lambda)}^G(V_\lambda)$.*

Proof. By Lemma 3.4.1, V can be considered as an irreducible $\overline{G}$-module where we set $\overline{G} = G/\overline{[N,N]}$. Let $\lambda \in \operatorname{Spec}_N(V)$ be arbitrary; since $\ker(\lambda)$ is a closed subgroup of N, we have $\overline{[N,N]} \subseteq \ker(\lambda)$, and thus λ determines uniquely a smooth character of the quotient group $\overline{N} = N/\overline{[N,N]}$. Moreover, in virtue of the natural isomorphism $N^\circ \simeq (N/\overline{[N,N]})^\circ$, we see that $\operatorname{Spec}_N(V) = \operatorname{Spec}_{\overline{N}}(V)$, and hence $\lambda \in \operatorname{Spec}_{\overline{N}}(V)$. Therefore, by Proposition 3.3.4, we conclude that V_λ is an irreducible $C_{\overline{G}}(\lambda)$-module and that $\operatorname{Spec}_{\overline{N}}(V) \subseteq \lambda^G$.

On the other hand, Theorem 3.3.2 implies that

$$V \simeq \operatorname{c-Ind}_{C_{\overline{G}(\lambda)}}^{\overline{G}}(V_\lambda),$$

and the proposition follows because $\overline{[N,N]} \subseteq C_G(\lambda)$ and $C_{\overline{G}}(\lambda) = C_G(\lambda)/\overline{[N,N]}$. □

Finally, we mention the following; the proof uses the same techniques as the previous one.

Theorem 3.4.4. *Let G be a second countable ℓ-group, and let N be a normal closed ℓ_c-subgroup of G. Let V be a smooth G-module such that $\operatorname{Spec}_N(V) \subseteq \lambda^G$, and suppose that the G-orbit $\lambda^G \subseteq N^\circ$ is closed. If the smooth $C_G(\lambda)$-module V_λ is admissible, then V is also admissible.*

3.5 Smooth Representations of Unitriangular Groups

As an application of Rodier Theory, we shall describe some irreducible representations of the unitriangular group over a non-Archimedean local field $\Bbbk$. In the case of a unitriangular group $U_n(q)$ over a finite field $\mathbb{F}_q$ (with q elements), some irreducible characters are known, see for example [And, Lemma 2].

Proposition 3.5.1. *Let $1 \leq i < j \leq n$ be natural numbers, and $\alpha \in \mathbb{F}_q^\times$. Let $U_{i,j}(q)$ be the subgroup of $U_n(q)$ consisting of the matrices $x \in U_n(q)$ such that $x_{i,k} = 0$ for all $i < k < j$, and let $\lambda_{i,j}(\alpha)\colon U_{i,j}(q) \to \mathbb{C}^\times$ be the map defined by*

$$\lambda_{i,j}(\alpha)(x) = \psi(\alpha x_{i,j}), \qquad x \in U_n(q),$$

where $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$ is an arbitrary fixed non-trivial linear character of the additive group $\mathbb{F}_q^+$.

Then $\lambda_{i,j}(\alpha)$ *is a linear character of* $U_{i,j}(q)$ *and the induced character*

$$\xi_{i,j}(\alpha) = \mathrm{Ind}_{U_{i,j}(q)}^{U_n(q)}(\lambda_{i,j}(\alpha))$$

is irreducible.

We shall prove that the analogous proposition holds in the case of a unitriangular group over a non-Archimedean local field $\Bbbk$. As above, we fix a non-trivial smooth linear character $\psi\colon \Bbbk \to \mathbb{C}^\times$ of the additive group of $\Bbbk^+$, and for every $1 \leq i < j \leq n$, we define the subgroup

$$U_n^{i,j}(\Bbbk) = \{x \in U_n(\Bbbk) \mid x_{i,k} = 0, i < k < j\}$$

of $U_n(\Bbbk)$; by way of example, the subgroup $U_6^{2,5}(\Bbbk)$ of $U_6(\Bbbk)$ consists of the matrices of the shape:

$$\begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & 0 & 0 & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad * \in \Bbbk$$

Furthermore, for every $1 \leq j \leq n$, we define the following subgroups of $U_n(\Bbbk)$:

- $U_n^{j,L}(\Bbbk) = \{x \in U_n(\Bbbk) \mid x_{k,t} = 0, k < t < j\}$;
- $U^{j,B}(\Bbbk) = \{x \in U_n(\Bbbk) \mid x_{k,t} = 0, j < k < t\}$;
- $U_n^j(\Bbbk) = U_n^{j,L}(\Bbbk) \cap U^{j,B}(\Bbbk)$;

in the example above, $U_6^5(\Bbbk)$ consists of all matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & * & * \\ 0 & 1 & 0 & 0 & * & * \\ 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad * \in \Bbbk$$

Lemma 3.5.2. *For every* $1 \leq j \leq n$, $U_n^j(\Bbbk)$ *is an abelian normal closed* ℓ_c*-subgroup of* $U_n(\Bbbk)$ *satisfying* $U_n^j(\Bbbk) \subseteq U_n^{i,j}(\Bbbk)$ *for all* $1 \leq i < j$.

Proof. Straightforward. $\square$

As above, for every $1 \leq i < j \leq n$ and every $\alpha \in \Bbbk$, we define the map $\lambda_{i,j}(\alpha)\colon U_n^{i,j}(\Bbbk) \to \mathbb{C}^\times$ by

$$\lambda_{i,j}(\alpha)(x) = \psi(\alpha x_{i,j}), \qquad x \in U_n^{i,j}(\Bbbk);$$

notice that $\lambda_{i,j}(\alpha)$ is a non trivial character of $U_n^{i,j}(\Bbbk)$.

Lemma 3.5.3. *For every $1 \leq i < j \leq n$ and every $\alpha \in \Bbbk$, the map $\lambda_{i,j}(\alpha)$ is a smooth character of $U_n^{i,j}(\Bbbk)$. Furthermore, $\lambda_{i,j}(\alpha)$ restricts to a smooth character $\lambda'_{i,j}(\alpha)$ of $U_n^{j}(\Bbbk)$ whose centralizer $C_{U_n(\Bbbk)}(\lambda'_{i,j}(\alpha))$ equals $U_n^{i,j}(\Bbbk)$.*

Proof. Straightforward. $\square$

Now, let $\mathbb{C}_{i,j}(\alpha)$ denote the one-dimensional smooth $U_n^{i,j}(\Bbbk)$-module whose underlying vector space is $\mathbb{C}$ and where $U_n^{i,j}(\Bbbk)$ acts via the character $\lambda_{i,j}(\alpha)$, that is,

$$x \cdot \beta = \lambda_{i,j}(\alpha)(x)\beta, \qquad x \in U_n^{i,j}(\Bbbk),\ \beta \in \mathbb{C}.$$

We can now give a description of some irreducible smooth representations of $U_n(\Bbbk)$.

Proposition 3.5.4. *For every $1 \leq i < j \leq n$ and every $\alpha \in \Bbbk^\times$, the smooth $U_n(\Bbbk)$-module*

$$V_{i,j}(\alpha) = \operatorname{Ind}_{U_n^{i,j}(\Bbbk)}^{U_n(\Bbbk)} \big(\mathbb{C}_{i,j}(\alpha)\big)$$

is irreducible and admissible; in particular, we have

$$\text{c-Ind}_{U_n^{i,j}(\Bbbk)}^{U_n(\Bbbk)} \big(\mathbb{C}_{i,j}(\alpha)\big) = \operatorname{Ind}_{U_n^{i,j}(\Bbbk)}^{U_n(\Bbbk)} \big(\mathbb{C}_{i,j}(\alpha)\big).$$

Proof. It follows from Theorem 3.3.2 that the smooth $U_n(\Bbbk)$-module

$$V'_{i,j}(\alpha) = \text{c-Ind}_{U_n^{i,j}(\Bbbk)}^{U_n(\Bbbk)} \big(\mathbb{C}_{i,j}(\alpha)\big)$$

is irreducible. Moreover, it is not hard to see that the $U_n(\Bbbk)$-orbit $\lambda_{i,j}(\alpha)^{U_n(\Bbbk)}$ consists of all smooth characters of $U_n^{j}(\Bbbk)$ with the form

$$\lambda_{i,j}(\alpha) \prod_{i<k<j} \prod_{\alpha_{i+1},\ldots,\alpha_{j-1} \in \Bbbk} \lambda_{k,j}(\alpha_k),$$

and so it is a closed subset of $U_n^{i,j}(\Bbbk)^\circ$. Therefore, by Theorem 3.3.5, we conclude that the smooth $U_n(\Bbbk)$-module $V'_{i,j}(\alpha)$ is admissible.

We now claim that $V'_{i,j}(\alpha) = V_{i,j}(\alpha)$, that is,

$$\text{c-Ind}_{U_n^{i,j}(\Bbbk)}^{U_n(\Bbbk)}\big(\mathbb{C}_{i,j}(\alpha)\big) = \text{Ind}_{U_n^{i,j}(\Bbbk)}^{U_n(\Bbbk)}\big(\mathbb{C}_{i,j}(\alpha)\big).$$

To see this, we observe that the smooth dual $\mathbb{C}_{i,j}(\alpha)^\vee$ equals $\mathbb{C}_{i,j}(-\alpha)$, and thus the Duality Theorem 2.5.3 implies that there are isomorphism of $U_n(\Bbbk)$-modules

$$\text{Ind}_{U_n^{i,j}(\Bbbk)}^{U_n(\Bbbk)}\big(\mathbb{C}_{i,j}(\alpha)\big) \simeq \Big(\text{c-Ind}_{U_n^{i,j}(\Bbbk)}^{U_n(\Bbbk)}\big(\mathbb{C}_{i,j}(-\alpha)\big)\Big)^\vee \simeq V'_{i,j}(-\alpha)$$

(we recall that both $U_n(\Bbbk)$ and $U_n^{i,j}(\Bbbk)$ are unimodular groups). Since $V_{i,j}(-\alpha)$ is irreducible and admissible (by above), we conclude that $V_{i,j}(-\alpha)^\vee$ is also irreducible (by Proposition 2.2.17), and so

$$V_{i,j}(\alpha) = \text{Ind}_{U_n^{i,j}(\Bbbk)}^{U_n(\Bbbk)}\big(\mathbb{C}_{i,j}(\alpha)\big)$$

is irreducible. Since $V'_{i,j}(\alpha)$ is a non-zero $U_n(\Bbbk)$ submodule of $V_{i,j}(\alpha)$, it follows that $V_{i,j}(\alpha) = V'_{i,j}(\alpha)$, as claimed. □

This proof can be repeated (with obvious modifications) in order to prove the following generalisation

Proposition 3.5.5. *Let $1 \leq i_1 < j_1 < i_2 < j_2 < \cdots < i_m < j_m \leq n$ (for some integer $m \geq 2$), and $\alpha_1, \cdots, \alpha_m \in \Bbbk^\times$ be arbitrary. Let $\mathcal{D} = \{(i_1, j_1), \cdots, (i_m, j_m)\}$, and let*

$$U_n^{\mathcal{D}}(\Bbbk) = \bigcap_{1 \leq k \leq m} U_n^{i_k, j_k}(\Bbbk).$$

Moreover, let $\widetilde{\alpha} = (\alpha_1, \cdots, \alpha_m) \in (\Bbbk^\times)^m$, and define the map $\lambda_{\mathcal{D}}(\widetilde{\alpha}) \colon U_n^{\mathcal{D}}(\Bbbk) \to \mathbb{C}^\times$ by

$$\lambda_{\mathcal{D}}(\widetilde{\alpha})(x) = \prod_{1 \leq k \leq m} \lambda_{i_k, j_k}(\alpha_k)(x), \quad x \in U_n^{\mathcal{D}}(\Bbbk).$$

Then, $\lambda_{\mathcal{D}}(\widetilde{\alpha})$ is a smooth character of $U_n^{\mathcal{D}}(\Bbbk)$ and, if $\mathbb{C}_{\mathcal{D}}(\widetilde{\alpha})$ denotes the one-dimensional smooth $U_n^{\mathcal{D}}(\Bbbk)$-module whose underlying vector space is $\mathbb{C}$ and where $U_n^{\mathcal{D}}(\Bbbk)$ acts by the character $\lambda_{\mathcal{D}}(\widetilde{\alpha})$, then the smooth $U_n(\Bbbk)$-module

$$V_{\mathcal{D}}(\widetilde{\alpha}) = \text{Ind}_{U_n^{\mathcal{D}}(\Bbbk)}^{U_n(\Bbbk)}\big(\mathbb{C}_{\mathcal{D}}(\widetilde{\alpha})\big)$$

is irreducible and admissible; in particular, we have

$$\text{c-Ind}_{U_n^{\mathcal{D}}(\Bbbk)}^{U_n(\Bbbk)}\big(\mathbb{C}_{\mathcal{D}}(\widetilde{\alpha})\big) = \text{Ind}_{U_n^{\mathcal{D}}(\Bbbk)}^{U_n(\Bbbk)}\big(\mathbb{C}_{\mathcal{D}}(\widetilde{\alpha})\big).$$

Chapter 4

Smooth representations of algebra groups and related subgroups

In this chapter, we discuss smooth irreducible representations of an arbitrary finite-dimensional algebra group P over a given non-Archimedean local field $\Bbbk$, and extend this study to the case where the given algebra $\mathcal{J}$ is endowed with an involution σ and $C_P(\sigma)$ consists of all σ-fixed elements of the corresponding algebra group $P = 1 + \mathcal{J}$ (these includes the maximal unipotent subgroups of the classical Borel subgroups of linear groups over $\Bbbk$). The first part is a resume of the results proved by M. Boyarchenko in the paper [Boy11]; for convenience we present some of the more relevant proofs. The second part is a generalisation of the work [And10] by C. André in the context of algebra groups over finite fields.

4.1 Algebra groups over non-Archimedean local fields

The main goal of this section is to sketch the proof of the following theorem; we note Boyarchenko's proof is also valid for algebra groups over finite fields, and in particular it gives an alternative proof of the corresponding result [Hal04, Theorem 1.2] by Z. Halasi.

Theorem 4.1.1 ([Boy11, Theorem 1.3]). *Let $\Bbbk$ be a non-Archimedean local field, and let P be a finite-dimensional algebra group over $\Bbbk$, and let V be an irreducible smooth P-module. Then, V is admissible and there exist an algebra subgroup H of P, and a one-dimensional smooth H-module W such that $V \simeq \mathrm{Ind}_H^P(W)$; in particular, we have $\text{c-}\mathrm{Ind}_H^P(W) = \mathrm{Ind}_H^P(W)$.*

The proof will be divided into two steps: firstly, we prove the existence of an algebra subgroup of P associated with each irreducible P-module in such a way that the Rodier theory may be applied, and then deduce the theorem using Rodier's results. To start with, we mention

the following fundamental theorem due to Z. Halasi whose proof can be found in [Hal04] (see also [Boy11, Theorem 3.3]).

Theorem 4.1.2 ([Hal04, Theorem 1.4]). *If $\mathcal{J}$ is an arbitrary nilpotent ring, then for every $m, n \in \mathbb{N}$, we have*

$$[1+\mathcal{J}^m, 1+\mathcal{J}^n] \leq [1+\mathcal{J}, 1+\mathcal{J}^{m+n-1}]$$

We are now able to state the following proposition. (as usual, if g and h are elements of a group, then we define $g^h = h^{-1}gh$ and $[g,h] = g^{-1}h^{-1}gh = g^{-1}g^h$.)

Proposition 4.1.3 ([Boy11, Proposition 3.1]). *Let $\Bbbk$ be a field, and let $\mathcal{J}$ be an associative nilpotent algebra over $\Bbbk$ without unity. For every $n \in \mathbb{N}$, $n \geq 2$, and define the map*

$$\varphi\colon (1+\mathcal{J}) \times (1+\mathcal{J}^{n-1}) \to (1+\mathcal{J}^n)/[1+\mathcal{J}, 1+\mathcal{J}^{n-1}]$$

by

$$\varphi(1+a, 1+b) = \overline{[1+a, 1+b]}, \qquad a, b \in \mathcal{J};$$

here, we use the bar notation to represent elements of $(1+\mathcal{J}^n)/[1+\mathcal{J}, 1+\mathcal{J}^{n-1}]$. Then, φ is well-defined and induces naturally a map

$$\overline{\varphi}\colon (1+\mathcal{J}/\mathcal{J}^2) \times (1+\mathcal{J}^{n-1}/\mathcal{J}^n) \to (1+\mathcal{J}^n)/[1+\mathcal{J}, 1+\mathcal{J}^{n-1}]$$

which is $\Bbbk$-bilinear in the sense that the following conditions hold for all $a, a_1, a_2 \in \mathcal{J}/\mathcal{J}^2$, all $b, b_1, b_2 \in \mathcal{J}^{n-1}/\mathcal{J}^n$ and $\alpha \in \Bbbk$:

(1) $\overline{\varphi}(1+a_1+a_2, 1+b) = \overline{\varphi}(1+a_1, 1+b)\,\overline{\varphi}(1+a_2, 1+b)$;

(2) $\overline{\varphi}(1+a, 1+b_1+b_2) = \overline{\varphi}(1+a, 1+b_1)\,\overline{\varphi}(1+a, 1+b_2)$;

(3) $\overline{\varphi}(1+\alpha a, 1+b) = \overline{\varphi}(1+a, 1+\alpha b)$.

Proof. The proof is very technical (and not hard), and can be found in [Boy11, Chapter 3.3]. We only observe that the previous theorem asserts that $[1+\mathcal{J}^2, 1+\mathcal{J}^{n-1}] \subset [1+\mathcal{J}, 1+\mathcal{J}^n]$, and so the map $\overline{\varphi}$ is well-defined. $\square$

In what follows, we fix a finite-dimensional nilpotent algebra $\mathcal{J}$ over a non-Archimedean local field $\Bbbk$, and consider the algebra group $P = 1+\mathcal{J}$. Moreover, let V be an arbitrary irreducible smooth P-module, and assume that $\dim V \geq 2$. Since $\mathcal{J}$ is nilpotent, there is an integer $n \geq 2$ such that $\mathcal{J}^n \neq 0$ and $\mathcal{J}^{n+1} = 0$; notice that $\mathcal{J}^2 \neq 0$ (otherwise, $P = 1+\mathcal{J}$ would be abelian, and hence V must be one-dimensional (see Schur's Lemma 2.2.14)). Since $1+\mathcal{J}^n$

lies in the centre of P, Schur's Lemma implies that $1+\mathcal{J}^n$ acts on V by scalar multiplications, and thus we may choose the smallest positive integer m for which there exists $\varsigma \in (1+\mathcal{J}^m)^\circ$ such that

$$g \cdot v = \varsigma(g)v, \qquad g \in 1+\mathcal{J}^m,\ v \in V.$$

We note that, since V is an irreducible smooth P-module with $\dim V \geq 2$, we must have $m \geq 2$; furthermore, since $[1+\mathcal{J}, 1+\mathcal{J}^{m-1}] \subseteq 1+\mathcal{J}^m$, the minimal choice of m implies that ς is not identically equal to 1 (otherwise, Schur's Lemma would imply that the subgroup $1+\mathcal{J}^{m-1}$ acts on V by scalar multiplications). Now, it is clear that

$$\mathcal{J}^{m-1} = \mathcal{L}_1 + \cdots + \mathcal{L}_t$$

for some ideals $\mathcal{L}_1, \cdots, \mathcal{L}_t$ of $\mathcal{J}$ satisfying

$$\mathcal{J}^m \subseteq \mathcal{L}_i \subseteq \mathcal{J}^{m-1} \qquad \text{and} \qquad \dim(\mathcal{L}_i/\mathcal{J}^m) = 1, \quad 1 \leq i \leq t.$$

By the minimal choice of m, we must have $\varsigma([1+\mathcal{J}, 1+\mathcal{L}_i]) \neq 1$ for some $1 \leq i \leq t$; otherwise, we would have $\varsigma([1+\mathcal{J}, 1+\mathcal{J}^{m-1}]) = 1$, and hence $1+\mathcal{J}^{m-1}$ would act on V by scalar multiplications. Henceforth, we fix such an ideal $\mathcal{L} = \mathcal{L}_i$, and let $Q = 1+\mathcal{L}$ (hence, Q is an ideal subgroup of P, and hence a normal subgroup); moreover, we set $N = 1+\mathcal{J}^m$, and note that the smooth character $\varsigma \in N^\circ$ is P-invariant (because $[P,N] \subseteq 1+\mathcal{J}^{m+1}$, and thus $\varsigma([P,N]) = 1$). We have the following (more general) result.

Proposition 4.1.4. *Let $\varsigma \in N^\circ$ be P-invariant, and define*

$$\mathcal{J}_\varsigma = \{a \in \mathcal{J} \mid \varsigma([1+a, 1+u]) = 1 \textit{ for all } u \in \mathcal{L}\}.$$

Then, $\mathcal{J}_\varsigma$ is a subalgebra of $\mathcal{J}$ satisfying $\mathcal{J}^2 \subseteq \mathcal{J}_\varsigma$ and $\dim \mathcal{J}_\varsigma \geq \dim \mathcal{J} - 1$. Furthermore, if we define the map $\varphi_\varsigma \colon P \to Q^\circ$ by the rule

$$\varphi_\varsigma(g)(h) = \varsigma\big([g,h]\big), \qquad g \in P,\ h \in Q,$$

then φ_ς is a group homomorphism with $\ker(\varphi_\varsigma) = 1+\mathcal{J}_\varsigma$ and $\varphi_\varsigma(P) \subseteq N^\perp$ where

$$N^\perp = \{\tau \in Q^\circ \mid N \subseteq \ker(\tau)\}$$

is the orthogonal subgroup of N in Q°; hence, φ_ς defines naturally a group homomorphism $\overline{\varphi}_\varsigma \colon P \to (Q/N)^\circ$.

Proof. We first observe that the map φ_ς is a well-defined group homomorphism. On the one

hand, we have

$$[P,Q] \subseteq [1+\mathcal{J}, 1+\mathcal{J}^{n-1}] \subseteq 1+\mathcal{J}^n = N.$$

On the other hand, since $[g,hk] = [g,k][g,h]^k$, we deduce that

$$\varphi_\varsigma(g)(hk) = \varsigma([g,k])\varsigma([g,h]) = \varphi_\varsigma(g)(h)\varphi_\varsigma(g)(k), \qquad g \in P,\ h,k \in Q;$$

we recall that ς is P-invariant. It follows that, for every $g \in P$, the map $\varphi_\varsigma(g)\colon Q \to \mathbb{C}^\times$ is indeed a smooth character of Q. Similarly, since $[gh,k] = [g,k]^h[h,k]$, we have

$$\varphi_\varsigma(gh)(k) = \varsigma([g,k])\varsigma([h,k]) = \varphi_\varsigma(g)(k)\varphi_\varsigma(h)(k), \qquad g,h \in P,\ k \in Q,$$

and so φ_ς is a group homomorphism.

Now, since $[P,N] \subseteq \ker(\varsigma)$ (because ς is P-invariant), the image $\varphi_\varsigma(P)$ clearly lies in $N^\perp$; moreover, it is obvious (by the definition) that

$$\ker(\varphi_\varsigma) = 1 + \mathcal{J}_\varsigma.$$

Let $a \in \mathcal{J}$ and $\alpha \in \Bbbk$ be arbitrary. It is straightforward to check that

$$[1+\alpha a, 1+u][1+a, 1+\alpha u]^{-1} \in 1+\mathcal{J}^{n+1}, \qquad u \in \mathcal{J}^{n-1};$$

indeed, Proposition 4.1.3 implies that

$$[1+\alpha a, 1+u]\,[1+a, 1+\alpha u]^{-1} \in [P, 1+\mathcal{J}^n], \qquad u \in \mathcal{J}^{n-1}.$$

Since ς is P-invariant, we conclude that

$$\varsigma([1+\alpha a, 1+u]) = \varsigma([1+a, 1+\alpha u]), \qquad u \in \mathcal{J}^{n-1} \tag{$\dagger$}$$

and this clearly implies that

$$\alpha a \in \mathcal{J}_\varsigma, \qquad \alpha \in \Bbbk,\ a \in \mathcal{J}_\varsigma.$$

On the other hand, Theorem 4.1.2 implies that

$$[1+\mathcal{J}^2, 1+\mathcal{L}] \subseteq [1+\mathcal{J}^2, 1+\mathcal{J}^{n-1}] \subseteq [1+\mathcal{J}, 1+\mathcal{J}^n] = [P,N],$$

and thus $\mathcal{J}^2 \subseteq \mathcal{J}_\varsigma$. Since

$$(1+u+v)^{-1}(1+u)(1+v) = 1 + (1+u+v)^{-1}uv \in 1 + \mathcal{J}^2,$$

we see that

$$u+v \in \mathcal{J}_\varsigma, \qquad u, v \in \mathcal{J}_\varsigma.$$

It follows that $\mathcal{J}_\varsigma$ is an ideal of $\mathcal{J}$ with $\mathcal{J}^2 \subseteq \mathcal{J}_\varsigma$.

