UNIVERSIDADE DE LISBOA
FACULDADE DE CIÊNCIAS

SUPERCHARACTER THEORIES FOR DISCRETE ALGEBRA GROUPS

"Documento definitivo"

Doutoramento em Matemática
Especialidade em Álgebra, Lógica e Fundamentos

Jocelyn Lochon

Tese orientada por:
Prof. Doutor Carlos Alberto Martins André

Documento especialmente elaborado para a obtenção do grau de doutor

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Abstract

The goal of this thesis is the extension of a construction of a supercharacter theory (first established for finite groups) to the context of infinite countable discrete groups, namely, for amenable countable discrete algebra groups. By an *algebra group* we mean a group of the form $G = 1 + A$ where A is an associative nil algebra over a field $\mathbb{K}$, which generalize the group $U_n(\mathbb{K})$ consisting of all unipotent uppertriangular $n \times n$-matrices over $\mathbb{K}$.

We may think about a supercharacter theory for a finite group as an approximation of the usual irreducible character theory, and it as been proved to provide a rich alternative to deal with the group representation theory. The success of supercharacter theories for finite groups motivates its generalization to infinite countable discrete groups, since there is a well defined character theory for these groups.

We develop a *standard* supercharacter theory that simultaneously extends the standard finite supercharacter theory, and allows us to deal with different types of algebra groups (depending on the the characteristic of $\mathbb{K}$ and on the $\mathbb{K}$-dimension of A) for which "typical" approaches do not work.

Our supercharacter theory translates into an ergodic framework, where supercharacters are defined by certain ergodic measures on the Pontryagin dual group of the abelian additive group A^+. This identification makes possible to present, not only integral expressions (over orbital closures) for supercharacter values, but also canonical unitary representations affording supercharacters.

We pay special attention to algebra groups realized as direct limits of finite algebra groups, which are locally nilpotent groups. For these groups, there is an innermost relationship with the finite standard supercharacter theory. Furthermore, our supercharacter theory establishes a link between the usual methods used when dealing either with nilpotent discrete groups or direct limits of finite groups. This is exemplified with the two infinite unitriangular groups of positive characteristic: the unitriangular group $U_n(\mathbb{F})$ over an algebraic closed field of prime characteristic, and the locally finite unitriangular group $U_\infty(\mathbb{F}_q)$ over a finite field.
Keywords: Unitary representation theory, discrete algebra group, positive definite function, character, supercharacter theory.

2010 AMS classification: Primary: 22D10, 43A35, 22D40; Secondary:43A15, 43A05.

Resumo

Esta tese pretende não só estender uma construção particular de uma teoria de supercaracteres (definida originalmente para grupos finitos) para grupos discretos contáveis, nomeadamente, para grupos álgebra discretos, contáveis e *mediáveis*. Grupos álgebra são grupos da forma $G = 1 + A$, onde A é uma álgebra nil sobre um corpo $\mathbb{K}$, e podem ser entendidos como uma generalizacção do grupo $U_n(\mathbb{K})$ das matrizes $n \times n$ triangulares superiores e unipotentes sobre $\mathbb{K}$.

Em geral, para um grupo discreto contável, o esquema tradicional da classificação de classes de equivalência de representações unitárias por meio da classificação das classes das representações irredutíveis não é possível. Tal deve-se à existência de representações do tipo *II* e *III*, cuja decomposição (a menos de equivalência) em representações irredutíveis não é única. Deste modo, a classificação é tipicamente feita via quasi-equivalência e, neste contexto, os caracteres classificam representações do tipo *I* e *II*. De certo modo, a teoria de caracteres generaliza a teoria de caracteres para grupos finitos onde caracteres indecomponíveis substituem os caracteres irredutíveis.

Para um grupo finito, uma teoria de supercaracteres visa ser uma *aproximação* da teoria de caracteres, já tendo sido provado serem uma alternativa viável para o estudo das suas representações, quando a tabela de caracteres irredutíveis não é conhecida. Este facto motiva, assim, a extensão da noção de uma teoria de supercaracteres a grupos topológicos discretos contáveis.