Finally, notice that $N^\perp \simeq (Q/N)^\circ$ and that

$$Q/N = (1+\mathcal{L})/(1+\mathcal{J}^n) \simeq 1 + (\mathcal{L}/\mathcal{J}^n) \simeq \Bbbk^+.$$

Therefore, we get an isomorphism of abelian groups $N^\perp \cong (\Bbbk^+)^\circ$, and hence $N^\perp$ acquires the structure of a vector space over $\Bbbk$ where the scalar multiplication is defined by

$$(\alpha\tau)(1+u) = \tau(1+\alpha u), \qquad \alpha \in \Bbbk,\ \tau \in N^\perp,\ u \in \mathcal{L}.$$

On the other hand, since $\Bbbk$ is a self-dual field, there is a $\Bbbk$-linear isomorphism $\Bbbk \cong (\Bbbk^+)^\circ$, and thus $N^\perp$ is one-dimensional. Furthermore, the argument used above can be repeated to show that the mapping $a \mapsto \varphi_\varsigma(1+a)$ defines a $\Bbbk$-linear homomorphism $\widehat{\varphi}_\varsigma \colon \mathcal{J} \to N^\perp$ with kernel $\mathcal{J}_\varsigma$; we note that (†) implies that

$$\alpha\widehat{\varphi}_\varsigma(a) = \widehat{\varphi}_\varsigma(\alpha a), \qquad \alpha \in \Bbbk,\ a \in \mathcal{J}.$$

It follows that

$$\dim \mathcal{J} - \dim \mathcal{J}_\varsigma \leq 1,$$

and this completes the proof. □

We next prove the following crucial result.

Proposition 4.1.5. *Let $\varsigma \in N^\circ$ be P-invariant, and let $\mathcal{J}_\varsigma \subseteq \mathcal{J}$ be defined as in Proposition 4.1.4. Then, $[Q,Q] \subseteq \ker(\varsigma)$, and there exists $\lambda \in Q^\circ$ such that $\lambda_N = \varsigma$; moreover, the following properties hold.*

(1) $C_P(\lambda') = 1 + \mathcal{J}_\varsigma$ for all $\lambda' \in Q^\circ$ such that $\lambda'_N = \varsigma$.

(2) If $C_P(\lambda) \neq P$ and if $\lambda' \in Q^\circ$ is such that $\lambda'_N = \varsigma$, then there exists $g \in P$ such that $\lambda' = \lambda^g$.

Proof. We start by observing that $[Q,Q] \subseteq \ker(\varsigma)$. Indeed, since $\dim \mathcal{L} = \dim \mathcal{J}^n + 1$, there exists $a \in \mathcal{L}$ such that $\mathcal{L} = \mathcal{J}^n \oplus \Bbbk a$, and hence

$$Q = (1 + \Bbbk a)N.$$

Since $[1 + \alpha a, 1 + \beta a] = 1$ for all $\alpha, \beta \in \Bbbk$, we see that

$$\varsigma([1 + \Bbbk a, 1 + \Bbbk a]) = 1,$$

and this clearly implies that $\varsigma([Q,Q]) = 1$ (because ς is P-invariant). Let $\mathbb{C}_\varsigma$ denote the (canonical) one-dimensional N-module associated with ς, and let W be an irreducible quotient of the smoothly induced Q-module $\mathrm{Ind}_N^Q(\mathbb{C}_\varsigma)$ (the existence of W is guaranteed by Proposition 2.4.6). Since N is a normal subgroup of Q and ς is Q-invariant, we have

$$x \cdot \phi = \varsigma(x)\phi, \qquad x \in N,\ \phi \in \mathrm{Ind}_N^Q(\mathbb{C}_\varsigma),$$

and thus

$$x \cdot w = \varsigma(x)w, \qquad x \in N,\ w \in W.$$

Since $[Q,Q] \subseteq \ker(\varsigma)$, it follows from Schur's Lemma that $\dim W = 1$, and thus W affords a character $\lambda \in Q^\circ$ which clearly satisfies $\lambda_N = \varsigma$.

In order to prove properties (1) and (2), we consider the group homomorphism $\varphi_\varsigma \colon P \to Q^\circ$ as defined in the Proposition 4.1.4; we recall that

$$\varphi_\varsigma(P) \subseteq N^\perp \qquad \text{and} \qquad \ker(\varphi_\varsigma) = 1 + \mathcal{J}_\varsigma$$

where $\mathcal{J}_\varsigma$ is an ideal of $\mathcal{J}$ satisfying $\mathcal{J}^2 \subseteq \mathcal{J}_\varsigma$ and $\dim \mathcal{J}_\varsigma \geq \dim \mathcal{J} - 1$. On the one hand, (1) follows because

$$C_P(\lambda') = \ker(\varphi_\varsigma) = 1 + \mathcal{J}_\varsigma$$

for all $\lambda' \in Q^\circ$ such that $\lambda'_N = \varsigma$. On the other hand, let us assume that $C_P(\lambda) \neq P$ (hence, $\ker(\varphi_\varsigma) \neq P$ and $\mathcal{J}_\varsigma \neq \mathcal{J}$), and let $x \in P$ be such that $\varphi_\varsigma(x) \in Q^\circ$ is not identically equal to 1. Let $a \in \mathcal{J}$ be such that $x = 1 + a$; then, (†) implies that

$$\varphi_\varsigma(1 + \alpha a) \in \varphi_\varsigma(P) = N^\perp, \qquad \alpha \in \Bbbk.$$

Moreover, since $\mathcal{J}/\mathcal{J}_\varsigma$ is one-dimensional, we conclude that the $\Bbbk$-linear map $\widehat{\varphi}_\varsigma \colon \mathcal{J} \to N^\perp$ (as defined in the proof of Proposition 4.1.4) is surjective, and thus the group homomorphism $\varphi_\varsigma \colon P \to N^\perp$ is also surjective and induces an isomorphism $P/C_P(\lambda) \cong N^\perp$; in particular, it

follows that the mapping $\alpha \mapsto \varphi_\varsigma(1+\alpha a)$ defines group isomorphism $\Bbbk^+ \cong N^\perp$, and that

$$N^\perp = \{\varphi_\varsigma(1+\alpha a) \mid \alpha \in \Bbbk\}.$$

To conclude the proof of (2), let $\lambda' \in Q^\circ$ be such that $\lambda'_N = \varsigma$, and consider the character $\lambda'\lambda^{-1} \in Q^\circ$. It is obvious that $\lambda'\lambda^{-1} \in N^\perp$, and thus there exists $\alpha \in \Bbbk$ such that

$$\lambda'\lambda^{-1} = \varphi_\varsigma(1+\alpha a)$$

If we set $g = 1+\alpha a$, then

$$\begin{aligned} \lambda'(x)\lambda(x)^{-1} &= \varsigma([g,x]) = \varsigma(g^{-1}x^{-1}gx) = \varsigma(gxg^{-1}x^{-1}) \\ &= \lambda(gxg^{-1}x^{-1}) = \lambda(gxg^{-1})\lambda(x)^{-1} \end{aligned}$$

(where the third equation follows from the P-invariance of ς), and hence

$$\lambda'(x) = \lambda(gxg^{-1}), \qquad x \in Q,$$

as required. $\square$

We are now able to prove the main result of this section.

Proof of Theorem 4.1.1. We proceed by induction on $\dim \mathcal{J}$, the result being obvious if $\dim \mathcal{J} = 1$. Therefore, we assume that $\dim \mathcal{J} \geq 2$, and that the result is true whenever $\mathcal{J}'$ is a subalgebra of $\mathcal{J}$ with $\dim \mathcal{J}' \lneq \dim \mathcal{J}$.

Let V be an arbitrary irreducible smooth P-module, and assume that $\dim V \geq 2$ (the case where $\dim V = 1$ is obvious). In this situation, as justified above, we may choose the smallest positive integer m for which there exists a P-invariant smooth character $\varsigma \in (1+\mathcal{J}^m)^\circ$ such that

$$g \cdot v = \varsigma(g)v, \qquad g \in 1+\mathcal{J}^m,\ v \in V;$$

we recall that, since V is an irreducible smooth P-module with $\dim V \geq 2$, we must have $m \geq 2$. Furthermore, by the minimal choice of m, there exists an ideal $\mathcal{L}$ of $\mathcal{J}$ satisfying

$$\mathcal{J}^m \subseteq \mathcal{L} \subseteq \mathcal{J}^{m-1} \qquad \text{and} \qquad \dim(\mathcal{L}/\mathcal{J}^m) = 1,$$

and such that $\varsigma([1+\mathcal{J}, 1+\mathcal{L}]) \neq 1$. Let $N = 1+\mathcal{J}^m$, and let $Q = 1+\mathcal{L}$.

By Proposition 2.4.6, we know that the smooth Q-module $\mathrm{Res}^P_Q(V)$ has an irreducible quotient V'. Since $[Q,Q] \subseteq \ker(\varsigma)$ (by Proposition 4.1.5), Schur's lemma implies that V' is one-

dimensional, and thus it affords a character $\lambda \in Q^\circ$. [Notice that the extreme case where $m = 2$ and $\dim \mathcal{J} = \dim \mathcal{J}^2 + 1$ cannot occur; indeed, in this situation, we must have $Q = P$, and hence $V' = V$ which contradicts the assumption $\dim V \geq 2$.] In particular, we conclude that $V_\lambda \neq 0$ (because, by definition, V_λ is the largest quotient of V where Q acts via the character λ). By Proposition 3.4.3, we know that V_λ is an irreducible $C_P(\lambda)$-module and that $V \simeq \text{c-Ind}_{C_P(\lambda)}^P(V_\lambda)$. Since N acts on V (hence, on V') via the character ς, we must have $\lambda_N = \varsigma$, and thus $C_P(\lambda) = 1 + \mathcal{J}_\varsigma$ for some subalgebra $\mathcal{J}_\varsigma$ of $\mathcal{J}$ (by Proposition 4.1.5). Since $\lambda([P,Q]) = \varsigma([P,Q]) \neq 1$, we must have $C_P(\lambda) \neq P$. Therefore, we have $\dim \mathcal{J}_\varsigma \lneqq \dim \mathcal{J}$, and thus it follows by induction that the smooth $C_P(\lambda)$-module V_λ is admissible and that there exists a subalgebra $\mathcal{J}'$ of $\mathcal{J}_\varsigma$ such that

$$V_\lambda \simeq \text{Ind}_{1+\mathcal{J}'}^{C_P(\lambda)}(W)$$

where W is a one-dimensional $(1+\mathcal{J}')$-module; in particular, we also conclude that

$$\text{c-Ind}_{1+\mathcal{J}'}^{C_P(\lambda)}(W) = \text{Ind}_{1+\mathcal{J}'}^{C_P(\lambda)}(W).$$

Now, by transitivity of c-induction (see [BZ76, Proposition 2.25]), we deduce that

$$V \simeq \text{c-Ind}_{C_P(\lambda)}^P \left(\text{c-Ind}_{1+\mathcal{J}'}^{C_P(\lambda)}(W)\right) \simeq \text{c-Ind}_{1+\mathcal{J}'}^P(W),$$

and so, in order to conclude the proof, it is enough to show that $\text{c-Ind}_{1+\mathcal{J}'}^P(W) = \text{Ind}_{1+\mathcal{J}'}^P(W)$. On the one hand, since P is unimodular, the Duality Theorem 2.5.3 implies that

$$V^\vee \simeq \left(\text{c-Ind}_{1+\mathcal{J}'}^P(W)\right)^\vee \simeq \text{Ind}_{1+\mathcal{J}'}^P(W^\vee).$$

On the other hand, it follows from Proposition 4.1.5 that $\varsigma = \lambda_N$ is a P-invariant smooth character of N, and that

$$\lambda^P = \{\lambda' \in Q^\circ \mid (\lambda')_N = \varsigma\};$$

in particular, λ^P is a closed subset of Q°. Since the smooth $C_P(\lambda)$-module V_λ is admissible, Theorem 3.4.4 implies that

$$V \simeq \text{c-Ind}_{C_P(\lambda)}^P(V_\lambda)$$

is also admissible, and thus its smooth dual $V^\vee$ is irreducible (by Proposition 2.2.17 because V is irreducible). Since $\text{c-Ind}_{1+\mathcal{J}'}^P(W)$ is a submodule of $\text{Ind}_{1+\mathcal{J}'}^P(W)$, we conclude that

$$V^\vee = \text{c-Ind}_{1+\mathcal{J}'}^P(W),$$

and thus
$$(V^\vee)^\vee \simeq \left(\text{c-Ind}_{1+\mathcal{J}'}^P(W^\vee)\right)^\vee \simeq \operatorname{Ind}_{1+\mathcal{J}'}^P((W)^\vee)^\vee$$
(again by the Duality Theorem). Since $(W^\vee)^\vee = W$ (because W is one-dimensional) and since $(V^\vee)^\vee = V$ (by Proposition 2.2.17 because V is admissible), we conclude that
$$V \simeq \operatorname{Ind}_{1+\mathcal{J}'}^P(W),$$
and this completes the proof. □

We next apply Theorem 4.1.1 to the study of unitarisable smooth representations (and also to unitary representations) of algebra groups over non-Archimedean local fields. A smooth G-module V is said to be *unitarisable* if V has a positive-definite Hermitian inner product invariant under the action of G. We also recall the (usual notion) of unitary representations (which are not necessarily smooth) of topological groups. By a *unitary representation* of a topological group G we mean a pair $(\pi, \mathcal{H})$ where $\mathcal{H}$ is an Hilbert vector space over $\mathbb{C}$ and $\pi\colon G \to \mathrm{U}(\mathcal{H})$ is a continuous group homomorphism from G to the group of unitary linear automorphisms of $\mathcal{H}$ equipped with the strong operator topology; in this case, the representation $(\pi, \mathcal{H})$ is said to be *irreducible* if $\mathcal{H} \neq 0$ and 0 and $\mathcal{H}$ are the only $\pi(G)$-invariant closed subspaces of $\mathcal{H}$. As in the case of smooth representations, we use the terminology "unitary G-module" with the obvious meaning: an Hilbert vector space $\mathcal{H}$ is said to be a *unitary G-module* if there is a unitary representation $\pi\colon G \to \mathrm{U}(\mathcal{H})$ (and we sometimes write $g \cdot v = \pi(g)v$ for $g \in G$ and $v \in \mathcal{H}$).

Not all smooth representations of an ℓ-group are unitarisable; for example, the multiplicative group of the p-adic field $\mathbb{Q}_p^\times$ has smooth characters with values outside the complex unit circle, and it is well-known that there does not exist a positive definite Hermitian invariant inner product in $\mathbb{C}$. However, we have the following result.

Proposition 4.1.6. *If $\mathbb{k}$ is a non-Archimedean local field and χ is a smooth character of $\mathbb{k}^\times$, then $\chi = \lambda\mu$ for some smooth characters λ and μ of $\mathbb{k}^\times$ with μ unitary.*

Proof. Since we have the decomposition $\mathbb{k}^\times = \langle \varpi \rangle \times \mathfrak{o}^\times$ where $\mathfrak{o} = \mathfrak{o}_\mathbb{k}$ is the ring of integers of $\mathbb{k}$ and $\varpi \in \mathfrak{o}$ is its prime element, we can define the character λ as the restriction of χ to $\langle \varpi \rangle$ and the character μ as the restriction of χ to $\mathfrak{o}^\times$. Since $\mathfrak{o}^\times$ is a compact subgroup of $\mathbb{k}^\times$, all values of μ lie in the unit circle, and thus μ is unitary. It is also clear that $\chi = \lambda\mu$ (where we extend λ and μ to characters of $\mathbb{k}^\times$ in the natural way). □

The functors of smooth induction and of compact induction do not preserve necessarily the unitarisability of smooth representations; however, the following result holds.

Proposition 4.1.7. *Let G be an ℓ-group, let H be a closed subgroup of G, and suppose that $\delta_{G/H}(h) = 1$. If W is a unitarisable smooth H-module, then the smooth G-module $\text{c-Ind}_H^G W$ is also unitarisable.*

Proof. Since $\delta_{G/H}(h) = 1$, [DE14, Theorem 1.5.3] guarantees that there exists a non-zero Radon measure $\mu = \mu_{H\backslash G}$ on the coset space $H\backslash G$ which is invariant for the action of G. Let $\langle\cdot,\cdot\rangle_W$ denote the G-invariant positive definite Hermitian inner product on W, and define

$$\langle \phi, \psi \rangle = \int_{G/H} \langle \phi(g), \psi(g) \rangle_W d\mu, \qquad \phi, \psi \in \text{c-Ind}_H^G(W);$$

notice that, if $X \subseteq G$ is a complete set of representatives of the cosets of G/H, then every function $\phi \in \text{c-Ind}_H^G(W)$ is completely determined by its values in X, and hence defines a function on G/H which clearly has compact support (notice also that the G-invariance of the inner product on W assures that the values of this function do not depend on the particular choice of the set X). It is straightforward to check that the formula above defines a positive definite Hermitian inner product on $\text{c-Ind}_H^G(W)$ which is G-invariant (because the measure μ is G-invariant). Therefore, we conclude that $V' \cong \text{c-Ind}_H^G(W)$ is unitarisable, as claimed. □

In particular, if both G and H are unimodular groups, then the condition $\delta_{G/H}(h) = 1$ is obviously satisfied, and thus this result applies to algebra groups (which we know to be unimodular). Therefore, we deduce the following corollary of Theorem 4.1.1.

Proposition 4.1.8. *Let $\Bbbk$ be a non-Archimedean local field, and let P a finite-dimensional algebra group over $\Bbbk$. Then, every irreducible smooth P-module is unitarisable.*

Proof. Let V be an irreducible smooth P-module, and let H be an algebra subgroup of P such that $V \simeq \text{c-Ind}_H^P(W)$ for some one-dimensional smooth H-module W. Then, since every smooth character of an algebra group is unitary (by Proposition 2.2.9, because any algebra group is an ℓ_c-group), the smooth H-module W is unitarisable, and thus V is also unitarisable (by the previous proposition). □

Finally, we briefly discuss unitary representations of algebra groups. Let P be an algebra group over a non-Archimedean local field $\Bbbk$, and let $\mathcal{H}$ be a unitary P-module. Given an arbitrary algebra subgroup P_0 of P, we define the unitary P-module $\text{u-Ind}_{P_0}^P(\mathcal{H}_0)$ (which is *unitarily induced* by $\mathcal{H}_0$) as follows: we fix a P-invariant Borel measure $\mu = \mu_{P_0\backslash P}$ on the coset space $P_0\backslash P$, and define $\text{u-Ind}_{P_0}^P(\mathcal{H}_0)$ to be the Hilbert space consisting of all measurable functions $\phi\colon P \to \mathcal{H}_0$ satisfying the conditions:

(1) $\phi(gh) = h \cdot \phi(g)$ for all $g \in P$ and all $h \in P_0$,

(2) $\displaystyle\int_{P_0\backslash P} \langle \phi(g), \phi(g)\rangle_0 \, d\mu_{P/P_0} < \infty,$

where $\langle\cdot,\cdot\rangle_0$ denotes the inner product on $\mathcal{H}_0$. Then, $\text{u-Ind}_{P_0}^{P}(\mathcal{H}_0)$ becomes an Hilbert vector space with respect to the inner product defined by

$$\langle \phi, \psi\rangle = \int_{P_0\backslash P} \langle \phi(g), \psi(g)\rangle_0 \, d\mu, \qquad \phi,\psi \in \text{u-Ind}_{P_0}^{P}(\mathcal{H}_0),$$

and it becomes indeed a unitary P-module with respect to the usual action defined by $(g \cdot \phi)(g') = \phi(g'g)$ for all $g, g' \in P$ and all $\phi \in \text{u-Ind}_{P_0}^{P}(\mathcal{H}_0)$.

As explained by M. Boyarchenko in [Boy11, §5.5], the proof of Theorem 4.1.1 can be adapted to the setting of unitary representations (using the classical Mackey theory; see [Fol95, Chapter 6]) in order to deduce the following.

Theorem 4.1.9 ([Boy11, Theorem 1.1]). *Let $\Bbbk$ be a non-Archimedean local field, let P an algebra group over $\Bbbk$, and let $\mathcal{H}$ be an irreducible unitary P-module. Then, there exist an algebra subgroup P_0 of P and a unitary character $\lambda \in P_0^{\circ}$ such that $\mathcal{H} \simeq \text{u-Ind}_{P_0}^{P}(\mathbb{C}_\lambda)$.*

4.2 Examples: unitriangular groups of small size

We conclude this chapter with two examples describing the irreducible smooth representations of unitriangular groups (of small size) over a non-Archimedean local field $\Bbbk$.

Example 4.2.1 ($U_3(\Bbbk)$)**.** From section 3.5, we know that the following are irreducible admissible smooth $U_3(\Bbbk)$-modules:

- For every $\alpha, \beta \in \Bbbk$, the one-dimensional smooth $U_3(\Bbbk)$-module

$$V_{1,2}(\alpha) \otimes V_{2,3}(\beta) = \mathbb{C}_{1,2}(\alpha) \otimes \mathbb{C}_{2,3}(\beta).$$

- For every $\alpha \in \Bbbk^\times$, the infinite-dimensional smooth $U_3(\Bbbk)$-module

$$V_{1,3}(\alpha) = \text{Ind}_{U_3^{1,3}(\Bbbk)}^{U_3(\Bbbk)}(\mathbb{C}_{1,3}(\alpha))$$

We claim that these are all the irreducible smooth $U_3(\Bbbk)$-modules. On the one hand, since

$$\overline{[U_3(\Bbbk), U_3(\Bbbk)]} = [U_3(\Bbbk), U_3(\Bbbk)] = \{x \in U_3(\Bbbk) \mid x_{1,2} = x_{1,3} = 0\},$$

we have $U_3/[U_3(\Bbbk), U_3(\Bbbk)] \simeq (\Bbbk^2)^+$, and thus every one-dimensional smooth representation is as described above (see Example 2.2.13).

On the other hand, let V be an irreducible $U_3(\Bbbk)$-module with $\dim V \geq 2$. We set $\mathcal{J} = U_3(\Bbbk) - 1$ and, as in the proof of the Theorem 4.1.1, we choose the smallest positive integer $m \in \mathbb{N}$ such that such that $1 + \mathcal{J}^m$ acts by scalar multiplications on V (or, equivalently, such that $1 + \mathcal{J}^{m+1}$ acts trivially on V). Since $\mathcal{J}^3 = 0$, either $m = 1$, or $m = 2$. If $m = 1$, then V is one-dimensional (by the definition of m), and thus we must have $m = 2$. Therefore, $1 + \mathcal{J}^2 = [U_3(\Bbbk), U_3(\Bbbk)]$ acts by non-trivial scalar multiplication on V, which means that there exists some non-trivial smooth character ς of $1 + \mathcal{J}^2$ such that

$$x \cdot v = \varsigma(x)v, \qquad x \in 1 + J^2,\ v \in V.$$

By the construction above, it is not hard to check that the ideal $\mathcal{L}$ may be chosen so that $1 + \mathcal{L} = U_3^{1,3}(\Bbbk)$ and that $\varsigma = \lambda_{1,3}(\alpha)_{1+\mathcal{J}^2}$ for some $\alpha \in \Bbbk^\times$. Therefore, in this case, we see that $C_{U_3(\Bbbk)}(\varsigma) = U_3^{1,3}(\Bbbk)$ and that $V \simeq V_{1,3}(\alpha)$.

Example 4.2.2 ($U_4(\Bbbk)$)**.** Also from Section 3.5, we known that the following are irreducible admissible smooth $U_4(\Bbbk)$-modules:

- For every $\alpha, \beta, \gamma \in \Bbbk$, the one-dimensional smooth $U_4(\Bbbk)$-module

$$V_{1,2}(\alpha) \otimes V_{2,3}(\beta) \otimes V_{3,4}(\gamma) = \mathbb{C}_{1,2}(\alpha) \otimes \mathbb{C}_{2,3}(\beta) \otimes \mathbb{C}_{3,4}(\gamma).$$

- For every $\alpha \in \Bbbk$, and every $\beta \in \Bbbk^\times$ the infinite-dimensional smooth $U_4(\Bbbk)$-module

$$V_{1,2}(\alpha) \otimes V_{2,4}(\beta) = \mathbb{C}_{1,2}(\alpha) \otimes \operatorname{Ind}_{U_4^{2,4}(\Bbbk)}^{U_4(\Bbbk)}(\mathbb{C}_{2,4}(\beta)).$$

- For every $\alpha \in \Bbbk^\times$ and every $\beta \in \Bbbk$, the infinite-dimensional smooth $U_4(\Bbbk)$-module

$$V_{1,3}(\alpha) \otimes V_{3,4}(\beta) = \operatorname{Ind}_{U_4^{1,3}(\Bbbk)}^{U_4(\Bbbk)}(\mathbb{C}_{1,3}(\alpha)) \otimes \mathbb{C}_{3,4}(\beta).$$

- For every $\alpha \in \Bbbk^\times$ and every $\beta \in \Bbbk$, the infinite-dimensional smooth $U_4(\Bbbk)$-module

$$V_{1,4}(\alpha) \otimes V_{2,3}(\beta) = \operatorname{Ind}_{U_4^{1,4}(\Bbbk)}^{U_4(\Bbbk)}(\mathbb{C}_{1,4}(\alpha)) \otimes \mathbb{C}_{2,3}(\beta).$$

Besides these four families, there is a fifth family of irreducible smooth $U_4(\Bbbk)$-modules which are constructed as follows; as before, we set $\mathcal{J} = U_4(\Bbbk) - 1$ and keep the notation as in the general case. Let $N = 1 + \mathcal{J}^2$, and consider the smooth character $\varsigma \in N^\circ$ given by

$$\varsigma = \lambda_{1,3}(\alpha)_N \lambda_{2,4}(\beta)_N$$

for arbitrarily fixed elements $\alpha, \beta \in \Bbbk^\times$. Moreover, let $\mathcal{L} = \mathcal{J}^2 + \Bbbk e_{1,2}$, let $\gamma \in \Bbbk$ be arbitrary, and define the smooth character

$$\lambda = \lambda_{1,2}(\gamma)_Q \lambda_{1,3}(\alpha)_Q \lambda_{2,4}(\beta)_Q$$

of $Q = 1 + \mathcal{L}$; hence,

$$\lambda(x) = \psi(\gamma x_{1,2} + \alpha x_{1,3} + \beta x_{2,4}), \qquad x \in Q;$$

recall that ψ is a non-trivial smooth character of $\Bbbk^+$. It is not hard to check that

$$C_P(\lambda) = 1 + (\mathcal{L} + \Bbbk e_{3,4}) = Q(1 + \Bbbk e_{3,4}),$$

and that λ extends to a smooth character $\vartheta \in C_P(\lambda)^\circ$; indeed, we may choose ϑ so that ϑ is trivial on $1 + \Bbbk e_{1,2}$ (however, we should mention our conclusion will not depend on this choice). Then, by Theorem 4.1.1, we get an irreducible smooth $U_4(\Bbbk)$-module

$$V = \mathrm{Ind}_{C_P(\lambda)}^{U_4(\Bbbk)}(\mathbb{C}_\vartheta)$$

which is non-isomorphic to any of the ones listed above. Furthermore, we can also prove that these five families exhaust the isomorphism classes of irreducible smooth $U_4(\Bbbk)$-modules.

4.3 Involutive Algebra Groups over non-Archimedean fields

Let $\Bbbk$ be an arbitrary field with characteristic different from 2, and let $\mathcal{A}$ be an associative algebra with identity over $\Bbbk$. We say that a map $\sigma : \mathcal{A} \to \mathcal{A}$ is an *involution* if the following conditions are satisfied for all $a, b \in \mathcal{A}$:

(1) $\sigma(a + b) = \sigma(a) + \sigma(b)$;

(2) $\sigma(ab) = \sigma(b)\sigma(a)$;

(3) $\sigma^2 = \mathrm{id}_{\mathcal{A}}$.

We note that an involution σ is not required to be $\Bbbk$-linear; however, we will assume that the field $\Bbbk = \Bbbk \cdot 1$ is preserved by σ. Then, σ defines a field automorphism of $\Bbbk$ which is either the identity or has order 2; we say that σ is of the *first kind* if σ fixes $\Bbbk$, and of the *second kind* if its restriction $\sigma_\Bbbk$ to $\Bbbk$ has order 2. In any case, we let

$$\Bbbk^\sigma = \{\alpha \in \Bbbk \mid \sigma(a) = a\}$$

be the σ-fixed subfield of $\Bbbk$, and consider that $\mathcal{A}$ is a finite-dimensional associative $\Bbbk^\sigma$-algebra; we observe that σ is of the second kind if and only if the field extension $\Bbbk^\sigma \subseteq \Bbbk$ has degree 2.

An important example occurs in the case where $\mathcal{A} = \mathcal{M}_n(\Bbbk)$ is endowed with the canonical *transpose involution* given by the mapping $a \mapsto a^T$ where a^T denotes the transpose of a matrix $a \in \mathcal{M}_n(\Bbbk)$; notice that the transpose involution is of the first kind. On the other hand, if $F : \Bbbk \to \Bbbk$ is a field automorphism of order 2, then it is naturally extended to an automorphism $F : \mathcal{M}_n(\Bbbk) \to \mathcal{M}_n(\Bbbk)$ (by the rule $F(a) = (F(a_{i,j}))$ for all $a = (a_{i,j}) \in \mathcal{M}_n(\Bbbk)$), and thus we may endow $\mathcal{M}_n(\Bbbk)$ with the involution σ of the second kind defined by

$$\sigma(a) = F(a)^T, \qquad a \in \mathcal{M}_n(\Bbbk).$$

The following classification of the involutions on $\mathcal{M}_n(\Bbbk)$ can be found in the book [Knu98] by M.A. Knus et al. (see, in particular, Propositions 2.19 and 2.20) where the complete classification of involutions is also given for arbitrary central $\Bbbk$-algebras (see Propositions 2.7 and 2.18).

Proposition 4.3.1. *If σ is an involution on $\mathcal{M}_n(\Bbbk)$, then:*

(1) σ is of the first kind if and only if there exists an invertible matrix $u \in GL_n(\Bbbk)$ satisfying $u^T = \pm u$ (and uniquely determined up to a factor in $\Bbbk^\times$) and such that $\sigma(a) = u^{-1}a^T u$ for all $a \in \mathcal{M}_n(\Bbbk)$.

(2) σ is of the second kind if and only if there exists an invertible matrix $u \in \mathrm{GL}_n(\Bbbk)$, and an automorphism $F : GL_n(\Bbbk) \to GL_n(\Bbbk)$ of order 2 satisfying $(F(u))^T = u$ (and uniquely determined up to a factor in $(\Bbbk^\sigma)^\times$) such that $\sigma(a) = u^{-1}(F(a))^T u$ for all $a \in \mathcal{M}_n(\Bbbk)$.

Following the standard terminology, we say that an involution σ on $\mathcal{M}_n(\Bbbk)$ is *sympletic* if σ is of first kind and $u^T = -u$, *orthogonal* if σ is of the first kind and $u^T = u$, and *unitary* if σ is of the second kind.

Henceforth, we let $\mathcal{A}$ be an arbitrary finite-dimensional associative algebra endowed with an involution σ, and let $\mathcal{A}^\times$ denote the unit group of $\mathcal{A}$. Then, σ acts in $\mathcal{A}^\times$ via the mapping

$$x \mapsto x^\sigma = \sigma(x)^{-1}, \qquad x \in \mathcal{A}^\times;$$

notice that this mapping defines an automorphism (of order two) of $\mathcal{A}^\times$. For every σ-invariant subgroup H of $\mathcal{A}^\times$ (that is, a subgroup H such that $H^\sigma = H$), we define the σ-fixed subgroup

$$C_H(\sigma) = \{h \in H \mid h^\sigma = h\}.$$

In the case where $\mathcal{A} = \mathcal{M}_n(\Bbbk)$ and $H = \mathrm{GL}_n(\Bbbk)$, we have a full description of the subgroups $C_{\mathrm{GL}_n(\Bbbk)}(\sigma)$ for every involution σ on $\mathcal{M}_n(\Bbbk)$: up to isomorphism, they correspond to one of the classical groups described as follows. If J_m denotes the matrix of size m with 1 in the anti-diagonal and zero elsewhere, then $C_{\mathrm{GL}_n(\Bbbk)}(\sigma)$ is one of the following groups:

- $\mathrm{Sp}_{2m}(\Bbbk)$, if $\sigma(a) = u^{-1}a^T u$ with $u = \begin{bmatrix} 0 & J_m \\ -J_m & 0 \end{bmatrix}$.
- $\mathrm{O}^+_{2m}(\Bbbk)$ or $\mathrm{O}^+_{2m+1}(\Bbbk)$, if $\sigma(a) = u^{-1}a^T u$ with $u = J_n$ with $n = 2m$ or $n = 2m+1$.
- $\mathrm{O}^-_{2m}(\Bbbk)$, if $\sigma(a) = u^{-1}a^T u$ with $\begin{bmatrix} 0 & 0 & J_m \\ 0 & c & 0 \\ J_m & 0 & 0 \end{bmatrix}$ and $c = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}$ with $\epsilon \in \Bbbk^\times \setminus (\Bbbk^\times)^2$
- $\mathrm{U}_n(\Bbbk)$, if $\sigma(a) = u^{-1}(F(a))^T u$ with $u = J_n$.

The purpose of this section is to study the irreducible representations of the group $C_P(\sigma)$ where $P = 1 + \mathcal{J}$ is an algebra group associated with a nilpotent σ-invariant subalgebra $\mathcal{J}$ of $\mathcal{A}$; we refer to P as an algebra subgroup of $\mathcal{A}^\times$. We also note that in the case where $\Bbbk$ is the finite field with q elements, $\mathcal{A} = \mathcal{M}_n(q)$ and $P = U_n(q)$, $C_P(\sigma)$ is a p-Sylow subgroup of $C_{\mathrm{GL}_n(q)}(\sigma)$ where p is the characteristic of $\Bbbk$. We also note that, in the case where $\Bbbk$ is a non-Archimedean local field, $C_P(\sigma)$ is an ℓ_c-group (because P is an ℓ_c-group).

One important tool is the *Cayley transform*: let $P = 1 + \mathcal{J}$ an algebra subgroup of $\mathcal{A}$. Then the Cayley transform is the map $\Psi\colon \mathcal{J} \to P$ defined by

$$\Psi(a) = (1-a)(1+a)^{-1}, \qquad a \in \mathcal{J}.$$

Since $(1+a)^{-1} = 1 - a + a^2 - a^3 \ldots$, we have

$$\Psi(a) = 1 - 2a + 2a^2 - 2a^3 \ldots, \qquad a \in \mathcal{J}$$

and thus (since $\Bbbk$ has characteristic not equal to 2) Ψ is a bijection with inverse $\Phi = \Psi^{-1}\colon P \to \mathcal{J}$ defined by

$$\Phi(x) = (x-1)(x+1)^{-1}, \qquad x \in P.$$

As a first application of the Cayley transform, we have the following lemma.

Lemma 4.3.2. *Let $\mathcal{J}$ be a σ-invariant nilpotent subalgebra of $\mathcal{A}$, and define*

$$C_{\mathcal{J}}(\sigma) = \{a \in \mathcal{J} \mid \sigma(a) = -a\}$$

Then, the Cayley transform defines a bijection between $C_{\mathcal{J}}(\sigma)$ and $C_P(\sigma)$, where $P = 1 + \mathcal{J}$. Furthermore, $C_{\mathcal{J}}(\sigma)$ is a Lie algebra over the σ-fixed subfield $\mathbb{k}^\sigma$ of $\mathbb{k}$ (with respect to the Lie product defined by $[a, b] = ab - ba$ for all $a, b \in C_{\mathcal{J}}(\sigma)$).

Proof. Note that

$$\Psi(-a) = a^{-1}, \quad a \in \mathcal{J}, \qquad \text{and} \qquad \Phi(x^{-1}) = -\Phi(x), \quad x \in P.$$

Therefore, we deduce

$$\sigma(\Psi(a)) = \Psi(\sigma(a)) = \Psi(-a) = \Psi(a)^{-1}, \qquad a \in C_{\mathcal{J}}(\sigma),$$

and a similar calculation for Φ shows that it is in fact a bijection between $C_{\mathcal{J}}(\sigma)$ and $C_P(\sigma)$. It is also easy to see that

$$[a, b] \in C_{\mathcal{J}}(\sigma), \qquad a, b \in C_{\mathcal{J}}(\sigma),$$

and so $C_{\mathcal{J}}(\sigma)$ is a Lie algebra over $\mathbb{k}^\sigma$. □

In the case where $\mathbb{k}$ is a finite field, C. André proved the following theorem.

Theorem 4.3.3 ([And10, Theorem 1.1]). *Let $\mathbb{k}$ be a finite field of odd characteristic, and let $\mathcal{A}$ be a finite-dimensional $\mathbb{k}$-algebra endowed with an involution σ. Let $\mathcal{J}$ be a σ-invariant nilpotent subalgebra of $\mathcal{A}$, let $P = 1 + \mathcal{J}$, and let χ be an irreducible character of $C_P(\sigma)$. Then, there exist a σ-invariant algebra subgroup of Q of P and a linear character λ of $C_Q(\sigma)$ such that $\chi = \operatorname{Ind}_Q^P(\lambda)$.*

Our goal is to generalise this theorem to the case where $\mathbb{k}$ is an arbitrary non-Archimedean local field.

Theorem 4.3.4. *Let $\mathbb{k}$ be a non-Archimedean local field of characteristic different from 2, and let $\mathcal{A}$ be a finite-dimensional $\mathbb{k}$-algebra endowed with an involution σ. Let $\mathcal{J}$ be a σ-invariant nilpotent subalgebra of $\mathcal{A}$, let $P = 1 + \mathcal{J}$, and let V be an irreducible smooth $C_P(\sigma)$-module. Then, V is admissible and there exist a σ-invariant subgroup Q of P and a one-dimensional smooth $C_Q(\sigma)$-module W such that*

$$V \simeq \operatorname{Ind}_{C_Q(\sigma)}^{C_P(\sigma)}(W);$$

in particular, we have

$$\text{c-}\operatorname{Ind}_{C_Q(\sigma)}^{C_P(\sigma)}(W) = \operatorname{Ind}_{C_Q(\sigma)}^{C_P(\sigma)}(W).$$

As in the previous section, we consider the sequence $\mathcal{J} \subseteq \mathcal{J}^2 \subseteq \mathcal{J}^3 \subseteq \cdots$ of ideals of $\mathcal{J}$; for every $n \in \mathbb{N}$, so set $P_n = 1 + \mathcal{J}^n$, so that we obtain a descending sequence

$$P = P_1 \supseteq P_2 \supseteq P_3 \supseteq \cdots$$

of normal subgroups of P. It is obvious that, for every $n \in \mathbb{N}$, the ideal $\mathcal{J}^n$ is σ-invariant, and hence the subgroup P_n is also σ-invariant. Therefore, we obtain a descending sequence

$$C_P(\sigma) = C_{P_1}(\sigma) \supseteq C_{P_2}(\sigma) \supseteq C_{P_3}(\sigma) \supseteq \cdots$$

of normal subgroups of $C_P(\sigma)$. We start by proving the following elementary result.

Lemma 4.3.5. *For every $n \in \mathbb{N}$, we have $[P, P_m] \cap C_P(\sigma) = [C_P, C_{P_m}(\sigma)]$.*

Proof. Let $[P, \sigma]$ be the subgroup of P generated by all the elements $g^{-1}g^{\sigma}$ for $g \in P$; recall that $g^{\sigma} = \sigma(g^{-1})$, and thus $[P, \sigma]$ is also generated by the set $\{g\sigma(g) \mid g \in P\}$. Then, the group P decomposes as the product $P = C_P(\sigma)[P, \sigma]$ and we clearly have $C_P(\sigma) \cap [P, \sigma] = 1$. Moreover, we have

$$h(g\sigma(g))h^{-1} = h(g\sigma(g))\sigma(h) = (hg)\sigma(hg) \in [P, \sigma], \qquad g \in P,\ h \in C_P(\sigma)$$

and this shows that $[P, \sigma]$ is a normal subgroup in P. Notice that this argument does not depend on P, and thus we also have a similar decomposition $P_n = C_{P_n}(\sigma)[P_n, \sigma]$ for all $n \in \mathbb{N}$.

Now, let $n \in \mathbb{N}$ be arbitrary. Let $g \in P$ and $h \in P_n$, and write $g = g_1 g_2$ and $h = h_1 h_2$ where $g_1 \in C_P(\sigma)$, $g_2 \in [P, \sigma]$, $h_1 \in C_{P_n}(\sigma)$ and $h_2 \in [P_n, \sigma]$ are uniquely determined. Then,

$$\begin{aligned} ghg^{-1}h^{-1} &= g_1 g_2 h_1 h_2 g_2^{-1} g_1^{-1} h_2^{-1} h_1^{-1} \\ &= (g_1 h_1 g_1^{-1} h_1^{-1})(h_1(g_1((h_1^{-1} g_2 h_1) h_2 g_2) g_1^{-1}) h_1^{-1}) \end{aligned}$$

Since

$$h_1(g_1((h_1^{-1} g_2 h_1) h_2 g_2) g_1^{-1}) h_1^{-1} \in [P, \sigma],$$

we conclude that

$$ghg^{-1}h^{-1} \in C_P(\sigma) \quad \Longleftrightarrow \quad ghg^{-1}h^{-1} = g_1 h_1 g_1^{-1} h_1^{-1}.$$

Using an iterative argument, we see that the analogous conclusion holds for any product of commutators, and thus the desired equality $[P, P_n] \cap C_P(\sigma) = [C_P(\sigma), C_{P_n}(\sigma)]$ follows. $\square$

In what follows, we let V be an arbitrary irreducible smooth $C_P(\sigma)$-module, and assume

that $\dim V \geq 2$. Let $n \in \mathbb{N}$ be such that $\mathcal{J} \neq 0$ and $\mathcal{J}^{n+1} = 0$; indeed, we must have $n \geq 2$, otherwise $C_P(\sigma)$ would be abelian, and hence V must be one-dimensional (by 2.2.14). Since $C_{P_n}(\sigma)$ lies in the centre of $C_P(\sigma)$, Schur's lemma implies that $C_{P_n}(\sigma)$ acts on V by scalar multiplications, and thus we may choose the smallest positive integer m for which there exists $\varsigma \in (C_{P_m}(\sigma))^{\circ}$ such that

$$g \cdot v = \varsigma(g)v, \qquad g \in C_{P_m}(\sigma), v \in V.$$

We note that, since V is an irreducible smooth $C_P(\sigma)$-module with $\dim V \geq 2$, we must have $m \geq 2$; furthermore, since $[C_P(\sigma), C_{P_{m-1}}(\sigma)] \subseteq C_{P_m}(\sigma)$, the minimal choice of m implies that ς is not identically equal to 1 (otherwise, Schur's lemma would imply that $C_{P_{m-1}}(\sigma)$ acts on V by scalar multiplications).

Following the Boyarchenko's construction in the case of algebra groups, we next show that there exists a σ-invariant ideal $\mathcal{L}$ of $\mathcal{J}$ satisfying $\mathcal{J}^m \subseteq \mathcal{L} \subseteq \mathcal{J}^{m-1}$ and $\dim(\mathcal{L}/\mathcal{J}^m) = 1$, and such that $\varsigma([C_P(\sigma), C_{1+\mathcal{L}}(\sigma)]) \neq 1$; notice that, in particular, $C_{P_m}(\sigma) \subsetneqq C_{1+\mathcal{L}}(\sigma) \subset C_{P_{m-1}}(\sigma)$. To see this, we first prove the following elementary lemma.

Lemma 4.3.6. *For every $m \in \mathbb{N}$, $m \geq 2$, there is an isomorphism of abelian groups*

$$C_{P_{m-1}}(\sigma)/C_{P_m}(\sigma) \simeq C_{\mathcal{J}^{m-1}}(\sigma)/C_{\mathcal{J}^m}(\sigma).$$

Proof. Firstly, we observe that the mapping $u \mapsto 1 + u$ clearly defines an isomorphism of abelian groups

$$\mathcal{J}^{m-1}/\mathcal{J}^m \simeq 1 + (\mathcal{J}^{m-1}/\mathcal{J}^m),$$

and thus we naturally get an isomorphism of abelian groups

$$\mathcal{J}^{m-1}/\mathcal{J}^m \simeq (1 + \mathcal{J}^{m-1})/(1 + \mathcal{J}^m) = P_{m-1}/P_m.$$

It is straightforward to check that this isomorphism is σ-invariant, and that it restricts to an isomorphism

$$(C_{\mathcal{J}^{m-1}}(\sigma) + \mathcal{J}^m)/\mathcal{J}^m \simeq (C_{P_{m-1}}(\sigma)P_m)/P_m.$$

Since

$$(C_{\mathcal{J}^{m-1}}(\sigma) + \mathcal{J}^m)/\mathcal{J}^m \simeq C_{\mathcal{J}^{m-1}}(\sigma)/C_{\mathcal{J}^m}(\sigma)$$

and

$$(C_{P_{m-1}}(\sigma)P_m/P_m \simeq C_{P_{m-1}}(\sigma)/C_{P_m}(\sigma)),$$

the required isomorphism follows. ☐

Now, by the minimal choice of m, we have $C_{P_{m-1}}(\sigma)/C_{P_m}(\sigma) \neq 1$, and thus

$$C_{\mathcal{J}^m-1}(\sigma)/C_{\mathcal{J}^m}(\sigma) \neq 0.$$

Since both $C_{\mathcal{J}^{m-1}}(\sigma)$ and $C_{\mathcal{J}^m}(\sigma)$ are $\Bbbk^\sigma$-vector subspaces of $C_{\mathcal{J}}(\sigma)$ (by Lemma 4.3.2), we conclude that

$$C_{\mathcal{J}^{m-1}}(\sigma) = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_t$$

for some $\Bbbk^\sigma$ subspaces $\mathcal{L}_1, \cdots, \mathcal{L}_t$ of $C_{\mathcal{J}}(\sigma)$ satisfying

$$C_{\mathcal{J}^m}(\sigma) \subseteq \mathcal{L} \subseteq C_{\mathcal{J}^{m-1}}(\sigma) \qquad \text{and} \qquad \dim(\mathcal{L}_i/C_{\mathcal{J}^m}(\sigma)) = 1, \quad 1 \leq i \leq t.$$

By the isomorphism above, we see that

$$C_{P_{m-1}}(\sigma) = (1+\mathcal{L}_1)\cdots(1+\mathcal{L}_t),$$

and so there must exist $1 \leq i \leq t$ such that $\varsigma([C_P(\sigma), 1+\mathcal{L}_i]) \neq 1$; otherwise, we would have $\varsigma([C_P(\sigma), C_{P_{m-1}}(\sigma)]) = 1$, and hence $C_{P_{m-1}}(\sigma)$ would act on V by scalar multiplications. Henceforth, we choose a non-zero vector $u \in \mathcal{L}_i$, and let

$$\mathcal{L} = \Bbbk u + \mathcal{J}^m;$$

notice that $C_{\mathcal{L}}(\sigma) = \Bbbk^\sigma u + \mathbb{C}_{\mathcal{J}^m}(\sigma) = \mathcal{L}_i$. We set $Q = 1 + \mathcal{L}$, and observe that Q is an ideal subgroup of P (hence, a normal subgroup) and that $C_Q(\sigma)$ is a normal subgroup of $C_P(\sigma)$; moreover, note that

$$Q = (1+\Bbbk u)P_m \qquad \text{and} \qquad C_Q(\sigma) = (1+\Bbbk^\sigma u)C_{P^m}(\sigma).$$

Furthermore, for simplicity, we write $N = P_m = 1 + \mathcal{J}^m$, and note that the smooth character $\varsigma \in C_N(\sigma)^\circ$ is $C_P(\sigma)$ invariant; indeed,

$$[C_P(\sigma), C_N(\sigma)] = [P, N] \cap C_P(\sigma) = C_{P^{m+1}}(\sigma),$$

and thus $\varsigma([C_P(\sigma), C_N(\sigma)]) = 1$.