Para a construção da teoria standard de supercaracteres para um grupo álgebra (amenejável, contável e discreto) $G = 1 + A$, adoptamos um ponto de vista essencialmente ergódico. O grupo $\mathbb{G} = G \times G$ actua em A por multiplicação à esquerda e à direita, induzindo uma acção no grupo dual de Pontryagin A° para a qual as respectivas medidas ergódicas determinam os supercaracteres. A ligação entre supercaracteres e medidas $\mathbb{G}$-ergódicas em A° permite, não só, obter uma expressão integral (sobre fechos orbitais) para os valores dos supercaracteres, mas também, definir, de um modo canónico, representações unitárias que determinam supercaracteres.

Deste modo, a teoria standard de supercracteres possibilita o estudo da teoria da representação de diferentes tipos de grupos álgebra $G = 1 + A$ (dependendo da característica de $\mathbb{K}$ e da $\mathbb{K}$-dimensão de A) para os quais os médodos "típicos" são difíceis (se não, impossíveis) de aplicar.

Damos especial atenção a grupos álgebra obtidos como limite directo de grupos álgebra finitos. Neste caso, a teoria de supercaracteres standard está intimamente ligada às teorias de supercaracteres standard dos grupos finitos envolvidos; de facto, cada supercarácter do grupo considerado pode ser aproximado por uma sequência de supercaracteres dos grupos finitos correspondentes.

Limites directos de grupos álgebra finitos são localmente nilpotentes e a teoria de supercaracteres standard generaliza, simultaneamente, as abordagens usuais no estudo da teoria da representação de grupos nilpotentes discretos e de limites directos de grupos finitos. Tal é exemplificado com os dois tipos de grupos unitriangulares infinitos de característica prima: o grupo unitriangular $U_n(\mathbb{F})$ sobre um corpo algebricamente fechado de característica prima, e o grupo unitriangular localmente finito $U_\infty(\mathbb{F}_q)$ sobre um corpo finito.

Palavras-chave: Teoria de representações unitárias, grupo álgebra, função definida positiva, carácter, teoria de supercaracteres.

List of symbols

A	countable associative nil algebra over a field $\mathbb{K}$
A°	Pontryagin dual of A^+
$\mathscr{A}$	the algebra $\mathbb{K} \oplus A$
$\mathscr{B}(X)$	the Borel σ-algebra of a topological space X
χ^λ	supercharacter associated with $\mathscr{O}^\lambda$
$C(X)$	the complex continuous functions on topological space X
$\mathrm{Cl}(G)$	the set consisting of all conjugacy classes of G
$\mathscr{E}_{\mathscr{K}}(G) = \mathscr{E}$	the set consisting of all standard supercharacters of an algebra group $G = 1 + A$
$\mathrm{Ex}(G)$	the set consisting of all indecomposable characters of an arbitrary group G
$\mathbb{F}_q$	a finite field of prime characteristic p and q elements
$\mathbb{F}$	the algebraic closure of $\mathbb{F}_q$
G	a (topological) group, most often an algebra group $G = 1 + A$
$\Delta\mathbb{G}$	the diagonal group $\{(g,g) \colon g \in G\}$
$\mathbb{G}$	the direct product $G \times G$
$\mathscr{G}$	an amenable discrete countable group
$\mathscr{H}^\mu$	the Hilbert space $L^2(A^\circ, \mu)$
$\mathrm{Irr}(G)$	the irreducible characters of the finite group G
K	a standard superclass of an algebra group
$\mathbb{K}$	a field, most often a countable discrete field
$\mathscr{K}$	the set consisting of all standard superclasses of an algebra group