We have the following result (cf. Proposition 4.1.4).

Proposition 4.3.7. *Let $\varsigma \in C_N(\sigma)^\circ$ be $C_P(\sigma)$ invariant, and define*

$$\mathcal{S}_\varsigma = \{a \in C_{\mathcal{J}}(\sigma) \mid \varsigma([\Psi(a), \Psi(b)]) = 1 \textit{ for all } b \in C_{\mathcal{L}}(\sigma)\}$$

where $\Psi\colon \mathcal{J} \to P$ is the Cayley transform. Then, $\mathcal{S}_\varsigma$ is a $\Bbbk^\sigma$-subalgebra of $C_{\mathcal{J}}(\sigma)$ satisfying $C_{\mathcal{J}^2}(\sigma) \subseteq \mathcal{S}_\varsigma$ and $\dim \mathcal{S}_\varsigma \geq \dim C_{\mathcal{J}}(\sigma) - 1$. Further, if $\varphi_\varsigma^\sigma\colon C_P(\sigma) \to C_Q(\sigma)^\circ$ is the map defined by

$$\varphi_\varsigma^\sigma(g)(h) = \varsigma([g,h]), \qquad g \in C_P(\sigma), h \in C_Q(\sigma),$$

then φ_ς^σ group homomorphism with

$$\ker(\varphi_\varsigma^\sigma) = \Psi(\mathcal{S}_\varsigma) \qquad \textit{and} \qquad \varphi_\varsigma^\sigma(C_P(\sigma)) \subseteq C_N(\sigma)^\perp$$

where $C_N(\sigma)^\perp$ is the orthogonal subgroup of $C_N(\sigma)$ in $C_Q(\sigma)^\circ$; hence,φ_ς^σ defines naturally a group homomorphism $\overline{\varphi_\varsigma}^\sigma : C_P(\sigma) \to (C_Q(\sigma)/C_N(\sigma))^\circ$.

Proof. We first observe that the map φ_ς^σ is a well-defined group homomorphism. On one hand, we have

$$[C_P(\sigma), C_Q(\sigma)] \subseteq [C_P(\sigma), C_{P_{m-1}}(\sigma)] \subseteq C_N(\sigma).$$

On the other hand, since $[g, hk] = [g, k][g, h]^k$, we deduce that

$$\varphi_\varsigma^\sigma(g)(gk) = \varsigma([g,k])\varsigma([g,h]) = \varphi_\varsigma^\sigma(g)(h)\varphi_\varsigma(g)(k), \quad g \in C_P(\sigma),\ h,k \in C_Q(\sigma);$$

we recall that ς is $C_P(\sigma)$-invariant. It follows that, for every $g \in C_P(\sigma)$, the map $\varphi_\varsigma^\sigma(g)\colon C_Q(\sigma) \to \mathbb{C}^\times$ is indeed a smooth character of $C_Q(\sigma)$. Similarly, since $[gh, k] = [g,k]^h[h,k]$, we have

$$\varphi_\varsigma^\sigma(gh)(k) = \varsigma([g,k])\varsigma([h,k]) = \varphi_\varsigma^\sigma(g)(k)\varphi_\varsigma^\sigma(h)(k), \quad g \in C_P(\sigma),\ h,k \in C_Q(\sigma),$$

and so $\varphi_\varsigma^\sigma\colon C_P(\sigma) \to C_Q(\sigma)^\circ$ is a group homomorphism.

Now, since $[C_P(\sigma), C_N(\sigma)] \subseteq \ker(\varsigma)$ (because ς is $C_P(\sigma)$-invariant), the image $\varphi_\varsigma^\sigma(C_P(\sigma))$ clearly lies in $C_N(\sigma)^\perp$; moreover, it is obvious (by the definition) that

$$\ker(\varphi_\varsigma^\sigma) = \Psi(\mathcal{S}_\varsigma).$$

Let $a \in C_{\mathcal{J}}(\sigma)$ and $\alpha \in \Bbbk^\sigma$ be arbitrary. We know claim that

$$[\Psi(\alpha a), \Psi(b)][\Psi(a), \Psi(\alpha b)]^{-1} \in C_{P^{m+1}}(\sigma), \qquad b \in C_{\mathcal{J}^{m-1}}(\sigma);$$

indeed, as in the previous case, Proposition 4.1.3 implies that

$$[\Psi(\alpha a), \Psi(b)][\Psi(a), \Psi(\alpha b)]^{-1} \in [P, N], \qquad b \in C_{\mathcal{J}^{m-1}}(\sigma),$$

and thus it follows from Lemma 4.3.6 that

$$[\Psi(\alpha a), \Psi(b)][\Psi(a), \Psi(\alpha b)]^{-1} \in [C_P(\sigma), C_N(\sigma)], \qquad b \in C_{\mathcal{J}^{m-1}}(\sigma).$$

Since ς is $C_P(\sigma)$ invariant, we conclude that

$$\varsigma([\Psi(\alpha a), \Psi(b)]) = \varsigma([\Psi(a), \Psi(\alpha b)]), \qquad b \in C_{\mathcal{J}^{m-1}}(\sigma), \tag{‡}$$

and this clearly implies that

$$\alpha a \in \mathcal{S}_\varsigma, \qquad \alpha \in \Bbbk^\sigma,\ a \in \mathcal{S}_\varsigma.$$

On the other hand, by Theorem 4.1.2, we know that $[P_2, Q] \subseteq [P_2, P_{m-1}] \subseteq [P, N]$, and thus

$$[C_{P_2}(\sigma), C_Q(\sigma)] \subseteq C_P(\sigma) \cap [P, N] \subseteq [C_P(\sigma), C_N(\sigma)]$$

which implies that $C_{\mathcal{J}^2}(\sigma) \subseteq \mathcal{S}_\varsigma$; notice that $C_Q(\sigma) = \Psi(C_{\mathcal{L}}(\sigma))$. Since $\ker(\varphi_\varsigma^\sigma) = \Psi(\mathcal{S}_\varsigma)$ and since

$$\Psi(a+b)^{-1}\Psi(a)\Psi(b) \in C_{P_2} \subseteq \Psi(\mathcal{S}_\varsigma),$$

we see that

$$\Psi(a+b) \in \ker(\varphi_\varsigma^\sigma), \qquad a, b \in \mathcal{S}_\varsigma.$$

It follows that $\mathcal{S}_\varsigma$ is a $\Bbbk^\sigma$-subalgebra of $C_{\mathcal{J}}(\sigma)$ with $C_{\mathcal{J}^2}(\sigma) \subseteq \mathcal{S}_\varsigma$.

Finally, notice that

$$C_N(\sigma)^\perp \simeq (C_Q(\sigma)/C_N(\sigma))^\circ \qquad \text{and} \qquad C_Q(\sigma)/C_N(\sigma) \simeq \Psi(\mathcal{L}/\mathcal{J}^m) \simeq (\Bbbk^\sigma)^+.$$

Therefore, since

$$C_P(\sigma)/\ker(\varphi_\varsigma^\sigma) \simeq \varphi_\varsigma^\sigma(C_P(\sigma)) \subseteq C_N(\sigma)^\perp,$$

we conclude that $\dim C_{\mathcal{J}}(\sigma) - \dim \mathcal{S}_\varsigma \leq 1$, and this completes the proof. □

Next, we prove the following crucial result.

Proposition 4.3.8. *Let $\varsigma \in C_N(\varsigma)^\circ$ be $C_P(\sigma)$-invariant, and define $\mathcal{S}_\varsigma \subseteq C_{\mathcal{J}}(\sigma)$ as in Proposition 4.3.7. Then, $[C_Q(\sigma), C_Q(\sigma)] \subseteq \ker(\varsigma)$, and there exists $\lambda \in C_Q(\sigma)^\circ$ such that $\lambda_{C_N(\sigma)} = \varsigma$; moreover, the following properties hold.*

(1) $C_{C_P(\sigma)}(\lambda') = \Psi(\mathcal{S}_\varsigma)$ for all $\lambda' \in C_Q(\sigma)^\circ$ such that $\lambda'_{C_N(\sigma)} = \varsigma$.

(2) If $C_{C_P(\sigma)} \neq C_P(\sigma)$ and if $\lambda' \in C_Q(\sigma)^\circ$ is such that $\lambda'_{C_N(\sigma)} = \varsigma$, then there exists $g \in C_P(\sigma)$ such that $\lambda' = \lambda^g$.

Proof. By construction, we have

$$C_{\mathcal{L}}(\sigma) = \Bbbk^\sigma \oplus C_{\mathcal{J}^m}(\sigma)$$

for some $u \in C_{\mathcal{L}}(\sigma)$, and hence

$$C_Q(\sigma) = (1 + \Bbbk^\sigma u) C_N(\sigma) = \Psi(\Bbbk^\sigma u) C_N(\sigma).$$

Since

$$[\Psi(\alpha u), \Psi(\beta u)] = 1, \qquad \alpha, \beta \in \Bbbk^\sigma,$$

we see that

$$\varsigma([\Psi(\alpha u), \Psi(\beta u)]) = 1,$$

and this clearly implies that $\varsigma([C_Q(\sigma), C_Q(\sigma)]) = 1$ (because ς is $C_P(\sigma)$-invariant).

Let $\mathbb{C}_\varsigma$ denote the (canonical) one-dimensional $C_N(\sigma)$-module associated with ς and let W be an irreducible quotient of the smoothly induced $C_Q(\sigma)$-module $\mathrm{Ind}_{C_N(\sigma)}^{C_Q(\sigma)}(\mathbb{C}_\varsigma)$ (the existence of W is guaranteed by Proposition 2.4.6). Since $C_N(\sigma)$ is a normal subgroup of $C_Q(\sigma)$ and ς is $C_Q(\sigma)$-invariant, we have

$$x\phi = \varsigma(x)\phi, \qquad x \in C_N(\sigma),\ \phi \in \mathrm{Ind}_{C_N(\sigma)}^{C_Q(\sigma)}(\mathbb{C}_\varsigma),$$

and thus

$$x \cdot w = \varsigma(x) w, \qquad x \in C_N(\sigma), w \in W.$$

Since $[C_Q(\sigma), C_Q(\sigma)] \subseteq \ker(\varsigma)$, it follows from Schur's lemma that $\dim W = 1$, and thus W affords a character $\lambda \in C_Q(\sigma)^\circ$ which clearly satisfies $\lambda_{C_N(\sigma)} = \varsigma$.

Next, we consider the group homomorphism $\varphi_\varsigma^\sigma : C_P(\sigma) \to C_Q(\sigma)^\circ$ as defined in the Proposition 4.3.7; we recall that

$$\varphi_\varsigma^\sigma(C_P(\sigma)) \subseteq C_N(\sigma)^\perp \qquad \text{and} \qquad \ker(\varphi_\varsigma^\sigma) = \Psi(\mathcal{S}_\varsigma)$$

where $\mathcal{S}_\varsigma$ is an ideal of $C_{\mathcal{J}}(\sigma)$ satisfying $C_{\mathcal{J}^2}(\sigma) \subseteq \mathcal{S}_\varsigma$ and $\dim \mathcal{S}_\varsigma \geq \dim C_{\mathcal{J}}(\sigma) - 1$. On the one hand, (1) follows because

$$C_{C_P(\sigma)}(\lambda') = \ker(\varphi_\varsigma^\sigma) = \Psi(\mathcal{S}_\varsigma)$$

for all $\lambda' \in C_Q(\sigma)^\circ$ such that $\lambda'_{C_N(\sigma)} = \varsigma$. On the other hand, assume that $C_{C_P(\sigma)}(\lambda) \neq C_P(\sigma)$ (hence, $\ker(\varphi^\sigma_\varsigma) \neq C_P(\sigma)$ and $\mathcal{S}_\varsigma \neq C_\mathcal{J}(\sigma)$), and let $x \in C_P(\sigma)$ be such that $\varphi^\sigma_\varsigma(x) \in C_Q(\sigma)^\circ$ is not identically equal to 1. Let $a \in C_\mathcal{J}(\sigma)$ be such that $x = \Psi(a)$; then, ‡ implies that

$$\varphi^\sigma_\varsigma(\Psi(\alpha a)) \in \varphi^\sigma_\varsigma(C_P(\sigma)) = C_N(\sigma)^\perp, \qquad \alpha \in \Bbbk^\sigma.$$

Since

$$C_N(\sigma)^\perp \simeq (C_Q(\sigma)/C_N(\sigma))^\circ \qquad \text{and} \qquad C_Q(\sigma)/C_N(\sigma) \simeq 1 + (\mathcal{L}/\mathcal{J}^m) \simeq (\Bbbk^\sigma)^+$$

(by Lemma 4.3.6), it is straightforward to show that the mapping $\alpha \mapsto \varphi^\sigma_\varsigma(\Psi(\alpha a))$ defines a group isomorphism $\Bbbk^+ \simeq C_N(\sigma)^\perp$ (we note that since $\Bbbk$ is a self-dual field, $\Bbbk^\sigma$ is also a self dual field). In particular, it follows that

$$C_N(\sigma)^\perp = \{\varphi^\sigma_\varsigma(\Psi(\alpha a)) \mid \alpha \in \Bbbk^\sigma\}$$

and so the map $\varphi^\sigma_\varsigma : C_P(\sigma) \to C_N(\sigma)^\perp$ is surjective and

$$C_P(\sigma)/C_{C_P(\sigma)}(\lambda) \simeq C_N(\sigma)^\perp \simeq (\Bbbk^\sigma)^+.$$

To conclude the proof of (ii)), let $\lambda' \in C_Q(\sigma)^\circ$ be such that $\lambda'_{C_N(\sigma)} = \varsigma$, and consider the character $\lambda'\lambda^{-1} \in C_Q(\sigma)^\circ$. It is obvious that $\lambda'\lambda^{-1} \in C_N(\sigma)^\perp$, and thus there exists $\alpha \in \Bbbk$ such that

$$\lambda'\lambda^{-1} = \varphi^\sigma_\varsigma(\Psi(\alpha a))$$

If we set $g = \Psi(\alpha a)^{-1}$, then

$$\begin{aligned} \lambda'(x)\lambda(x)^{-1} &= \varsigma([g^{-1}, x^{-1}]) = \varsigma(gxg^{-1}x^{-1}) \\ &= \lambda(gxg^{-1}x^{-1}) = \lambda(gxg^{-1})\lambda(x) \end{aligned}$$

and hence $\lambda'(x) = \lambda(gxg^{-1})$ for all $x \in C_Q(\sigma)$, as required. □

We are now able to prove the main result of this section.

Proof of Theorem 4.3.4. We proceed by induction on $\dim \mathcal{J}$, the result being obvious if $\dim \mathcal{J} = 1$. Therefore, we assume that $\dim \mathcal{J} \geq 2$, and that the result is true whenever $\mathcal{J}'$ is a subalgebra of $\mathcal{J}$ with $dim \mathcal{J}' \lneq \dim \mathcal{J}$.

Let V be an arbitrary irreducible smooth $C_P(\sigma)$-module, and assume that $\dim V \geq 2$ (the case where $\dim V = 1$ is obvious). In this situation, as justified above, we may choose the

smallest positive integer m for which there exists a $C_P(\sigma)$-invariant smooth character $\varsigma \in C_{(1+Jm)}(\sigma)^\circ$ such that

$$g \cdot v = \varsigma(g)v, \qquad g \in 1 + \mathcal{J}^m,\ v \in V;$$

we recall that, since V is an irreducible smooth $C_P(\sigma)$-module with $\dim V \geq 2$, we must have $m \geq 2$. Furthermore, by the minimal choice of m, there exists an ideal $\mathcal{L}$ of $\mathcal{J}$ satisfying

$$\mathcal{J}^m \subseteq \mathcal{L} \subseteq \mathcal{J}^{m-1} \qquad \text{and} \qquad \dim(C_{\mathcal{L}}(\sigma)/C_{\mathcal{J}^m}(\sigma)) = 1,$$

and such that

$$\varsigma([C_P(\sigma), C_{1+\mathcal{L}}(\sigma)]) \neq 1.$$

Let $N = 1 + \mathcal{J}^m$, and let $Q = 1 + \mathcal{L}$.

By Proposition 2.4.6, we know that the smooth $C_Q(\sigma)$-module $\mathrm{Res}^{C_P(\sigma)}_{C_Q(\sigma)}(V)$ has an irreducible quotient V'. Since $[C_Q(\sigma), C_Q(\sigma)] \subseteq \ker(\varsigma)$ (by Proposition 4.3.8), Schur's Lemma implies that V' is one-dimensional, and thus it affords a character $\lambda \in C_Q(\sigma)^\circ$. [Notice that the extreme case where $m = 2$ and $\dim \mathcal{J} = \dim \mathcal{J}^2 + 1$ cannot occur; indeed, in this situation, we must have $Q = P$ (and $C_Q(\sigma) = C_P(\sigma)$), and hence $V' = V$ which contradicts the assumption $\dim V \geq 2$.] In particular, we conclude that $V_\lambda \neq 0$ (because, by definition, V_λ is the largest quotient of V where $C_Q(\sigma)$ acts via the character λ). By Proposition 3.4.3, we know that V_λ is an irreducible $C_{C_P(\sigma)}(\lambda)$-module and that

$$V \simeq \text{c-Ind}^{C_P(\sigma)}_{C_{C_P(\sigma)}(\lambda)}(V_\lambda).$$

Since $C_N(\sigma)$ acts on V (hence, on V') via the character ς, we must have $\lambda_{C_N(\sigma)} = \varsigma$, and thus

$$C_{C_P(\sigma)}(\lambda) = \Psi(\mathcal{S}_\varsigma)$$

for some subalgebra $\mathcal{J}_\sigma$ of $\mathcal{J}$ (by Proposition 4.3.8). Since

$$\lambda([C_P(\sigma), C_Q(\sigma)]) = \varsigma([C_P(\sigma), C_Q(\sigma)]) \neq 1,$$

we must have $C_{C_P(\sigma)}(\lambda) \neq C_P(\sigma)$ which means that $\dim_{\Bbbk^\sigma} \mathcal{S}_\varsigma \lneq \dim_{\Bbbk^\sigma} C_{\mathcal{J}}(\sigma)$.

Now, let $\hat{\mathcal{S}}_\sigma$ be the $\Bbbk$-vector subspace of $\mathcal{J}$ spaced by $\mathcal{S}_\varsigma$, and define

$$\mathcal{J}_\varsigma = \hat{\mathcal{S}}_\varsigma + \mathcal{J}^2;$$

it is clear that $\mathcal{J}_\varsigma$ is a $\Bbbk$-algebra (in fact, an ideal) of $\mathcal{J}$ with $\dim \mathcal{J}_\varsigma = \dim \mathcal{J} - 1$, and that

$C_{\mathcal{J}_\varsigma}(\sigma) = \mathcal{S}_\varsigma$. Therefore we may apply the induction hypothesis which asserts that the smooth $C_{C_P(\sigma)}(\lambda)$ module V_λ is admissible and that there exists a subalgebra $\mathcal{J}'$ of $\mathcal{J}_\varsigma$ such that

$$V_\lambda \simeq \mathrm{Ind}_{C_{1+\mathcal{J}'}(\sigma)}^{C_{C_P(\sigma)}(\lambda)}(W)$$

where W is a one-dimensional $C_{1+\mathcal{J}'}(\sigma)$-module; in particular, we also conclude that

$$\text{c-Ind}_{C_{1+\mathcal{J}'}(\sigma)}^{C_{C_P(\sigma)}(\lambda)}(W) = \mathrm{Ind}_{C_{1+\mathcal{J}'}(\sigma)}^{C_{C_P(\sigma)}(\lambda)}(W).$$

Finally, by transitivity of c-induction, we deduce that

$$V \simeq \text{c-Ind}_{C_{C_P(\sigma)}(\lambda)}^{C_P(\sigma)}\left(\text{c-Ind}_{C_{1+\mathcal{J}'}(\sigma)}^{C_{C_P(\sigma)}(\lambda)}(W)\right) \simeq \text{c-Ind}_{C_{1+\mathcal{J}'}(\sigma)}^{C_P(\sigma)}(W),$$

and so, in order to conclude the proof, it is enough to show that $\text{c-Ind}_{C_{1+\mathcal{J}'}}^{C_P(\sigma)}(W) = \mathrm{Ind}_{C_{1+\mathcal{J}'}}^{C_P(\sigma)}(W)$. On the one hand, since $C_P(\sigma)$ is unimodular, the duality theorem (Proposition 2.5.3) implies that

$$V^\vee \simeq \left(\text{c-Ind}_{C_{1+\mathcal{J}'}}^{C_P(\sigma)}(W)\right)^\vee \simeq \mathrm{Ind}_{C_{1+\mathcal{J}'}}^{C_P(\sigma)}(W^\vee).$$

On the other hand, it follows from Proposition 4.3.8 that $\varsigma = \lambda_{C_N(\sigma)}$ is a $C_P(\sigma)$-invariant smooth character of $C_N(\sigma)$, and that

$$\lambda^{C_P(\sigma)} = \{\lambda' \in C_Q(\sigma)^\circ \mid (\lambda')_{C_N(\sigma)} = \varsigma\};$$

in particular, $\lambda^{C_P(\sigma)}$ is a closed subset of $C_Q(\sigma)^\circ$. Since the smooth $C_{C_P(\sigma)}(\lambda)$-module V_λ is admissible, Theorem 3.4.4 implies that

$$V \simeq \text{c-Ind}_{C_{C_P(\sigma)}(\lambda)}^{C_P(\sigma)}(V_\lambda)$$

is also admissible, and thus its smooth dual $V^\vee$ is irreducible (by Proposition 2.2.17 because V is irreducible). Since $\text{c-Ind}_{C_{1+\mathcal{J}'}(\sigma)}^{C_P(\sigma)}(W)$ is a submodule of $\mathrm{Ind}_{C_{1+\mathcal{J}'}(\sigma)}^{C_P(\sigma)}(W)$, we conclude that

$$V^\vee = \text{c-Ind}_{C_{1+\mathcal{J}'}(\sigma)}^{C_P(\sigma)}(W),$$

and thus

$$(V^\vee)^\vee \simeq \left(\text{c-Ind}_{C_{1+\mathcal{J}'}(\sigma)}^{C_P(\sigma)}(W^\vee)\right)^\vee \simeq \mathrm{Ind}_{C_{1+\mathcal{J}'}(\sigma)}^{C_P(\sigma)}((W)^\vee)^\vee$$

(again by the Duality Theorem). Since $(W^\vee)^\vee = W$ (because W is one-dimensional) and since

$(V^\vee)^\vee = V$ (by Proposition 2.2.17 because V is admissible), we conclude that

$$V \simeq \mathrm{Ind}_{C_{1+\mathcal{J}'}(\sigma)}^{C_P(\sigma)}(W),$$

and this completes the proof. $\square$

Repeating the proof of Proposition 4.1.8, we obtain the following consequence of Theorem 4.3.4.

Proposition 4.3.9. *Let $\Bbbk$ be a non-Archimedean local field of characteristic different from 2, let $\mathcal{A}$ be a finite-dimensional $\Bbbk$-algebra endowed with an involution σ, and let $\mathcal{J}$ be a σ-invariant nilpotent subalgebra of $\mathcal{A}$. Then, every irreducible smooth $C_{1+\mathcal{J}}(\sigma)$-module is unitarisable.*

We also mention that, as in the case of algebra groups, the classical Mackey theory (see [Fol95, Chapter 6]) can be applied to deduce the following.

Theorem 4.3.10. *Let $\Bbbk$ be a non-Archimedean local field of characteristic different from 2, let $\mathcal{A}$ be a finite-dimensional $\Bbbk$-algebra endowed with an involution σ. Let $\mathcal{J}$ be a σ-invariant nilpotent subalgebra of $\mathcal{A}$, let $P = 1 + \mathcal{J}$, and let $\mathcal{H}$ be an irreducible unitary $C_P(\sigma)$-module. Then, there exist an algebra subgroup P_0 of P and a unitary character $\lambda \in (P_0)^\circ$ such that $\mathcal{H} \simeq \text{u-}\mathrm{Ind}_{P_0}^{P}(\mathbb{C}_\lambda)$.*

Chapter 5

Smooth representations of unit groups of split basic algebras

A theorem due to Z. Halasi ([Hal06, Theorem 1.3]) asserts that, if $G = \mathcal{A}^\times$ is the unit group of a finite-dimensional split basic algebra $\mathcal{A}$ over a finite field $\Bbbk$, then every irreducible character of G is induced from a linear character of the unit group $H = \mathcal{B}^\times$ of some subalgebra $\mathcal{B}$ of $\mathcal{A}$. The main goal of this chapter is to extend this result to the case where $\Bbbk$ is an arbitrary non-Archimedean local field; we should mention that our proof (which is strongly based in the methods introduced by M. Boyarchenko) is still valid in the finite field case, and hence provides an alternative to Halasi's proof. (The content of this chapter is essentially [AD19].)

Before we delve into the statement and proof of the theorem, we present a small example which illustrates some of the ideas to be used in the proof.

5.1 Representation Theory of the Mirabolic Group

In this section, we describe the irreducible smooth representations of the mirabolic group of order 2:

$$M = M_2 = \left\{ \begin{bmatrix} r & s \\ 0 & 1 \end{bmatrix} \mid r \in \Bbbk^\times,\ s \in \Bbbk \right\};$$

Note that M is the semidirect product $M = T \ltimes N$ where

$$T = \left\{ \begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix} \mid r \in \Bbbk^\times \right\} \qquad \text{and} \qquad N = \left\{ \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \mid s \in \Bbbk \right\};$$

moreover, we clearly have $T \simeq \Bbbk^\times$ and $N \simeq \Bbbk^+$.

Lemma 5.1.1. *Let V be a smooth N-module where N does not act trivially. Then, there exists*

a non-trivial smooth character $\lambda \in N^\circ$ such that $V_\lambda \neq 0$.

Proof. By Proposition 2.4.6, V has an irreducible quotient W which must be one-dimensional because N is abelian. Therefore, there exists $\lambda \in N^\circ$ such that

$$n \cdot w = \lambda(n)w, \qquad n \in N,\ w \in W,$$

and so we must have $V_\lambda \neq 0$ (because V_λ is the largest quotient of V where N acts via λ). Furthermore, since N acts non-trivially on V, we may choose λ to be not-trivial. □

Next, we observe that the M-action on N° (given by conjugation) has two orbits: one consisting only on the trivial character of N, and the other on all the non-trivial smooth characters. Moreover, it is clear that the stabiliser $C_M(\lambda)$ of any non-trivial $\lambda \in N^\circ$ equals N.

Proposition 5.1.2. *Let V be an irreducible smooth M-module. Then, either V is one-dimensional and N acts trivially on V (in which case, V is uniquely determined by a smooth character of $T \simeq M/N$), or V is infinite-dimensional and*

$$V \simeq \text{c-Ind}_N^M(W)$$

for some one-dimensional N-module W where N acts via a smooth non-trivial character $\lambda \in N^\circ$ (that is, $nw = \lambda(n)w$ for all $n \in N$ and all $w \in W$).

Proof. We start by recalling that, since N is an ℓ_c-group, Proposition 2.4.2 implies that

$$V = V^N \oplus V(N)$$

where $V^N = \{v \in V \mid nv = v \text{ for all } n \in N\}$ and $V(N)$ is the vector subspace of V spanned by the set $\{v - nv \mid v \in V, n \in N\}$.

On the one hand, if $V(N) = 0$, then $V = V^N$, and so N acts trivially on V. Therefore, if this is the case, then V may be naturally considered as a smooth T-module (because $T \simeq M/N$), and thus V must be one-dimensional (by the Schur's lemma because T is abelian).

On the other hand, suppose that $V(N) \neq 0$. Then, since $V(N)$ is a submodule of V (because N is a normal subgroup) and V is irreducible, we must have $V(N) = V$, and thus $V^N = 0$ which implies that N does not act trivially on V. By the previous lemma, we conclude that there exists a non-trivial smooth character $\lambda \in N^\circ$ such that $V_\lambda \neq 0$, and so Theorem 3.3.2 guarantees that V_λ is an irreducible smooth $C_M(\lambda)$-module and

$$V \simeq \text{c-Ind}_{C_M(\lambda)}^M(V_\lambda).$$

Since λ is non-trivial, we have $C_M(\lambda) = N$, and thus V_λ is one-dimensional (because N is abelian).

The result follows. $\square$

Finally, we remark that, if W is a one-dimensional smooth N-module, then every function $f \in \mathrm{Ind}_N^M(W)$ is completely determined by the values it takes on T. In fact, we have $f \in \text{c-}\mathrm{Ind}_N^M(W)$ if and only if the restriction of f to T has compact support, and thus we conclude that $\mathrm{Ind}_N^M(W)$ cannot be irreducible. We also note that

$$(\text{c-}\mathrm{Ind}_N^M(W))^\vee \simeq \mathrm{Ind}_N^M(W^\vee)$$

(by the Duality Theorem), and hence we get an example of an irreducible smooth M-module whose smooth dual is not irreducible. (A situation of this type cannot happen in the representation theory of a finite group.)

5.2 Split Basic algebras over a local field

Let $\mathcal{A}$ be a finite-dimensional algebra over an arbitrary field $\Bbbk$, an let $\mathcal{J} = \mathcal{J}(\mathcal{A})$ denote the Jacobson radical of $\mathcal{A}$; notice that, since $\mathcal{A}$ is finite-dimensional, every element of $\mathcal{J}$ is nilpotent, but there may exist nilpotent elements which do not lie in $\mathcal{J}$. We say that $\mathcal{A}$ is a *split basic* $\Bbbk$*-algebra* if $\mathcal{J}$ equals the set consisting of all nilpotent elements of $\mathcal{A}$ (in [Sze96], B. Szegedy refers to $\mathcal{A}$ as an *N-algebra* over $\Bbbk$), and the semisimple $\Bbbk$-algebra $\mathcal{A}/\mathcal{J}$ is isomorphic to a (finite) direct sum of isomorphic copies of the base field $\Bbbk$ (in the terminology of [Sze96], $\mathcal{A}$ is referred to as a *DN-algebra* over $\Bbbk$). In particular, if $\mathcal{A}$ is a split basic $\Bbbk$-algebra, then the quotient $\Bbbk$-algebra $\mathcal{A}/\mathcal{J}$ does not have nilpotent elements, and so it follows from Wedderburn's theorem that there exists a set $\{\overline{e}_1, \ldots, \overline{e}_n\}$ of non-zero mutually orthogonal minimal idempotents of $\mathcal{A}/\mathcal{J}$ such that

$$\mathcal{A}/\mathcal{J} = \Bbbk\overline{e}_1 \oplus \cdots \oplus \Bbbk\overline{e}_n.$$

Therefore, by the process of lifting idempotents, we know that there exists a set $\{e_1, \ldots, e_n\}$ of non-zero mutually orthogonal minimal idempotents of $\mathcal{A}$ such that

$$\mathcal{A} = (\Bbbk e_1 \oplus \cdots \oplus \Bbbk e_n) \oplus \mathcal{J};$$

we refer to $\mathcal{D} = \Bbbk e_1 \oplus \cdots \oplus \Bbbk e_n$ as a *diagonal subalgebra* of $\mathcal{A}$.

A classical example is the Borel subalgebra $\mathcal{A} = \mathcal{B}_n(\Bbbk)$ of $\mathcal{M}_n(\Bbbk)$ consisting of all upper-

triangular matrices. In this case, the Jacobson radical $\mathcal{J} = \mathcal{J}(\mathcal{A})$ is the nilpotent ideal of $\mathcal{A}$ consisting of all upper-triangular matrices with zeroes on the main diagonal. For each $1 \leq i \leq n$, the idempotent $e_i \in \mathcal{A}$ can be chosen to be the elementary matrices $e_i = e_{i,i}$ having a unique non-zero entry (equal to 1) in the position (i, i), and thus $\mathcal{D} = \Bbbk e_1 \oplus \cdots \oplus \Bbbk e_n$ is the subalgebra of $\mathcal{A}$ consisting of all diagonal matrices. Notice that $e_i \mathcal{J} e_i = 0$ for all $1 \leq i \leq n$; however, this is not true in general for arbitrary split basic $\Bbbk$-algebras. For example, the subalgebra $\mathcal{A}'$ of $\mathcal{B}_n(\Bbbk)$ consisting of all the matrices $a = (a_{i,j}) \in \mathcal{B}_n(\Bbbk)$ satisfying $a_{1,1} = a_{2,2}$ has idempotents $e_1 + e_2, e_3, \ldots, e_n$, and

$$(e_1 + e_2)\mathcal{J}(\mathcal{A}')(e_1 + e_2) = e_1 \mathcal{J}(\mathcal{A}') e_2 \neq 0$$

(notice that $\mathcal{J}(\mathcal{A}') = \mathcal{J}$).

The following easy observation is crucial for inductive arguments; a proof in the case where $\Bbbk$ is a finite field can be found in [Sze96, Lemmas 2.2 and 2.3] (the proof given in this paper uses counting arguments which obviously cannot be used in the case where $\Bbbk$ is infinite).

Lemma 5.2.1. *Let $\mathcal{A}$ be a finite-dimensional split basic $\Bbbk$-algebra, and let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$ which contains the identity. Then, $\mathcal{B}$ is also a split basic $\Bbbk$-algebra.*

Proof. Let $\mathcal{J}(\mathcal{A})$ and $\mathcal{J}(\mathcal{B})$ denote the Jacobson radicals of $\mathcal{A}$ and $\mathcal{B}$, respectively. Then, $\mathcal{J}(\mathcal{B}) = \mathcal{B} \cap \mathcal{J}(\mathcal{A})$, and so $\mathcal{B}$ is a basic $\Bbbk$-algebra. On the other hand, the $\Bbbk$-algebra $\mathcal{B}/\mathcal{J}(\mathcal{B})$ is naturally isomorphic to the subalgebra $\big(\mathcal{B} + \mathcal{J}(\mathcal{A})\big)/\mathcal{J}(\mathcal{A})$ of the semisimple $\Bbbk$-algebra $\mathcal{A}/\mathcal{J}(\mathcal{A})$. Therefore, without loss of generality, we may assume that $\mathcal{A}$ is a split basic semisimple $\Bbbk$-algebra. Since $\mathcal{B}$ is a basic semisimple $\Bbbk$-algebra, there are nonzero orthogonal idempotents $e'_1, \ldots, e'_m \in \mathcal{B}$ such that

$$\mathcal{B} = \Bbbk_1 e'_1 \oplus \cdots \oplus \Bbbk_n e'_m$$

where $\Bbbk_1, \ldots, \Bbbk_n$ are finite field extensions of $\Bbbk$. On the other hand, let $e_1, \ldots, e_n \in \mathcal{A}$ be nonzero orthogonal idempotents such that

$$\mathcal{A} = \Bbbk e_1 \oplus \cdots \oplus \Bbbk e_n.$$

It is straightforward to check that there exists a subset partition $I_1, \ldots, I_m$ of $\{1, \ldots, n\}$ such that

$$e'_j = \sum_{i \in I_j} e_i, \qquad 1 \leq j \leq m,$$

and thus $e'_j e_i = e_i$ for all $i \in I_j$ and all $1 \leq j \leq m$. It follows that

$$\Bbbk_j e'_j e_i = \Bbbk_j e_i \subseteq \mathcal{A} e_i = \Bbbk e_i, \qquad i \in I_j,\ 1 \leq j \leq m,$$

which clearly implies that $\Bbbk_j = \Bbbk$ for all $1 \leq j \leq m$. $\square$

In the following result we list some elementary properties which will be used repeatedly throughout the paper; a detailed proof (which does not depend on the finiteness of the field $\Bbbk$) can be found in [Hal06, Lemma 2.1].

Lemma 5.2.2. *Let $\mathcal{A}$ be a finite-dimensional split basic $\Bbbk$-algebra, let $\mathcal{D}$ be a diagonal subalgebra of $\mathcal{A}$, and let $e_1, \ldots, e_n \in \mathcal{A}$ be nonzero orthogonal idempotents such that*

$$\mathcal{D} = \Bbbk e_1 \oplus \cdots \oplus \Bbbk e_n.$$

The following properties hold for an arbitrary $\mathcal{D}$-module $\mathcal{V}$.

(1) $\mathcal{V}$ decomposes as a direct sum of the (non-zero) homogeneous sub-bimodules $e_i \mathcal{V} e_j$ for $1 \leq i, j \leq n$.

(2) For every sub-bimodule $\mathcal{V}_1$ of $\mathcal{V}$, there exists a sub-bimodule $\mathcal{V}_2$ of $\mathcal{V}$ such that $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$.

(3) $\mathcal{V}$ decomposes as a direct sum of one-dimensional sub-bimodules.

(4) If $\mathcal{V}$ is one-dimensional and $v \in \mathcal{V}$ is such that $\mathcal{V} = \Bbbk v$, then there exist uniquely determined $1 \leq i, j \leq n$ such that $v = e_i v e_j$.

Proof. (1) Since $1 = e_1 + \cdots + e_n$, we clearly have

$$\mathcal{V} = \sum_{1 \leq i,j \leq n} e_i \mathcal{V} e_j;$$

moreover, this sum is direct because the idempotents $e_1, \ldots, e_n$ are mutually orthogonal. It is also clear that for every $1 \leq i, j \leq n$, the vector space is left and right $\mathcal{D}$-invariant, and hence it is a sub-bimodule of $\mathcal{V}$.

(2) For every $1 \leq i, j \leq n$, it is obvious that $e_i \mathcal{V}_1 e_j$ is a sub-bimodule of $e_i \mathcal{V} e_j$. Since every vector subspace of a homogeneous sub-bimodule is also a homogeneous sub-bimodule, we see that $e_i \mathcal{V} e_j = e_i \mathcal{V}_1 e_j \oplus \mathcal{V}_{2,i,j}$ for some homogeneous sub-bimodule $\mathcal{V}_{2,i,j}$ of $e_i \mathcal{V} e_j$, and thus $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ where

$$\mathcal{V}_2 = \sum_{1 \leq i,j \leq n} V_{2,i,j}.$$

(3) It is enough to observe that $\mathcal{V}$ decomposes as the direct sum of the homogeneous sub-bimodules $e_i \mathcal{V} e_j$, for $1 \leq i, j \leq n$, and that every vector subspace of a homogeneous sub-bimodule is also a homogeneous sub-bimodule.

(4) If V is one-dimensional, then $\mathcal{V} = e_i\mathcal{V}e_j$ for unique $1 \leq i, j \leq n$, and so $v = e_ive_j$ for all $v \in \mathcal{V}$. $\square$

Let $G = \mathcal{A}^\times$ denote the unit group of a split basic $\Bbbk$-algebra $\mathcal{A}$. For any (nilpotent) subalgebra $\mathcal{U}$ of $\mathcal{J}(\mathcal{A})$, the set $1 + \mathcal{U}$ is a subgroup of G to which we refer as an *algebra subgroup* of G; similarly, if $\mathcal{I} \subseteq \mathcal{J}(\mathcal{A})$ is an ideal of $\mathcal{A}$, we refer to $1 + \mathcal{I}$ as an *ideal subgroup* of G. In the particular case where $\mathcal{I} = \mathcal{J}(\mathcal{A})$, it is clear that $P = 1 + \mathcal{J}(\mathcal{A})$ is a normal subgroup of G; furthermore, G is the semidirect product $G = T \ltimes P$ where the subgroup T of G is isomorphic to the unit group of $\mathcal{A}/\mathcal{J}(\mathcal{A})$. Since $\mathcal{A}$ is a split basic $\Bbbk$-algebra, T is isomorphic to a direct product $\Bbbk^\times \times \cdots \times \Bbbk^\times$ of $n = \dim \mathcal{A}/\mathcal{J}(\mathcal{A})$ copies of the multiplicative group $\Bbbk^\times$ of $\Bbbk$. In fact, $T = \mathcal{D}^\times$ is the unit group of a diagonal subalgebra $\mathcal{D}$ of $\mathcal{A}$; we will refer to T as a *diagonal subgroup* of G.

As a standard example, let $\mathcal{A} = \mathcal{B}_n(\Bbbk)$ denote the Borel subalgebra of $\mathcal{M}_n(\Bbbk)$ consisting of all upper-triangular matrices. Then, $G = \mathcal{A}^\times$ is the standard Borel subgroup $B_n(\Bbbk)$ of the general linear group $\mathrm{GL}_n(\Bbbk)$. In this case, T is the standard torus $T_n(\Bbbk)$ consisting of all diagonal matrices, and $P = 1 + \mathcal{J}(\mathcal{A})$ is the standard unitriangular group $U_n(\Bbbk)$.

The main goal of this chapter is to study smooth modules for the unit group $G = \mathcal{A}^\times$ of an arbitrary finite-dimensional split basic $\Bbbk$-algebra $\mathcal{A}$, where $\Bbbk$ is an arbitrary non-Archimedean local field; in particular, we aim to establish that every irreducible smooth G-module may be obtained by induction with compact supports from a one-dimensional smooth module for the unit group $H = \mathcal{B}^\times$ of some subalgebra $\mathcal{B}$ of $\mathcal{A}$ (thus extending Halasi's theorem in the finite field case). More precisely, our aim is to prove the following result.

Theorem 5.2.3. *Let $\mathcal{A}$ be a finite-dimensional split basic algebra over a non-Archimedean local field $\Bbbk$, let $G = \mathcal{A}^\times$ be the unit group of $\mathcal{A}$, and let V be an irreducible smooth G-module. Then, there exist a subalgebra $\mathcal{B}$ of $\mathcal{A}$ and a smooth character of the unit group $H = \mathcal{B}^\times$ such that*

$$V \simeq \text{c-Ind}_H^G(\mathbb{C}_\vartheta).$$

The proof of this theorem relies on a refinement of the general techniques used by M. Boyarchenko in the paper [Boy11] (see Section 5.2). Henceforth, we will fix the following notation which we will use repeatedly, without always recalling their meaning.

- $\Bbbk$ is a non-Archimedean local field.
- $\mathcal{A}$ is a finite-dimensional split basic $\Bbbk$-algebra.
- $G = \mathcal{A}^\times$ is the unit group of $\mathcal{A}$.
- $\mathcal{J} = \mathcal{J}(\mathcal{A})$ is the Jacobson radical of $\mathcal{A}$, and $P = 1 + \mathcal{J}$;

- $\mathcal{D}$ is a diagonal subalgebra of $\mathcal{A}$, and $T = \mathcal{D}^\times$ is a diagonal subgroup of G.

We recall that G is a second countable ℓ-group, and that P is a normal ℓ_c-subgroup of G; moreover, G is the semidirect product $G = T \ltimes P$.

5.3 Proof of Theorem 5.2.3

In this section, we shall prove that, for every irreducible smooth G-module V, there exist an ideal subgroup Q of G and a smooth character $\lambda \in Q^\circ$ such that $V_\lambda \neq 0$. We begin by proving the following auxiliary result.

Lemma 5.3.1. *Let V be a smooth G-module, and let Q be an algebra subgroup of G. Then, the restriction $V_Q = \operatorname{Res}^G_Q(V)$ of V to Q has an irreducible quotient.*

Proof. By Theorem 4.1.1 every irreducible smooth Q-module is admissible. Since Q is also an ℓ_c-group, V_Q has an irreducible quotient by Proposition 2.4.6. □

Next, we deal with the case where $\operatorname{Res}^G_P(V)$ has a one-dimensional quotient.

Lemma 5.3.2. *Let V be an irreducible smooth G-module, and let W be an irreducible quotient of $\operatorname{Res}^G_P(V)$. Suppose that W is one-dimensional, and let $\lambda \in P^\circ$ be the character afforded by W. Then, λ is G-invariant if and only if V is one-dimensional.*

Proof. It is clear that λ is G-invariant whenever V is one-dimensional. Conversely, suppose that λ is G-invariant. Then, $V(\lambda)$ is a G-invariant vector subspace of V, and hence either $V(\lambda) = 0$ or $V(\lambda) = V$ (because V is irreducible). Since P is an ℓ_c-group, Proposition 3.4.7 implies that $W_\lambda = W/W(\lambda)$ is an epimorphic image of $V_\lambda = V/V(\lambda)$, and thus $V_\lambda \neq 0$ (because $W(\lambda) = 0$, and hence $W \simeq W_\lambda$). Therefore, we must have $V(\lambda) = 0$, and thus

$$xv = \lambda(x)v, \qquad x \in P,\ v \in V.$$

It follows that the restriction $\operatorname{Res}^G_T(V)$ of V to the diagonal subgroup T of G is irreducible; indeed, if V' is a T-submodule of $\operatorname{Res}^G_T(V)$, then V' is also a G-submodule of V (because every vector subspace of V is P-invariant), and so either $V' = 0$ or $V' = V$. Since T is an abelian ℓ-group, Schur's lemma implies that V is one-dimensional, and this completes the proof. □

We next prove that Theorem 5.2.3 is valid in the case where the restriction to P of the given irreducible smooth G-module has an irreducible one-dimensional quotient. Thus, in what follows, we let V be an irreducible smooth G-module, and assume that $\dim V \geq 2$. On the other hand, we let W be an irreducible quotient of $\operatorname{Res}^G_P(V)$, and suppose that W is one-dimensional.

Let $\lambda \in P^\circ$ be the character afforded by W. Since $\dim V \geq 2$, we have $C_G(\lambda) \neq G$ (by the previous lemma); moreover, by Proposition 3.4.2, V_λ is an irreducible smooth $C_G(\lambda)$-module and

$$V \simeq \text{c-Ind}_{C_G(\lambda)}^G(V_\lambda).$$

As in the proof of Lemma 5.3.2, we conclude that V_λ is one-dimensional; notice that W_λ is a one-dimensional irreducible quotient of $\mathrm{Res}_P^{C_G(\lambda)}(V_\lambda)$. Therefore, Theorem 5.2.3 holds in this situation once we prove that $C_G(\lambda)$ is the unit group of some subalgebra of $\mathcal{A}$; we observe that $C_G(\lambda)$ is the semidirect product $C_G(\lambda) = C_T(\lambda) \ltimes P$.

Proposition 5.3.3. *For every ideal subgroup Q of G and every $\lambda \in Q^\circ$, the centraliser $C_T(\lambda)$ is the unit group of a subalgebra of $\mathcal{D}$.*

Proof. If $\mathcal{J} \subseteq \mathcal{J}(\mathcal{A})$ is an ideal of $\mathcal{A}$ and $\lambda \in (1+\mathcal{J})^\circ$, then it is straightforward to check that

$$\mathcal{D}_\lambda = \{d \in \mathcal{D} \mid \lambda(1+ad) = \lambda(1+da) \text{ for all } a \in \mathcal{J}\}$$

is a subalgebra of $\mathcal{D}$ with $\mathcal{D}_\lambda^\times = C_T(\lambda)$. By the way of example, let $d, d' \in \mathcal{D}_\lambda$, and let $a \in \mathcal{J}$ be arbitrary. Then, since $a(1+da)^{-1} = (1+ad)^{-1}a$, we deduce that

$$\begin{aligned}
\lambda(1+da+d'a) &= \lambda(1+d'a(1+da)^{-1})\lambda(1+da) \\
&= \lambda(1+a(1+da)^{-1}d')\lambda(1+ad) \\
&= \lambda(1+ad)\lambda(1+(1+ad)^{-1}ad') \\
&= \lambda(1+ad+ad'),
\end{aligned}$$

and thus $d+d' \in \mathcal{D}_\lambda$. $\square$

As we mentioned above, this completes the proof of the following particular case of Theorem 5.2.3.

Proposition 5.3.4. *Let V be an irreducible smooth G-module, and let W be an irreducible quotient of $\mathrm{Res}_P^G(V)$. Suppose that W is one-dimensional, and let $\lambda \in P^\circ$ be the character afforded by W. Then, $C_G(\lambda)$ is the unit group of some subalgebra of $\mathcal{A}$, and V_λ is a one-dimensional smooth $C_G(\lambda)$-module such that*

$$V \simeq \text{c-Ind}_{C_G(\lambda)}^G(V_\lambda).$$

Proof. We have already proved that the smooth $C_G(\lambda)$-module V_λ is one-dimensional and that $V = \text{c-Ind}_{C_G(\lambda)}^G(V_\lambda)$. If $\mathcal{D}$ is a diagonal subalgebra of $\mathcal{A}$ and $T = \mathcal{D}^\times$, then the previous

proposition assures that $C_T(\lambda)$ is the unit group of some subalgebra $\mathcal{D}_\lambda$ of $\mathcal{D}$. Since $C_G(\lambda) = C_T(\lambda)P$ and since $\mathcal{J} = \mathcal{J}(\mathcal{A})$ is an ideal of $\mathcal{A}$, it follows that $\mathcal{A}_\lambda = \mathcal{D}_\lambda \oplus \mathcal{J}$ is a subalgebra of $\mathcal{A}$ and that $C_G(\lambda) = (\mathcal{A}_\lambda)^\times$ is the unit group of $\mathcal{A}_\lambda$. $\square$

The proof of Theorem 5.2.3 will be proceed by induction on $\dim \mathcal{A}$. By the results above, the inductive step depends on the existence of an ideal subgroup Q of G such that $\mathrm{Spec}_Q(V) \neq \emptyset$ where V is an arbitrary irreducible smooth G-module; moreover, we are reduced to the case where $\dim W \geq 2$ for every irreducible quotient W of $\mathrm{Res}^G_P(V)$ (which obviously implies that $\dim V \geq 2$).

In what follows, we fix the following notation which we will use repeatedly in the subsequent results (without always recalling their meaning). Let $m \geq 2$ be an integer such that $\mathcal{J}^m \neq 0$, and write $N = 1 + \mathcal{J}^m$. Since $\mathcal{J}^m \subseteq \mathcal{J}^{m-1}$ are ideals of $\mathcal{A}$, it follows from Lemma 5.2.2 that there exists an ideal $\mathcal{L}$ of $\mathcal{A}$ such that

$$\mathcal{J}^m \subseteq \mathcal{L} \subseteq \mathcal{J}^{m-1} \qquad \text{and} \qquad \dim \mathcal{L} = \dim \mathcal{J}^m + 1.$$

We fix such an ideal, and set $Q = 1 + \mathcal{L}$. Furthermore, we recall (from Lemma 4.1.4) that, for every P-invariant smooth character $\varsigma \in N^\circ$,

$$\mathcal{J}_\varsigma = \{a \in \mathcal{J} \mid \varsigma([1+a, 1+u]) = 1 \text{ for all } u \in \mathcal{L}\}$$

is a subalgebra of $\mathcal{J}$ satisfying

$$\mathcal{J}^2 \subseteq \mathcal{J}_\varsigma \qquad \text{and} \qquad \dim \mathcal{J}_\varsigma \geq \dim \mathcal{J} - 1.$$

On the other hand, if $\varphi_\varsigma \colon P \to Q^\circ$ is the map defined by

$$\varphi_\varsigma(g)(h) = \varsigma\big([g,h]\big), \qquad g \in P,\ h \in Q,$$

then φ_ς is a group homomorphism with $\ker(\varphi_\varsigma) = 1 + \mathcal{J}_\varsigma$ and $\varphi_\varsigma(P) \subseteq N^\perp$ (hence, φ_ς defines naturally a group homomorphism $\overline{\varphi}_\varsigma \colon P \to (Q/N)^\circ$).

The following result is essentially a particular case of [Hal06, Lemma 3.4]; for convenience, we include a proof which avoids the finiteness of the base field $\Bbbk$.

Lemma 5.3.5. *Let $\varsigma \in N^\circ$ be G-invariant, let $\mathcal{I}$ be an ideal of $\mathcal{A}$ with $\mathcal{J}^2 \subseteq \mathcal{I} \subseteq \mathcal{J}$, and let $\mathcal{J}_\varsigma \subseteq \mathcal{J}$ be defined as above. Then, $\mathcal{I}_\varsigma = \mathcal{I} \cap \mathcal{J}_\varsigma$ is an ideal of $\mathcal{A}$.*

Proof. The result is obvious in the case where $\mathcal{J}_\varsigma = \mathcal{J}$; hence, we assume that $\mathcal{J}_\varsigma \neq \mathcal{J}$, so that $\dim \mathcal{J}_\varsigma = \dim \mathcal{J} - 1$ (by the Lemma 4.1.4). The result is also clearly true in the case where

$\mathcal{I} \subseteq \mathcal{J}_\varsigma$; thus, we may assume that $\mathcal{J}_\varsigma + \mathcal{I} = \mathcal{J}$, which implies that $\dim \mathcal{I}_\varsigma = \dim \mathcal{I} - 1$. We now proceed by induction on $\dim \mathcal{I}$, the result being obvious if $\dim \mathcal{I} = \dim \mathcal{J}^2 + 1$; indeed, since $\mathcal{J}^2 \subseteq \mathcal{I} \cap \mathcal{J}_\varsigma \subseteq \mathcal{I}$, either $\mathcal{I} \cap \mathcal{J}_\varsigma = \mathcal{J}^2$, or $\mathcal{I} \cap \mathcal{J}_\varsigma = \mathcal{I}$. Therefore, we may assume that $\dim \mathcal{I} \geq \dim \mathcal{J}^2 + 2$, and that the result is true whenever $\mathcal{I}'$ is a ideal of $\mathcal{A}$ with $\mathcal{J}^2 \subseteq \mathcal{I}' \subseteq \mathcal{J}$ and $\dim \mathcal{I}' < \dim \mathcal{I}$.

Let $\mathcal{I}'_\varsigma$ be the unique ideal of $\mathcal{A}$ which is maximal with respect to the condition $\mathcal{I}'_\varsigma \subseteq \mathcal{I}_\varsigma$; hence, we must prove that $\mathcal{I}'_\varsigma = \mathcal{I}_\varsigma$. Since $\mathcal{I}'_\varsigma$ is clearly a $\mathcal{D}$-bimodule, Lemma 5.2.2 assures that

$$\mathcal{I} = \mathcal{I}'_\varsigma \oplus \mathcal{V}$$

for some sub-bimodule $\mathcal{V}$ of $\mathcal{I}$. Let $\mathcal{V}_\varsigma = \mathcal{V} \cap \mathcal{J}_\varsigma$, and note that $\mathcal{I}_\varsigma = \mathcal{I}'_\varsigma \oplus \mathcal{V}_\varsigma$; hence, $\mathcal{I}'_\varsigma = \mathcal{I}_\varsigma$ if and only if $\mathcal{V}_\varsigma = 0$. By the way of contradiction, we assume that $\mathcal{V}_\varsigma \neq 0$; notice that $\mathcal{V}_\varsigma \neq \mathcal{V}$ (otherwise, $\mathcal{I}_\varsigma = \mathcal{I}$). Since Q is a T-invariant subgroup of G (because it is an ideal subgroup of G), we deduce that

$$\varsigma([1 + t^{-1}at, h]) = \varsigma([1 + a, tht^{-1}]^t) = \varsigma([1 + a, tht^{-1}]) = 1, \quad a \in \mathcal{V}_\varsigma,\ t \in T,\ h \in Q.$$

Since $\mathcal{V}_\varsigma \subseteq \mathcal{J}_\varsigma$ (and since $\mathcal{V}$ is clearly T-invariant), it follows that $\mathcal{V}_\varsigma$ is a T-invariant vector subspace of $\mathcal{V}$.

On the other hand, let $\mathcal{V}' \neq 0$ be a proper sub-bimodule of $\mathcal{V}$, and let $\mathcal{I}' = \mathcal{I}'_\varsigma + \mathcal{V}'$. Then, $\mathcal{I}'$ is an ideal of $\mathcal{A}$ with $\mathcal{I}' \subsetneq \mathcal{I}$, and thus $\mathcal{I}' \cap \mathcal{J}_\varsigma$ is an ideal of $\mathcal{A}$ (by the inductive hypothesis). Since

$$\mathcal{I}'_\varsigma \subseteq \mathcal{I}' \cap \mathcal{J}_\varsigma \subseteq \mathcal{I} \cap \mathcal{J}_\varsigma = \mathcal{I}_\varsigma,$$

we conclude that $\mathcal{I}' \cap \mathcal{J}_\varsigma = \mathcal{I}'_\varsigma$ (by the maximality of $\mathcal{I}'_\varsigma$), and thus

$$\mathcal{V}_\varsigma \cap \mathcal{V}' = (\mathcal{J}_\varsigma \cap \mathcal{I}') \cap \mathcal{V} = \mathcal{I}'_\varsigma \cap \mathcal{V} = 0.$$

Therefore, the vector subspaces $\mathcal{V}$ and $\mathcal{V}_\varsigma$ of $\mathcal{I}$ satisfy the assumptions of [Hal06, Lemma 2.2] (we note that the proof of this result holds for an arbitrary field). In particular, if $e_1, \ldots, e_n \in \mathcal{D}$ are non-zero orthogonal idempotents such that $\mathcal{D} = \Bbbk e_1 \oplus \cdots \oplus \Bbbk e_n$, then

$$\dim e_r\mathcal{V} \leq 1 \quad \text{and} \quad \dim \mathcal{V}e_r \leq 1$$

for every $1 \leq r \leq n$.

Next, we consider the ideal subgroup $Q = 1 + \mathcal{L}$ of G, and the group homomorphism

$\varphi_\varsigma \colon P \to Q^\circ$ (as defined above); we recall that

$$\ker(\varphi_\varsigma) = 1 + \mathcal{J}_\varsigma.$$

Since $\mathcal{L}$ is an ideal of $\mathcal{A}$ with $\dim \mathcal{L} = \dim \mathcal{J}^n + 1$, we have

$$\mathcal{L} = \mathcal{J}^n \oplus \Bbbk u$$

where $u = e_i u e_j$ for some $1 \leq i, j \leq n$; hence, $Q = (1 + \Bbbk u)N$.

Firstly, suppose that $e_j \mathcal{V} = \mathcal{V} e_i = 0$, and let $v \in \mathcal{V}$ be arbitrary. Then, $uv = ue_j v = 0$ and $vu = ve_i u = 0$, and so

$$[1 + v, 1 + \alpha u] = 1, \qquad \alpha \in \Bbbk.$$

It follows that

$$1 + v \in \ker(\varphi_\varsigma) = 1 + \mathcal{J}_\varsigma,$$

and thus $\mathcal{V} \subseteq \mathcal{J}_\varsigma$. Therefore, in this case, we have $\mathcal{V} \subseteq \mathcal{I} \cap \mathcal{J}_\varsigma = \mathcal{I}_\varsigma$, and hence

$$\mathcal{I} = \mathcal{I}'_\varsigma + \mathcal{V} \subseteq \mathcal{I}_\varsigma$$

which implies that $\mathcal{I}_\varsigma = \mathcal{I}$ is an ideal of $\mathcal{A}$.

Now, suppose that $e_j \mathcal{V} \neq 0$, and let $v \in \mathcal{V}$ be such that $e_j \mathcal{V} = \Bbbk v$; since $\mathcal{V}$ has a $\Bbbk$-basis consisting of vectors $w \in \mathcal{V}$ satisfying

$$\mathcal{D} w = w \mathcal{D} = \Bbbk w,$$

it is clear that $v = ve_k$ for some $1 \leq k \leq n$ (see Lemma 5.2.2). Then,

$$\mathcal{V} = \mathcal{V}' \oplus \Bbbk v$$

for some sub-bimodule $\mathcal{V}'$ of $\mathcal{V}$; in particular, we have $e_j \mathcal{V}' = \mathcal{V}' e_k = 0$. On the one hand, suppose that $\mathcal{V}' e_i = 0$. Then, the argument above shows that $\mathcal{V}' \subseteq \mathcal{I}_\varsigma$, and so

$$\mathcal{V}' \subseteq \mathcal{V} \cap \mathcal{I}_\varsigma = \mathcal{V}_\varsigma.$$

It follows that $\mathcal{V}' = 0$ (because $\mathcal{V}' \subsetneq \mathcal{V}$), and thus $\mathcal{V} = \Bbbk v$. By the definition of $\mathcal{V}$, we conclude that

$$\mathcal{I} = \mathcal{I}'_\varsigma \oplus \Bbbk v,$$

and hence $\dim \mathcal{I} = \dim \mathcal{I}'_\varsigma + 1$. Since $\mathcal{I}'_\varsigma \subseteq \mathcal{I}_\varsigma$ and $\dim \mathcal{I}_\varsigma = \dim \mathcal{I} - 1$, we must have $\mathcal{I}'_\varsigma = \mathcal{I}_\varsigma$,

and hence $\mathcal{I}_\varsigma$ is an ideal of $\mathcal{A}$. If $k = i$, then

$$\mathcal{V}e_i = \mathcal{V}'e_i \oplus \Bbbk v.$$

Since $\dim(\mathcal{V}e_i) \leq 1$, we get $\mathcal{V}'e_i = 0$, which is the previously handled case.

Since $k \neq i$ (otherwise, $v = ve_i \in \mathcal{V}e_i = \Bbbk w$), we have $vu = 0$, and thus

$$[1 + v, 1 + u] = 1 - u'v$$

where $u' \in \mathcal{J}$ is such that $(1 + u)^{-1} = 1 - u'$. Since $u'v \in u\mathcal{A}v$, we see that

$$(u'v)^2 \in (u\mathcal{A}v)^2 = 0,$$

and thus $S = 1 + \Bbbk(u'v)$ is a T-invariant algebra subgroup of N; indeed, we have $\mathcal{D}(u'v)\mathcal{D} = \Bbbk u'v$ (because $u' = e_i u'$ and $v = ve_k$). Let $\alpha \in \Bbbk^\times$ be arbitrary, and choose $t \in T$ such that

$$t^{-1}u' = \alpha u' \qquad \text{and} \qquad vt = v;$$

notice that, since $i \neq k$, it is enough to choose $t \in T$ satisfying $te_i = \alpha^{-1}e_i$ and $te_k = e_k$. It follows that

$$[1 + v, 1 + u]^t = (1 - u'v)^t = 1 - t^{-1}u'vt = 1 - \alpha u'v,$$

and thus the restriction ς_S of ς to S is a (smooth) character of S which is constant on $S \setminus \{1\}$ (because ς is T-invariant). Therefore, ς_S must be the trivial character, and so $\varsigma([1+v, 1+u]) = 1$. It follows that $1 + v \in 1 + \mathcal{J}_\varsigma$, and thus $v \in \mathcal{V} \cap \mathcal{J}_\varsigma = \mathcal{V}_\varsigma$. Since $\Bbbk v \neq \mathcal{V}$ and $\mathcal{D}v = v\mathcal{D} = \Bbbk v$, we must have $\Bbbk v = 0$, a contradiction.

The case where $\mathcal{V}e_i \neq 0$ is analogous, and hence the proof is complete. $\square$

We are now able to prove the following crucial result.

Proposition 5.3.6. *Let $\varsigma \in N^\circ$ be P-invariant, and let $\lambda \in Q^\circ$ be such that $\lambda_N = \varsigma$. Then, $C_G(\lambda)$ is the unit group of some subalgebra of $\mathcal{A}$.*

Proof. In the case where λ is P-invariant, the result follows by Proposition 5.3.3: hence, we assume that $C_P(\lambda) \neq P$. Let $\mathcal{D}$ be a diagonal subalgebra of $\mathcal{A}$, and let $T = \mathcal{D}^\times$. By Proposition 5.3.3, we know that $C_T(\varsigma)$ is the unit group of some subalgebra $\mathcal{D}_\varsigma$ of $\mathcal{D}$; similarly, $C_T(\lambda)$ is the unit group of some subalgebra $\mathcal{D}_\lambda$ of $\mathcal{D}$.

Since ς is P-invariant, we see that $C_G(\varsigma)$ is the unit group of the subalgebra $\mathcal{A}_\varsigma = \mathcal{D}_\varsigma \oplus \mathcal{J}$ of $\mathcal{A}$; indeed, we have $C_G(\varsigma) = C_T(\varsigma)P$. Let $\mathcal{J}_\varsigma$ be the ideal of $\mathcal{J}$ defined as in Lemma 4.1.4,

and note that

$$1 + \mathcal{J}_\varsigma = C_P(\lambda)$$

is the centraliser of λ in P (by the Proposition 4.1.5, because $\lambda_N = \varsigma$). Since ς is $C_G(\varsigma)$-invariant, Lemma 5.3.5 implies that $\mathcal{J}_\varsigma$ is an ideal of $\mathcal{A}_\varsigma$, and thus $C_T(\varsigma)C_P(\lambda)$ is the unit group of the subalgebra $\mathcal{B}_\varsigma = \mathcal{D}_\varsigma \oplus \mathcal{J}_\varsigma$ of $\mathcal{A}_\varsigma$ (and hence of $\mathcal{A}$). We also observe that $C_G(\lambda) \subseteq C_G(\varsigma)$ (because $\lambda_N = \varsigma$).

Since $\mathcal{J}_\varsigma$ is an ideal of $\mathcal{J}$ with $\dim \mathcal{J}_\varsigma = \dim \mathcal{J} - 1$, there exists $a \in \mathcal{J}$ such that

$$\mathcal{J} = \mathcal{J}_\varsigma \oplus \Bbbk a \qquad \text{and} \qquad \mathcal{D}a = a\mathcal{D} = \Bbbk a$$

(see Lemma 5.2.2). On the one hand, suppose that $[\mathcal{D}_\varsigma, \mathcal{J}] = 0$ (so that $da = ad$ for all $d \in \mathcal{D}$); in particular, we see that $[C_T(\varsigma), P] = 1$ (because $C_T(\varsigma) = (\mathcal{D}_\varsigma)^\times$). Let $y \in C_G(\lambda)$ be arbitrary, and write $y = tx$ for $t \in C_T(\varsigma)$ and $x \in P$ (notice that this decomposition exists because $C_G(\lambda) \subseteq C_G(\varsigma) = C_T(\varsigma)P$). For every $g \in Q$, we deduce that

$$\lambda(g) = \lambda(ygy^{-1}) = \lambda(txgx^{-1}t^{-1}) = \lambda(xgx^{-1})$$

(where the last equality holds because $[C_T(\varsigma), P] = 1$, and hence $txgx^{-1} = xgx^{-1}t$). It follows that $x \in C_G(\lambda)$, and thus $t = yx^{-1} \in C_T(\lambda)$. It follows that $y = tx \in C_T(\lambda)C_P(\lambda)$, and so $C_G(\lambda) = C_T(\lambda)C_P(\lambda)$ is the unit group of the subalgebra $\mathcal{B}_\lambda = \mathcal{D}_\lambda \oplus \mathcal{J}_\varsigma$ of $\mathcal{A}$. On the other hand, suppose that $[\mathcal{D}_\varsigma, \mathcal{J}] \neq 0$, and observe that the above element $a \in \mathcal{J}$ can be chosen such that $[\mathcal{D}_\varsigma, a] \neq 0$; indeed, if $[\mathcal{D}_\varsigma, a] = 0$, then we may replace a by an element $a + b$ where $b \in \mathcal{J}_\varsigma$ satisfies $\mathcal{D}b = b\mathcal{D}$ and is such that $[\mathcal{D}_\varsigma, b] \neq 0$.

Now, since

$$C_G(\lambda) \subseteq C_G(\varsigma) = C_T(\varsigma)P \qquad \text{and} \qquad C_P(\lambda) \subseteq C_G(\lambda),$$

for every element $g \in C_G(\lambda)$ there exist unique elements $t \in C_T(\varsigma)$ and $x \in 1 + \Bbbk a$ such that $g \in txC_P(\lambda)$. In fact, for every $t \in C_T(\varsigma)$, there is a unique element $x(t) \in 1 + \Bbbk a$ such that $tx(t) \in C_G(\lambda)$. To see this, let $t \in C_T(\varsigma)$ be arbitrary. Then, $\lambda^t \in Q^\circ$ satisfies $(\lambda^t)_N = \varsigma^t = \varsigma$, and thus Proposition 4.1.5 implies that $\lambda^t = \lambda^x$ for some $x \in P$. Therefore, $\lambda^{tx^{-1}} = \lambda$, and hence $tx^{-1} \in C_G(\lambda)$. Since

$$P = (1 + \Bbbk a)(1 + \mathcal{J}_\varsigma) = (1 + \Bbbk a)C_P(\lambda)$$

and $C_P(\lambda) \subseteq C_G(\lambda)$, we have $x^{-1} \in x(t)C_P(\lambda)$ for some $x(t) \in 1 + \Bbbk a$, and so

$$\lambda^{tx(t)} = \lambda^{xx(t)} = \lambda;$$

notice that $x(t)$ is uniquely determined by $t \in C_T(\varsigma)$.

Suppose that $C_T(\varsigma) = C_T(\lambda)$. If this is the case, then $\lambda^{x(t)} = \lambda^{tx(t)} = \lambda$, and thus

$$x(t) \in C_G(\lambda) \cap P = C_P(\lambda) \qquad t \in C_T(\varsigma).$$

By the above, we conclude that $C_G(\lambda) = C_T(\varsigma)C_P(\lambda)$ is the unit group of the subalgebra $\mathcal{B}_\varsigma$ of $\mathcal{A}$. Therefore, we henceforth assume that $C_T(\varsigma) \neq C_T(\lambda)$.

For every $t \in C_T(\varsigma)$, let $\alpha(t) \in \Bbbk$ be such that

$$x(t) = 1 + \alpha(t)a.$$

It is straightforward to check that the mapping $tC_T(\lambda) \mapsto \alpha(t)$ defines an injective map

$$\alpha\colon C_T(\varsigma)/C_T(\lambda) \to \Bbbk.$$

Since $C_T(\varsigma) = (\mathcal{D}_\varsigma)^\times$ of $\mathcal{D}$, the stabiliser

$$C_{C_T(\varsigma)}(a) = \{t \in C_T(\varsigma) \mid t^{-1}at = a\}$$

is the unit group of the subalgebra

$$(\mathcal{D}_\varsigma)_a = \{d \in \mathcal{D}_\varsigma \mid da = ad\}$$

of $\mathcal{D}_\varsigma$. Moreover, since $[\mathcal{D}_\varsigma, a] \neq 0$, the mapping $d \mapsto da - ad$ defines a surjective $\Bbbk$-linear map $\mathcal{D}_\varsigma \to \Bbbk a$ with kernel $(\mathcal{D}_\varsigma)_a$, and so

$$\dim(\mathcal{D}_\varsigma)_a = \dim \mathcal{D}_\varsigma - 1.$$

On the other hand, it is straightforward to check that α induces (by restriction) a group homomorphism

$$\widetilde{\alpha}\colon \big((C_T(\varsigma))_a C_T(\lambda)\big)/C_T(\lambda) \to \Bbbk^+.$$

Since

$$\big((C_T(\varsigma))_a C_T(\lambda)\big)/C_T(\lambda) \simeq (C_T(\varsigma))_a/(C_T(\lambda) \cap (C_T(\varsigma))_a)$$

is, either the trivial group, or isomorphic to the direct product of a finite number of copies of the multiplicative group $\Bbbk^{\times}$ (because $C_{C_T(\varsigma)}(a)$ and $C_T(\lambda) \cap (C_T(\varsigma))_a$ are unit groups of subalgebras of $\mathcal{D}$), we must have

$$C_{C_T(\varsigma)}(a) = C_T(\lambda) \cap C_{C_T(\varsigma)}(a);$$

indeed, if we choose a root of unity $\zeta \in \Bbbk^{\times}$ of order coprime to the characteristic of the residue field of $\Bbbk$, then we must have $\widetilde{\alpha}(\zeta) = 0$. Therefore, we conclude that $C_{C_T(\varsigma)}(a) \subseteq C_T(\lambda)$, and so

$$(\mathcal{D}_\varsigma)_a \subseteq \mathcal{D}_\lambda \subsetneq \mathcal{D}_\varsigma.$$

Since $\dim(\mathcal{D}_\varsigma)_a = \dim \mathcal{D}_\varsigma - 1$, it follows that $\mathcal{D}_\lambda = (\mathcal{D}_\varsigma)_a$, and thus

$$C_T(\lambda) = C_{C_T(\varsigma)}(a) \qquad \text{and} \qquad C_T(\varsigma)/C_T(\lambda) \simeq \Bbbk^{\times}.$$

Since $C_P(\lambda)$ is an ideal subgroup of $C_G(\varsigma)$ and

$$C_P(\lambda) \subseteq C_G(\lambda) \subseteq C_G(\varsigma),$$

it is also a normal subgroup of $C_G(\lambda)$, and thus the mapping $t \mapsto (tx(t))C_P(\lambda)$ defines a bijection

$$\beta \colon C_T(\varsigma) \to C_G(\lambda)/C_P(\lambda).$$

Since P is a normal subgroup of G, we have

$$\big(tt'x(tt')\big)\big(t'x(t')\big)^{-1}\big(tx(t)\big)^{-1} \in P \cap C_G(\lambda) = C_P(\lambda),$$

and so

$$\beta(tt') = \beta(t)\beta(t'), \qquad t, t' \in C_T(\varsigma).$$

It follows that β is a group isomorphism, and hence $C_G(\lambda)/C_P(\lambda)$ is an abelian group. Therefore, $\big(C_T(\lambda)C_P(\lambda)\big)/C_P(\lambda)$ is a normal subgroup of $C_G(\lambda)/C_P(\lambda)$, and thus $C_T(\lambda)C_P(\lambda)$ is a normal subgroup of $C_G(\lambda)$. Since

$$\beta(C_T(\lambda)) = \big(C_T(\lambda)C_P(\lambda)\big)/C_P(\lambda),$$

we see that β induces naturally a group isomorphism

$$\widetilde{\beta} \colon C_T(\varsigma)/C_T(\lambda) \to C_G(\lambda)/\big(C_T(\lambda)C_P(\lambda)\big).$$

Now, for every $t \in C_T(\varsigma)$, we have $tat^{-1} \in \Bbbk a$, and hence there is $\xi(t) \in \Bbbk^{\times}$ such that

$tat^{-1} = \xi(t)a$. The mapping $t \mapsto \xi(t)$ defines a group homomorphism $\xi\colon C_T(\varsigma) \to \Bbbk^\times$ with

$$\ker(\xi) = C_{C_T(\varsigma)}(a) = C_T(\lambda).$$

On the other hand, since $\mathcal{J} = \mathcal{J}_\varsigma \oplus \Bbbk a$, for every $x \in P$ there exists $\zeta(x) \in \Bbbk$ such that $x \in (1+\zeta(x)a)C_P(\lambda)$, and the mapping $x \to \zeta(x)$ defines a group homomorphism $\zeta\colon P \to \Bbbk^+$ with $\ker(\zeta) = C_P(\lambda)$. Since every element $g \in C_T(\varsigma)P$ is uniquely written as a product $g = tx$ for $t \in C_T(\varsigma)$ and $x \in P$, we may define a map $\psi\colon C_T(\varsigma)P \to \mathrm{GL}_2(\Bbbk)$ by the rule

$$\psi(xt) = \begin{bmatrix} \xi(t) & \zeta(x) \\ 0 & 1 \end{bmatrix}, \qquad t \in C_T(\varsigma),\ x \in P.$$

(Since $C_T(\varsigma)P = PC_T(\varsigma)$, this is a well-defined map.) Since

$$(xt)(x't') = (xtx't^{-1})(tt') \qquad \text{and} \qquad \zeta(xtx't^{-1}) = \xi(t)\zeta(x') + \zeta(x),$$

we see that

$$\psi((xt)(x't')) = \psi(xt)\psi(x't'), \qquad t, t' \in C_T(\varsigma),\ x, x' \in P,$$

which means that ψ is a group homomorphism. It is clear that

$$\ker(\psi) = C_T(\lambda)C_P(\lambda),$$

and so ψ induces a group isomorphism

$$\widetilde{\psi}\colon \big(C_T(\varsigma)P\big)/\big(C_T(\lambda)C_P(\lambda)\big) \to M_2$$

where M_2 denotes the mirabolic subgroup

$$M_2 = \left\{ \begin{bmatrix} r & s \\ 0 & 1 \end{bmatrix} \mid r \in \Bbbk^\times,\ s \in \Bbbk \right\}$$

of $\mathrm{GL}_2(\Bbbk)$.

Finally, consider the image $M_2' = \widetilde{\psi}\big(C_G(\lambda)/\big(C_T(\lambda)C_P(\lambda)\big)\big)$; recall that $C_G(\lambda)$ is a subgroup of $C_T(\varsigma)P$. Since there are group isomorphisms

$$C_G(\lambda)/\big(C_T(\lambda)C_P(\lambda)\big) \simeq C_T(\varsigma)/C_T(\lambda) \simeq \Bbbk^\times,$$

we conclude that $M_2' \simeq \Bbbk^\times$ is a commutative subgroup of M_2. For every $t \in C_T(\varsigma)$, we have

$tx(t) \in C_G(\lambda)$, and

$$\psi(x(t)t) = \begin{bmatrix} \xi(t) & \zeta(x(t)) \\ 0 & 1 \end{bmatrix};$$

moreover, notice that the matrix $\psi(x(t)t)$ is semisimple for all $t \in C_T(\varsigma) \setminus C_T(\lambda)$. Therefore, since M_2' is commutative and consists of semisimple matrices, there exists $y \in \mathrm{GL}_2(\Bbbk)$ such that

$$y \begin{bmatrix} \xi(t) & \zeta(x(t)) \\ 0 & 1 \end{bmatrix} y^{-1} = \begin{bmatrix} \xi(t) & 0 \\ 0 & 1 \end{bmatrix}$$

for all $t \in C_T(\varsigma)$; in fact, we may choose $y \in M_2$. Let $g \in C_T(\varsigma)P$ be such that $\psi(g) = y$. Then,

$$\psi(gC_G(\lambda)g^{-1}) = xM_2'x^{-1} = \left\{ \begin{bmatrix} \xi(t) & 0 \\ 0 & 1 \end{bmatrix} \mid t \in C_T(\varsigma) \right\} = \psi(C_T(\varsigma)),$$

and thus $gC_G(\lambda)g^{-1} = C_T(\varsigma)C_P(\lambda)$ is the unit group of the subalgebra $\mathcal{B}_\varsigma$ of $\mathcal{A}$. It follows that $C_G(\lambda)$ is the unit group of the subalgebra $g^{-1}\mathcal{B}_\varsigma g$ of $\mathcal{A}$, and this completes the proof. □

We are now able to prove our main result.

Proof of Theorem 5.2.3. We proceed by induction on $\dim \mathcal{A}$, the result being obvious if $\dim \mathcal{A} = 1$. Therefore, we assume that $\dim \mathcal{A} \geq 2$, and that the result is true whenever $\mathcal{A}'$ is a subalgebra $\mathcal{A}$ with $\dim \mathcal{A}' \lneq \dim \mathcal{A}$.

Let V be an arbitrary irreducible smooth G-module, and let V' be an irreducible quotient of $\mathrm{Res}_P^G(V)$ (the existence of which is guaranteed by Lemma 5.3.1). In spite of Proposition 5.3.4, we may assume that $\dim V' \geq 2$. In this situation, there is an integer $n \geq 2$ such that $\mathcal{J}^n \neq 0$ and $\mathcal{J}^{n+1} = 0$; notice that $\mathcal{J}^2 \neq 0$ (otherwise, $P = 1 + \mathcal{J}$ is abelian, and hence V' must be one-dimensional). Since $1 + \mathcal{J}^m$ lies in the centre of P, Schur's lemma implies that $1 + \mathcal{J}^m$ acts on V' by scalar multiplications, and thus we may choose the smallest positive integer m for which there exists $\varsigma \in (1 + \mathcal{J}^m)^\circ$ such that

$$g \cdot v' = \varsigma(g)v', \qquad g \in 1 + \mathcal{J}^m,\ v' \in V';$$

recall that by construction ς is P-invariant. We note that, since V' is an irreducible smooth P-module with $\dim V' \geq 2$, we must have $m \geq 2$; furthermore, since $[1+\mathcal{J}, 1+\mathcal{J}^{m-1}] \subseteq 1+\mathcal{J}^m$, the minimal choice of m implies that ς is not identically equal to 1 (otherwise, Schur's lemma would imply that the subgroup $1 + \mathcal{J}^{m-1}$ acts on V' by scalar multiplications). Since $\mathcal{J}^{m-1}$ and $\mathcal{J}^m$ are ideals of $\mathcal{A}$, Lemma 5.2.2 implies that

$$\mathcal{J}^{m-1} = \mathcal{L}_1 + \cdots + \mathcal{L}_t$$

for some ideals $\mathcal{L}_1, \ldots, \mathcal{L}_t$ of $\mathcal{A}$ satisfying

$$\mathcal{J}^m \subseteq \mathcal{L}_i \subseteq \mathcal{J}^{m-1} \qquad \text{and} \qquad \dim\left(\mathcal{L}_i/\mathcal{J}^m\right) = 1, \quad 1 \leq i \leq t.$$

By the minimal choice of m, we must have $\varsigma[1+\mathcal{J}, 1+\mathcal{L}_i] \neq 1$ for some $1 \leq i \leq t$ (otherwise, we would have $[1+\mathcal{J}, 1+\mathcal{J}^{n-1}] \subseteq \ker(\varsigma)$, and hence $1+\mathcal{J}^{m-1}$ would act on V' by scalar multiplications).

Let $N = 1+\mathcal{J}^m$, and let $Q = 1+\mathcal{L}$ where we set $\mathcal{L} = \mathcal{L}_i$. The argument used in the proof of Lemma 5.3.1 shows that the smooth Q-module $\mathrm{Res}^P_Q(V')$ has an irreducible quotient V''. Since $[Q, Q] \subseteq \ker(\varsigma)$ (by Proposition 4.1.5), Schur's lemma implies that V'' is one-dimensional, and thus it affords a character $\lambda \in Q^\circ$. [Notice that the extreme case where $n = 2$ and $\dim \mathcal{J} = \dim \mathcal{J}^2 + 1$ cannot occur; indeed, in this situation, we must have $Q = P$, and hence $V'' = V'$ which contradicts the assumption $\dim V' \geq 2$.] In particular, we have $V'_\lambda \neq 0$, and thus $V_\lambda \neq 0$ (by Proposition 3.4.7 because P is an ℓ_c-group). By Proposition 3.4.2, V_λ is an irreducible $C_G(\lambda)$-module and we have

$$V = \text{c-Ind}^G_{C_G(\lambda)}(V_\lambda).$$

Since N acts on V' (hence, on V'') via the character ς, we must have $\lambda_N = \varsigma$, and thus $C_G(\lambda)$ is the unit group of some subalgebra $\mathcal{A}'$ of $\mathcal{A}$ (by Proposition 5.3.6). Since

$$\lambda([P, Q]) = \varsigma([P, Q]) \neq 1,$$

we must have $C_P(\lambda) \neq P$, and thus $C_G(\lambda) \neq G$. Therefore, we have $\dim \mathcal{A}' \lneq \dim \mathcal{A}$, and thus it follows by induction that there exists a subalgebra $\mathcal{B}$ of $\mathcal{A}'$ such that

$$V_\lambda \simeq \text{c-Ind}^{C_G(\lambda)}_H(W)$$

where $H = \mathcal{B}^\times$ and W is a one-dimensional H-module. By transitivity of c-induction, we conclude that

$$V \simeq \text{c-Ind}^G_{C_G(\lambda)}\left(\text{c-Ind}^{C_G(\lambda)}_H(W)\right) \simeq \text{c-Ind}^G_H(W),$$

as required. $\square$

5.4 Admissibility and unitarisability

As before, let $\Bbbk$ be a non-Archimedean local field, let $\mathcal{A}$ is a finite-dimensional split basic $\Bbbk$-algebra, and let $G = \mathcal{A}^\times$ be the unit group of $\mathcal{A}$. Contrary to what happens in the case of

algebra groups, not all irreducible smooth G-modules are admissible (nor unitarisable). In fact, an irreducible smooth G-module is admissible if and only if, in a certain sense, it is "induced" from an algebra subgroup, as Theorem 5.4.2 below clarifies. Firstly, we illustrate this situation with the following example.

Example 5.4.1. Let $\Bbbk$ be an arbitrary non-Archimedean local field, and let $\mathcal{A} = \mathcal{B}_2(\Bbbk)$ be the standard Borel algebra of $\mathcal{M}_2(\Bbbk)$; hence, $G = \mathcal{A}^\times$ is the standard Borel subgroup $B_2(\Bbbk)$ of $\mathrm{GL}_2(\Bbbk)$ consisting of all upper-triangular invertible matrices. It is obvious that G is the semidirect product $G = T \ltimes P$ where $T \simeq \Bbbk^\times \times \Bbbk^\times$ is the subgroup of G consisting of all diagonal matrices and $P \simeq \Bbbk^+$ is the abelian normal subgroup of G consisting of all unipotent matrices.

It is clear that there are exactly two G-orbits on P° (under the conjugation action of G on P°), namely: the singleton $\{1_P\}$ consisting of the trivial character of P, and its complement $P^\circ \setminus \{1_P\}$. Both these G-orbits are locally closed, and thus by Rodier's Theorem 3.4.2 there are two distinct families of irreducible smooth G-modules, each one corresponding to one of these two G-orbits. On the one hand, one has the family consisting of one-dimensional G-modules corresponding to the smooth characters $\lambda \in G^\circ$ which satisfy $P \subseteq \ker(\lambda)$; indeed, P lies is the kernel of every smooth character of G.

On the other hand, there is a family corresponding to a fixed non-trivial smooth character $\lambda \in P^\circ$. In this case, the centraliser $C_G(\lambda)$ is the (internal) direct product $C_G(\lambda) = ZP$ where $Z = Z(G)$ is the center of G; notice that $Z \simeq \Bbbk^\times$, whereas $P \simeq \Bbbk^+$, and hence $C_G(\lambda) \simeq \Bbbk^\times \times \Bbbk^+$. Since $C_G(\lambda)$ is abelian, every irreducible smooth $C_G(\lambda)$-module is one-dimensional. By Rodier's result, for every smooth character $\mu \in C_G(\lambda)^\circ$ satisfying $\mu_P = \lambda$, the c-induced smooth $C_G(\lambda)$-module $\text{c-Ind}_{C_G(\lambda)}^G(\mathbb{C}_\mu)$ is irreducible (and clearly of dimension ≥ 2); moreover, the mapping $\mu \mapsto \text{c-Ind}_{C_G(\lambda)}^G(\mathbb{C}_\mu)$ defines a one-to-one correspondence between smooth characters of $C_G(\lambda)$ satisfying $\mu_P = \lambda$ and irreducible smooth G-modules with dimension ≥ 2.

Now, let $\mathfrak{o} = \mathfrak{o}_\Bbbk$ denote the ring of algebraic integers of $\Bbbk$, let

$$T_0 = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \mid \alpha, \beta \in \mathfrak{o}^\times \right\}$$

where $\mathfrak{o}^\times$ denote the unit group of $\mathfrak{o}$, and consider the subgroup $G_0 = T_0 C_G(\lambda)$ of G; notice that G_0 is an open (hence, also a closed) subgroup of G. Let $\mu \in C_G(\lambda)^\circ$ be an arbitrary smooth character of $C_G(\lambda)$ satisfying $\mu_P = \lambda$, and consider the c-induced G-module $V = \text{c-Ind}_{C_G(\lambda)}^G(\mathbb{C}_\mu)$. By the transitivity of c-induction, we have $V = \text{c-Ind}_{G_0}^G(V_0)$ where $V_0 = \text{c-Ind}_{C_G(\lambda)}^{G_0}(\mathbb{C}_\mu)$. Since V is irreducible, it is obvious that V_0 is an irreducible smooth

G_0-module. Since G_0 is an open subgroup of G, it follows from Proposition 2.3.5 that V_0 is naturally embedded as a vector subspace of V and that we have a direct sum decomposition

$$V = \bigoplus_{g \in \mathcal{G}} gV_0$$

where $\mathcal{G}$ is a complete set of representatives of the coset space G/G_0.

Let K be a sufficiently small open compact subgroup of G_0 such that $(V_0)^K \neq 0$ (hence, $V^K \neq 0$); note that we can choose K to be the compact open subgroup consisting on the matrices $x \in G_0$ such that $x_{1,1}, x_{2,2} \in \mathfrak{o}^\times$ and $x_{1,2} \in \varpi^r \mathfrak{o}$ for a suitable $r \in \mathbb{Z}$ (here, $\varpi \in \mathfrak{o}$ is the prime element of $\mathfrak{o}$, so that $\varpi\mathfrak{o}$ is the unique maximal ideal of $\mathfrak{o}$). Let

$$g_m = \begin{bmatrix} \varpi^m & 0 \\ 0 & \varpi^{-m} \end{bmatrix}, \qquad m \in \mathbb{N};$$

then, for every $m \in \mathbb{N}$, we have $g_m \notin G_0$ and $g_m K g_m^{-1} \subseteq K$. It follows that $(g_m V_0)^K = g_m {V_0}^K \neq \{0\}$ for every $m \in \mathbb{N}$, and thus

$$V^K = \bigoplus_{g \in \mathcal{G}} (gV_0)^K$$

has infinite dimension. Therefore, the smooth G-module V is not admissible, and hence it follows from Rodier's Theorem that an irreducible smooth G-module is admissible if and only if it is one-dimensional.

We now characterise the admissible irreducible smooth G-modules; we use the same notation as before.

Theorem 5.4.2. *Let V be an irreducible smooth G-module, and let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$ such that $V \simeq \text{c-Ind}_H^G(W)$ where $H = \mathcal{B}^\times$ is the unit group of $\mathcal{B}$ and W is a one-dimensional smooth H-module. Then, the following are equivalent.*

(1) H contains a diagonal subgroup T of G.

(2) The smooth G-module V is admissible.

(3) There is an isomorphism of G-modules $V \simeq \mathrm{Ind}_H^G(W)$.

Proof. For simplicity of reading, we consider several (independent) steps; in the first step, we establish that both (2) and (3) imply (1).

Step 1. *Assume that, either V is admissible, or $V \simeq \mathrm{Ind}_H^G(W)$. Then, H contains a diagonal subgroup T of G.*

Proof. Let $\mathcal{D}'$ be a diagonal subalgebra of $\mathcal{B}$, and let $T' = (\mathcal{D}')^\times$ be the unit group of $\mathcal{D}'$ (hence, T' is a diagonal subgroup of H). On the other hand, let $\mathcal{D}$ be a diagonal subalgebra of $\mathcal{A}$ which contains $\mathcal{D}'$; to see that such diagonal subalgebra exists, it is enough to consider the inclusion map $(\mathcal{D}' + \mathcal{J})/\mathcal{J} \hookrightarrow (\mathcal{D} + \mathcal{J})/\mathcal{J}$, where $\mathcal{D}$ is a diagonal algebra of $\mathcal{A}$, and use it to extend the basis of orthogonal idempotents of $\mathcal{D}'$ to a basis of orthogonal idempotents in $\mathcal{D}$. Note that $\mathcal{D}$ is clearly a $\mathcal{D}'$-bimodule, and thus Lemma 5.2.2 implies that $\mathcal{D} = \mathcal{D}' \oplus \mathcal{D}''$ for some sub-bimodule $\mathcal{D}''$ of $\mathcal{D}$. Then, $T = T'T''$ where $T'' = (\mathcal{D}'')^\times$ is the unit group of $\mathcal{D}''$, and G decomposes as the semidirect product $G = G'T''$ where $G' = T'P$; moreover, we have $\mathcal{B} = \mathcal{D}' \oplus \mathcal{J}(\mathcal{B})$ where $\mathcal{J}(\mathcal{B})$ denotes the Jacobson radical of $\mathcal{B}$, and thus $H \subseteq G'$. Let $V' = \text{c-Ind}_H^{G'}(W)$, so that

$$V \simeq \text{c-Ind}_H^G(W) \simeq \text{c-Ind}_{G'}^G(V').$$

On the one hand, assume that $V \simeq \text{Ind}_H^G(W)$. Then, $\text{c-Ind}_H^G(W) = \text{Ind}_H^G(W)$ (because V is irreducible), and so $\text{Ind}_H^{G'}(W)$ is irreducible. This implies that $V' = \text{Ind}_H^{G'}(W)$, and thus $\text{Ind}_H^G(W) \simeq \text{Ind}_{G'}^G(V')$; moreover, we conclude that $\text{c-Ind}_{G'}^G(V') = \text{Ind}_{G'}^G(V')$. Now, since $G = G'T''$ is a semidirect product, every element of G is uniquely factorised as a product gt for $g \in G'$ and $t \in T''$, and hence every function $\phi \in \text{Ind}_{G'}^G(V')$ is uniquely determined by the rule

$$\phi(gt) = g\phi(t), \qquad g \in G',\ t \in T'';$$

in particular, a function $\phi \in \text{Ind}_{G'}^G(V')$ lies in $\text{c-Ind}_{G'}^G(V'))$ if and only if its restriction to T'' has compact support. Since $T'' \simeq (\Bbbk^\times)^r$ for some nonnegative integer $r \geq 0$, we conclude that

$$\text{Ind}_{G'}^G(V') = \text{c-Ind}_{G'}^G(V') \quad \iff \quad T'' = 1$$

(that is, if and only if $r = 0$); in other words, we have $\text{Ind}_{G'}^G(V') = \text{c-Ind}_{G'}^G(V')$ if and only if $G = G'$ which occurs if and only if $T \subseteq H$. Finally, since $\text{Ind}_{G'}^G(V') = \text{c-Ind}_{G'}^G(V')$, we conclude that $T \subseteq H$, as required.

On the other hand, suppose that $V \simeq \text{c-Ind}_H^G(W)$ is admissible. Let $\delta_G \colon G \to \mathbb{R}_+^\times$ and $\delta_H \colon H \to \mathbb{R}_+^\times$ be the modular characters of G and H, respectively. If $V^\vee$ denotes the smooth dual of V, then the Duality Theorem 2.5.3 implies that there is a natural isomorphism

$$V^\vee \simeq (\text{c-Ind}_H^G(W))^\vee \simeq \text{Ind}_H^G(\delta_{G/H} \otimes W^\vee)$$

where $\delta_{G/H} = (\delta_H)^{-1}(\delta_G)_H$ and where the smooth H-module $\delta_{G/H} \otimes W^\vee$ has underlying vector

space equal to $W^\vee$ and H-action defined by

$$h \cdot w^\vee = \delta_{G/H}(h)(hw^\vee), \qquad h \in H,\ w^\vee \in W^\vee.$$

Since V is admissible, the smooth dual $V^\vee$ is irreducible (by Proposition 2.2.17), and so the smooth G-module $\mathrm{Ind}_H^G(\delta_{G/H} \otimes W^\vee)$ is irreducible. By the above, we conclude that $T \subseteq H$, and this completes the proof. □

We next prove that (1) implies (2) in the particular situation where the subalgebra $\mathcal{B}$ has codimension one in $\mathcal{A}$.

Step 2. *Let $\mathcal{J}_0 = \mathcal{J}(\mathcal{B})$ denote the Jacobson radical of $\mathcal{B}$, and suppose that $\dim \mathcal{J}_0 = \dim \mathcal{J} - 1$ where $\mathcal{J} = \mathcal{J}(\mathcal{A})$. Moreover, assume that H contains a diagonal subgroup T of G. Then, the smooth G-module V is admissible.*

Proof. Firstly, we note that $\mathcal{J}^2 \subseteq \mathcal{J}_0$; otherwise, since $\dim \mathcal{J} = \dim \mathcal{J}_0 + 1$, we must have $\mathcal{J}_0 + \mathcal{J}^2 = \mathcal{J}$, and so $\mathcal{J}_0 = \mathcal{J}$ (see [Isa95, Lemma 3.1]). Now, let $P_0 = 1 + \mathcal{J}_0$, and note that P_0 is an ideal subgroup of G with $H = P_0 T$ (because $T \subseteq H$). Let $\mu \in (P_0)^\circ$ be the character of P_0 afforded by the one-dimensional smooth P_0-module $\mathrm{Res}_{P_0}^H(W)$, and consider the irreducible smooth $C_G(\mu)$-module V_μ; notice that

$$V \simeq \text{c-Ind}_{C_G(\mu)}^G(V_\mu).$$

Since H is clearly a subgroup of $C_G(\mu)$, there are isomorphisms

$$V \simeq \text{c-Ind}_H^G(W) \simeq \text{c-Ind}_{C_G(\mu)}^G(\text{c-Ind}_H^{C_G(\mu)}(W));$$

in particular, we conclude that the smooth $C_G(\mu)$-module $\text{c-Ind}_H^{C_G(\mu)}(W)$ is irreducible. Since

$$x\phi = \mu(x)\phi, \qquad x \in P_0,\ \phi \in \text{c-Ind}_H^{C_G(\mu)}(W),$$

it follows from Proposition 3.4.3 that

$$\text{c-Ind}_H^{C_G(\mu)}(W) \simeq V_\mu.$$

Since

$$xv = \mu(x)v, \qquad x \in P_0,\ v \in V_\mu,$$

it follows that $\mathrm{Res}_T^{C_G(\mu)}(V_\mu)$ is irreducible, and hence V_μ is one-dimensional (by Schur's Lemma); in particular, we conclude that $H = C_G(\mu)$ and that $W \simeq V_\mu$.

Since $\dim V \geq 2$, the proof of Theorem 5.2.3 guarantees that there exists an ideal $\mathcal{L}$ of $\mathcal{A}$ satisfying $\mathcal{J}^2 \subseteq \mathcal{L} \subseteq \mathcal{J}$ and $\dim(\mathcal{L}/\mathcal{J}^2) = 1$, and such that

$$V \simeq \text{c-Ind}_{C_G(\lambda)}^G(V_\lambda)$$

for some smooth character $\lambda \in (1+\mathcal{L})^\circ$. Since $\mu([1+\mathcal{J}, 1+\mathcal{J}_0]) \neq 1$ (because otherwise $P = 1+\mathcal{J}$ would be contained in $C_G(\lambda) = H$), we see from the construction that we may choose $\mathcal{L} \subseteq \mathcal{J}_0$ and $\lambda = \mu_{1+\mathcal{L}}$; furthermore, it follows from Proposition 4.1.5 that $\varsigma = \lambda_{1+\mathcal{J}^2}$ is a P-invariant smooth character of $1+\mathcal{J}^2$ such that $C_P(\lambda) = 1+\mathcal{J}_\varsigma$ where

$$\mathcal{J}_\varsigma = \{a \in \mathcal{J} \mid \varsigma([1+a, 1+u]) = 1 \text{ for all } u \in \mathcal{L}\}$$

is an ideal of $\mathcal{A}$ with $\dim \mathcal{J}_\varsigma = \dim \mathcal{J} - 1$ (see also Proposition 4.1.4). Since $\varsigma = \mu_{1+\mathcal{J}^2}$ and since $1+\mathcal{J}_0 \subseteq C_P(\mu)$, we must have $\mathcal{J}_0 \subseteq \mathcal{J}_\varsigma$, and thus $\mathcal{J}_0 = \mathcal{J}_\varsigma$ (because $\dim \mathcal{J}_0 = \dim \mathcal{J} - 1 = \dim \mathcal{J}_\varsigma$). On the other hand, we recall from Proposition 4.1.5 that

$$\lambda^P = \{\lambda' \in (1+\mathcal{L})^\circ \mid (\lambda')_{1+\mathcal{J}^2} = \varsigma\};$$

in particular, λ^P is a closed subset of $(1+\mathcal{L})^\circ$. Since $T \subseteq H = C_G(\mu)$, we conclude that

$$C_G(\lambda) = T(1+\mathcal{J}_0) = H = C_G(\mu),$$

and so $\lambda^G = \lambda^P$ is a closed subset of $(1+\mathcal{L})^\circ$. Finally, we notice that $V_\lambda = V_\mu$ (by the choice of $\lambda = \mu_{1+\mathcal{L}}$) and that V_μ is an admissible smooth $C_G(\mu)$-module (because it is one-dimensional). Therefore, it follows from Theorem 3.4.4 that the smooth G-module

$$V \cong \text{c-Ind}_{C_G(\lambda)}^G(V_\lambda) \cong \text{c-Ind}_{C_G(\mu)}^G(V_\mu)$$

is also admissible, and this completes the proof. □

In the next step we establish that (1) implies (2).

Step 3. *Assume that H contains a diagonal subgroup T of G. Then, the smooth G-module V is admissible.*

Proof. As in the previous proof, we argue by induction on the dimension of $\mathcal{A}$, the result being obvious in the case where $\mathcal{A}$ is one-dimensional; indeed, the result is obvious in the case where V is one-dimensional, which clearly includes the case where the $\Bbbk$-algebra $\mathcal{A}$ is semisimple (because, if this is the case, then $\mathcal{A}$ must be commutative). Therefore, we may assume that $\dim V \geq 2$; in particular, the Jacobson radical $\mathcal{J} = \mathcal{J}(\mathcal{A})$ of $\mathcal{A}$ must be non-zero.

Let $\mathcal{J}(\mathcal{B})$ denote the Jacobson radical of $\mathcal{B}$. If $\mathcal{J}(\mathcal{B})+\mathcal{J}^2 = \mathcal{J}$, then $\mathcal{J}(\mathcal{B}) = \mathcal{J}$ (see [Isa95, Lemma 3.1]), and thus $H = T(1+\mathcal{J}) = G$ and $V = W$ is one-dimensional. It follows that $\mathcal{J}(\mathcal{B}) + \mathcal{J}^2 \subsetneq \mathcal{J}$, and so there exists an ideal $\mathcal{J}'$ of $\mathcal{A}$ such that

$$\mathcal{J}(\mathcal{B}) + \mathcal{J}^2 \subseteq \mathcal{J}' \subsetneq \mathcal{J} \qquad \text{and} \qquad \dim \mathcal{J}' = \dim \mathcal{J} - 1.$$

Let $P' = 1+\mathcal{J}'$, and let $G' = TP'$; notice that P' is an ideal subgroup of G, and that $G' = (\mathcal{A}')^\times$ is the unit group of the subalgebra $\mathcal{A}' = \mathcal{D} \oplus \mathcal{J}'$ of $\mathcal{A}$ where $\mathcal{D}$ is the diagonal subalgebra of $\mathcal{A}$ such that $T = \mathcal{D}^\times$. Since

$$H \subseteq T(1+\mathcal{J}(\mathcal{B})) \subseteq G',$$

it is obvious that

$$V' = \text{c-Ind}_H^{G'}(W)$$

is an irreducible smooth G'-module; indeed, by the transitivity of compact induction, we see that

$$V \simeq \text{c-Ind}_{G'}^{G}(V').$$

Since $\mathcal{A}'$ is a proper subalgebra of $\mathcal{A}$, we know by induction that V' is admissible.

If V' is one-dimensional, then it follows from Step 2 that V is admissible; thus, we may assume that $\dim V' \geq 2$. In this situation, we repeat step-by-step the proof of Theorem 5.2.3 (but applied to the sequence of ideals $\mathcal{J}' \supseteq \mathcal{J}^2 \supseteq \mathcal{J}^3 \supseteq \cdots$ and to the commutator subgroups $[1+\mathcal{J}', 1+\mathcal{J}^m]$ for $m \in \mathbb{N}$) to construct an ideal subgroup $Q = 1+\mathcal{L}$ where $\mathcal{L} \subsetneq \mathcal{J}'$ is an ideal of $\mathcal{A}$ satisfying

$$\mathcal{J}^n \subseteq \mathcal{L} \subseteq \mathcal{J}^{n-1} \qquad \text{and} \qquad \dim(\mathcal{L}/\mathcal{J}^n) = 1$$

for a suitable integer $n \geq 2$, and such that

$$V' \simeq \text{c-Ind}_{C_{G'}(\lambda)}^{G'}(V'_\lambda)$$

for some smooth character $\lambda \in Q^\circ$. Indeed, the proof of Theorem 5.2.3 shows that

$$V \simeq \text{c-Ind}_{C_G(\lambda)}^{G}(V_\lambda);$$

we claim that $V_\lambda = \text{c-Ind}_{C_{G'}(\lambda)}^{C_G(\lambda)}\left(V'_\lambda\right)$. On the one hand, we note that there are isomorphisms

$$V \simeq \text{c-Ind}_{G'}^{G}(V') \simeq \text{c-Ind}_{G'}^{G}\left(\text{c-Ind}_{C_{G'}(\lambda)}^{G'}\left(V'_\lambda\right)\right)$$

$$\simeq \text{c-Ind}_{C_{G'}(\lambda)}^{G}\left(V'_\lambda\right) \simeq \text{c-Ind}_{C_G(\lambda)}^{G}\left(\text{c-Ind}_{C_{G'}(\lambda)}^{C_G(\lambda)}\left(V'_\lambda\right)\right),$$

and hence $\text{c-Ind}_{C_{G'}(\lambda)}^{C_G(\lambda)}\left(V'_\lambda\right)$ is an irreducible smooth $C_G(\lambda)$-module. On the other hand, since

$$x \cdot \phi = \lambda(x)\phi, \qquad x \in Q,\ \phi \in \text{c-Ind}_{C_{G'}(\lambda)}^{C_G(\lambda)}\left(V'_\lambda\right),$$

it follows from Proposition 3.4.3 that

$$\text{c-Ind}_{C_{G'}(\lambda)}^{C_G(\lambda)}\left(V'_\lambda\right) \simeq V_\lambda,$$

which is precisely what we claimed.

Now, we know from Proposition 5.3.6 that $C_{G'}(\lambda)$ is the unit group of some subalgebra $\mathcal{A}'_\lambda$ of $\mathcal{A}'$; moreover, by Theorem 5.2.3, we know that there is a subalgebra $\mathcal{B}'$ of $\mathcal{A}'_\lambda$ such that

$$V'_\lambda \simeq \text{c-Ind}_{H'}^{C_{G'}(\lambda)}(W')$$

where $H' = (\mathcal{B}')^\times$ is the unit group of $\mathcal{B}'$ and W' is a one-dimensional smooth H'-module. [Notice that $H' \subseteq C_{G'}(\lambda)$; however, in the general situation, we are not assuming that $H \subseteq C_{G'}(\lambda)$.] Since V' is admissible, the smooth $C_{G'}(\lambda)$-module is also admissible, and hence H' contains a diagonal subgroup T' of G (by Step 1). Since

$$V_\lambda \simeq \text{c-Ind}_{H'}^{C_G(\lambda)}(W')$$

(by the transitivity of c-induction), we conclude that the smooth $C_G(\lambda)$-module V_λ is admissible (by induction, because $C_G(\lambda)$ is the unit group of a proper subalgebra of $\mathcal{A}$). Finally, we recall from Proposition 4.1.5 that $\varsigma = \lambda_{1+\mathcal{J}^n}$ is a P-invariant smooth character of $1 + \mathcal{J}^n$, and that

$$\lambda^P = \{\lambda' \in Q^\circ \mid \lambda'_{1+\mathcal{J}^n} = \varsigma\};$$

in particular, λ^P is a closed subset of Q°. Since $T' \subseteq H' \subseteq C_G(\lambda)$ and since $G = T'P$ (because T' is a diagonal subgroup of G), it follows that $\lambda^G = \lambda^P$ is a closed subset of Q°, and thus we conclude that the smooth G-module

$$V \simeq \text{c-Ind}_{C_G(\lambda)}^{G}(V_\lambda)$$

is admissible (by Theorem 3.4.4), as required. $\square$

Finally, we prove that (1) implies (3).

Step 4. *Assume that H contains a diagonal subgroup T of G. Then, there is an isomorphism of smooth G-modules $V \simeq \mathrm{Ind}_H^G(W)$.*

Proof. Let $\delta_G \colon G \to \mathbb{R}_+^\times$ and $\delta_H \colon H \to \mathbb{R}_+^\times$ be the modular characters of G and H, respectively; we claim that $(\delta_G)_H = \delta_H$. To see this, we observe that there is a chain of closed subgroups

$$H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$$

such that H_{i-1} is normal in H_i for all $1 \leq i \leq n$; for each $1 \leq i \leq n$, let δ_i denote the modular character of H_i. By [DE14, Corollary 1.5.5(a)], we have

$$(\delta_i)_{H_{i-1}} = \delta_{i-1}, \qquad 1 \leq i \leq n;$$

in particular, we conclude that $(\delta_G)_H = \delta_H$ as claimed. It follows that the map $\delta_{G/H} = (\delta_H)^{-1}(\delta_G)_H$ is identically equal to 1, and thus the Duality Theorem 2.5.3 implies that

$$V^\vee \simeq (\text{c-Ind}_H^G(W))^\vee \simeq \mathrm{Ind}_H^G(W^\vee).$$

On the other hand, since $T \subseteq H$, the smooth G-module V is admissible (by Step 3), and thus its smooth dual $V^\vee$ is also irreducible (by Proposition 2.2.17). Since $\text{c-Ind}_H^G(W^\vee)$ is a submodule of $\mathrm{Ind}_H^G(W^\vee)$, we conclude that

$$V^\vee \simeq \text{c-Ind}_H^G(W^\vee),$$

and thus

$$(V^\vee)^\vee \simeq \left(\text{c-Ind}_H^G(W^\vee)\right)^\vee \simeq \mathrm{Ind}_H^G\left((W^\vee)^\vee\right)$$

(again by the Duality Theorem). Since $(W^\vee)^\vee \simeq W$ (because W is one-dimensional) and since $(V^\vee)^\vee \simeq V$ (by Proposition 2.2.17 because V is admissible), we conclude that

$$V \simeq \mathrm{Ind}_H^G(W),$$

as required. □

The proof of Theorem 2.2.17 is now complete. □

We finish the section with the following result concerning the unitarisability of an arbitrary irreducible smooth G-module.

Theorem 5.4.3. *Let V be an irreducible smooth G-module. Then, there is an isomorphism of G-modules*

$$V \cong V' \otimes V''$$

where V' is a unitarisable irreducible smooth G-module and V'' is a one-dimensional smooth G-module, and where the G-action on $V' \otimes V''$ is given by

$$g(v' \otimes v'') = (gv') \otimes (gv''), \qquad g \in G,\ v' \in V',\ v'' \in V''.$$

Proof. Let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$ such that $V \cong \text{c-Ind}_H^G(W)$ where $H = \mathcal{B}^\times$ is the unit group of $\mathcal{B}$ and W is a one-dimensional smooth H-module; moreover, let $\mu \in H^\circ$ be the smooth character of H afforded by W. Since $\mathcal{B}$ is a split basic $\Bbbk$-algebra, the group H decomposes as the semidirect product $H = SQ$ where S is the unit group of a diagonal algebra of $\mathcal{B}$ and where Q is the ideal subgroup of H which corresponds to the Jacobson radical of $\mathcal{B}$; notice that Q is a normal subgroup of H, and that S is isomorphic to a finite direct product of copies of $\Bbbk^\times$.

Let $\mu^* \in H^\circ$ be defined by

$$\mu^*(sx) = \mu^{-1}(s), \qquad s \in S,\ x \in Q;$$

we note that

$$(\mu^*\mu)(sx) = (\mu^{-1}\mu_S)(s)\mu(x) = \mu(x), \qquad s \in S,\ x \in Q,$$

and thus $\mu^*\mu$ is a unitary character of H (because Q is an ℓ_c-group, and hence μ_Q is a unitary character of Q; see Proposition 2.2.9). Let $W' \cong \mathbb{C}_{\mu^*}$ be a one-dimensional smooth H-module which affords the character μ^*, and consider the tensor product $W' \otimes W$. We note that $W' \otimes W$ is one-dimensional and affords the unitary character $\mu^*\mu \in H^\circ$; hence, the smooth H-module $W' \otimes W$ is unitarisable. As in the proof of Step 4 above, we see that $(\delta_G)_H = \delta_H$ where δ_G and δ_H are the modular characters of G and H, respectively, and thus it follows from Proposition 4.1.7 that the c-induced smooth G-module

$$V' \cong \text{c-Ind}_H^G(W' \otimes W)$$

is also unitarisable.

Finally, as in the proof of Theorem 5.4.2 (Step 1), we see that there exists a subgroup T' of T such that T decomposes as a direct product $T = ST'$, and thus the smooth character $\sigma = (\tau^*)_S = (\tau^{-1})_S \in S^\circ$ can be extended to a smooth character $\sigma' \in T^\circ$ of T; moreover, since G is the semidirect product $G = T \ltimes P$ where $P = 1 + \mathcal{J}(\mathcal{A})$, there is a smooth character $\vartheta \in G^\circ$ such that

$$\vartheta(tx) = \sigma'(t), \qquad t \in T,\ x \in P.$$

Let $W'' \cong \mathbb{C}_\vartheta$ be a one-dimensional smooth G-module which affords the character ϑ, and consider the tensor product $W'' \otimes \text{c-Ind}_H^G(W)$. For every $w'' \in W''$ and every $\phi \in \text{c-Ind}_H^G(W)$,

we define the function $\psi(w'' \otimes \phi) \colon G \to W'' \otimes W$ by the rule

$$\psi(w'' \otimes \phi)(g) = (gw'') \otimes \phi(g)) = \vartheta(g)(w'' \otimes \phi(g)), \qquad g \in G;$$

it is easy to check that the mapping $w'' \otimes \phi \mapsto \psi(w'' \otimes \phi)$ defines an isomorphism of G-modules

$$W'' \otimes \text{c-Ind}_H^G(W) \cong \text{c-Ind}_H^G \left(\text{Res}_H^G(W'') \otimes W \right).$$

Since we clearly have $\text{Res}_H^G(W'') \cong W'$ (as H-modules), we conclude that

$$W'' \otimes \text{c-Ind}_H^G(W) \cong \text{c-Ind}_H^G \left(W' \otimes W \right),$$

and thus

$$V \cong \text{c-Ind}_H^G(W) \cong V'' \otimes \text{c-Ind}_H^G \left(W' \otimes W \right) \cong V' \otimes V''$$

where $V'' = (W'')^\vee \cong \mathbb{C}_{\vartheta^{-1}}$. The result follows. □

5.5 Examples: triangular groups of small size

We conclude this chapter by illustrating our results in the case of triangular groups (of sizes 2 and 3) over a non-Archimedean local field $\Bbbk$. (The case where $n = 4$ is very similar; larger sizes may be studied at least partially, provided that we have "good" information about the irreducible representations of the corresponding unitriangular subgroup which however is known to be quite intractable.)

Example 5.5.1 ($B_2(\Bbbk)$)**.** Let $G = B_2(\Bbbk)$ denote the group of 2×2 uppertriangular invertible matrices over $\Bbbk$, let $P = U_2(\Bbbk)$ be the unitriangular subgroup of G, and let $T = T_2(\Bbbk)$ be the standard diagonal subgroup of G. Let V be an irreducible smooth G-module, and let W be an irreducible quotient of the restriction $\text{Res}_P^G(V)$ of V to P. Since P is abelian (in fact, $P \simeq \Bbbk^+$), W is one-dimensional, and hence W affords a character $\lambda_{1,2}(\alpha)$ for some $\alpha \in \Bbbk$ (see Proposition 3.5.1).

- If $\alpha = 0$, then Lemma 5.3.2 implies that V is one-dimensional, and so it affords a smooth character ϑ of G given by

  $$\vartheta(g) = \vartheta_1(g_{1,1})\vartheta_2(g_{2,2}), \qquad g \in B_2(\Bbbk),$$

  where ϑ_1 and ϑ_2 are smooth characters of the multiplicative group $\Bbbk^\times$.

- If $\alpha \neq 0$, then the stabiliser $H = C_G(\lambda_{1,2}(\alpha))$ of $\lambda_{1,2}(\alpha)$ consists of all matrices $g \in G$ satisfying $g_{1,1} = g_{2,2}$. The G-orbit of $\lambda_{1,2}(\alpha)$ is

$$\lambda_{1,2}(\alpha)^G = \{\lambda_{1,2}(\beta) \mid \beta \in \Bbbk^\times\},$$

and so without loss of generality we may assume that $\alpha = 1$. We can extend $\lambda_{1,2} = \lambda_{1,2}(1)$ to a smooth character $\mu_{1,2}(\vartheta_1)$ of H satisfying

$$\mu_{1,2}(\vartheta_1)(t) = \vartheta_1(t_{1,1}), \qquad t \in T,$$

where ϑ_1 is a smooth character of $\Bbbk^\times$. By Theorem 5.2.3 and its proof, we conclude that

$$V \simeq \text{c-Ind}_H^G(\mathbb{C}_{\mu_{1,2}(\vartheta_1)})$$

for some $\vartheta_1 \in (\Bbbk^\times)^\circ$.

Example 5.5.2 ($B_3(\Bbbk)$)**.** Let $G = B_3(\Bbbk)$ denote the group of 3×3 uppertriangular invertible matrices over $\Bbbk$, let $P = U_3(\Bbbk)$ be the unitriangular subgroup of G, and let $T = T_3(\Bbbk)$ be the standard diagonal subgroup of G. Let V be an irreducible smooth G-module, and let W be an irreducible quotient of the restriction $\mathrm{Res}_P^G(V)$ of V to P. According to Example 4.2.1, we have three different possibilities: W is the trivial P-module, W is a non-trivial one-dimensional P-module, or W is infinite-dimensional.

- If W is the trivial P-module, then Lemma 5.3.2 implies that V is one-dimensional, and so it affords a character ϑ of G given by

$$\vartheta(g) = \vartheta_1(g_{1,1})\vartheta_2(g_{2,2})\vartheta_3(g_{3,3}), \qquad g \in B_3(\Bbbk),$$

where ϑ_1, ϑ_2 and ϑ_3 are smooth characters of $\Bbbk^\times$.

- If W is one-dimensional and affords a non-trivial smooth character λ of P, then we must have $\lambda = \lambda_{1,2}(\alpha_1)\lambda_{2,3}(\alpha_2)$ for some $(\alpha_1, \alpha_2) \in (\Bbbk \times \Bbbk) \setminus \{(0,0\}$.

  - If $\alpha_1 = 0$ (the case $\alpha_2 = 0$ is similar), then the stabiliser $H = C_G(\lambda)$ consists of all matrices $g \in G$ satisfying $g_{2,2} = g_{3,3}$. We can extend λ to a smooth character $\mu(\vartheta_1, \vartheta_2)$ of H satisfying

$$\mu(\vartheta_1, \vartheta_2)(t) = \vartheta_1(t_{1,1})\vartheta_2(t_{2,2}), \qquad t \in T \cap H,$$

where ϑ_1 and ϑ_2 are a smooth characters of $\Bbbk^\times$. Then, by Theorem 5.2.3 and its

proof, we conclude that

$$V \simeq \text{c-Ind}_H^G(\mathbb{C}_{\mu(\vartheta_1,\vartheta_2)})$$

for some $\vartheta_1, \vartheta_2 \in (\Bbbk^\times)^\circ$.

– If $\alpha_1\alpha_2 \neq 0$, then the stabiliser $H = C_G(\lambda)$ consists of all matrices $g \in G$ satisfying $g_{1,1} = g_{2,2} = g_{3,3}$, and we can extend λ to a smooth character $\mu(\vartheta_1)$ of H satisfying

$$\mu(\vartheta_1)(t) = \vartheta_1(t_{1,1}), \qquad t \in T \cap H,$$

where ϑ_1 is a smooth character of $\Bbbk^\times$. Therefore, in this case, we conclude that

$$V \simeq \text{c-Ind}_H^G(\mathbb{C}_{\mu(\vartheta_1)})$$

for some $\vartheta_1 \in (\Bbbk^\times)^\circ$.

- If W has infinite dimension, then we have $W \simeq V_{1,3}(\alpha)$ for some $\alpha \in \Bbbk^\times$ (see Example 4.2.1). Let $Q \subseteq P$ be the subgroup consisting of $g \in P$ such that $g_{1,2} = 0$, then we have that $V_{\lambda_{1,3}(\alpha)} \neq 0$ where $\lambda_{1,3}(\alpha)$ is viewed as a smooth character of Q. The stabiliser $H = C_G(\lambda_{1,3}(\alpha))$ consists of all matrices $g \in G$ satisfying $g_{1,2} = 0$ and $g_{1,1} = g_{3,3}$, and we can extend $\lambda_{1,3}(\alpha)$ to a smooth character $\mu(\vartheta_1, \vartheta_2)$ of H satisfying

$$\mu(\vartheta_1,\vartheta_2)(t) = \vartheta_1(t_{1,1})\vartheta_2(t_{2,2}), \qquad t \in T \cap H,$$

where ϑ_1 and ϑ_2 are a smooth characters of $\Bbbk^\times$. In this case, it follows that

$$V \simeq \text{c-Ind}_H^G(\mathbb{C}_{\mu(\vartheta_1,\vartheta_2)})$$

for some $\vartheta_1, \vartheta_2 \in (\Bbbk^\times)^\circ$.

Chapter 6

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