$\mathscr{M}(X)$	the set consisting of all finite, Borel, complex measures on the topological space X
$\mathscr{M}^+(X)$	the set consisting of all probability Borel measures on the topological space X
$\Omega = \{\mathscr{O}^\lambda : \lambda \in A^\circ\}$	the set consisting of all orbit closures
ω_λ	the $\mathbb{G}$-ergodic measure on A° associated with $\mathscr{O}^\lambda$
O^λ	the closure of $\Delta\mathbb{G}\cdot\lambda$
$\mathscr{O}^\lambda$	the closure of $\mathbb{G}\cdot\lambda$
ψ^λ	the Kirillov function associated with O^λ
$\mathbb{Q}$	the field of rational numbers
ρ	the (left) regular character
$\mathbb{R}$	the field of real numbers
$\mathbb{R}_0^+$	the non-negative real numbers
S_n	the finite symmetric group
S_∞	the infinite symmetric group
$\mathrm{SCl}_{\mathscr{K}}(G)$	the set consisting of all superclass functions on G
$\mathrm{SCl}^+_{\mathscr{K}}(G)$	the set consisting of all superclass characters of G
$U_n(\mathbb{K})$	the finite dimensional unitriangular group over $\mathbb{K}$
$U_\infty(\mathbb{K})$	the locally finite dimensional unitriangular group over $\mathbb{K}$
w_λ	the $\Delta\mathbb{G}$-ergodic measure associated with O^λ

SUPERCHARACTER THEORIES FOR DISCRETE ALGEBRA GROUPS

Contents

Chapter 1

Introduction

This thesis is concerned with the unitary representation theory of infinite amenable countable discrete algebra groups. An *algebra group* over a field $\mathbb{K}$ is a group of the form $G = 1 + A$, where A is an associative nil $\mathbb{K}$-algebra, and the multiplication is defined as

$$(1+a)(1+b) = 1+a+b+ab \qquad a,b \in A.$$

The most prominent examples, and main prototypes, of algebra groups are the unitriangular groups over $\mathbb{K}$:

- the unitriangular group of degree n, $U_n(\mathbb{K}) = 1_n + \mathfrak{u}_n(\mathbb{K})$, where $\mathfrak{u}_n(\mathbb{K})$ is the $\mathbb{K}$-algebra consisting of all $n \times n$ strictly upper triangular matrices over $\mathbb{K}$ and 1_n stands for the $n \times n$ identity matrix;

- the unitriangular group of locally finite degree, $U_\infty(\mathbb{K}) = 1 + \mathfrak{u}_\infty(\mathbb{K})$, where $\mathfrak{u}_\infty(\mathbb{K})$ denotes the subalgebra of all infinite strictly upper triangular matrices over $\mathbb{K}$ consisting of all matrices having only a finite number of non-zero entries, and where 1 is the infinite identity matrix;

The representation theoretical object of our main focus are *characters* (which are defined as positive definite complex functions on the group, constant on the conjugacy classes, and whose value at the identity is 1) as they serve as analogues of the usual trace of finite dimensional representations of compact groups. Moreover, the study of characters is motivated by the fact that, for infinite discrete group, the usual scheme of classification of unitary representations *via* the classification of irreducible representations no longer holds. In following paragraphs, we clarify what we mean by this.

Let G be an arbitrary topological group (that we assume to be locally compact, Hausdorff and second countable) and let $(\pi, \mathscr{H})$ be a unitary representation of G. The von Neumman algebra W generated by the unitary operators $\{\pi(g)\colon g \in G\}$ admits a unique decomposition $W = W_I \oplus W_{II} \oplus W_{III}$ in which W_I, W_{II} and W_{III} are (possibly trivial) von Neumman algebras of types I, II and III, respectively (for all details, we refer to [16, Proposition III.1.4.7]). Such a decomposition translates into a decomposition of π as a direct sum of representations $\pi_I \oplus \pi_{II} \oplus \pi_{III}$; if π coincides with either π_I, π_{II} or π_{III}, the representation is said to be type I, II or III, respectively, and the group G is said to be *tame* if all of its representations are type I, and *wild* otherwise. Any representation admits a decomposition as a direct integral of irreducible representations and, for representations of type I, this decomposition is unique (up to equivalence of representations); however, this is not the case in types II or III (see for example [38, 71]).

As shown by Thoma in [89], infinite countable discrete groups are usually wild: a countable discrete group is tame if and only if it has an abelian subgroup of finite index. For this reason, it is customary to classify representations up to *quasi-equivalence*, where two representations are quasi-equivalent if and only if the corresponding von Neumman algebras are isomorphic; in this case a suitable decomposition theory can be obtained (for more details we refer to [38] and references therein). Thus, the classification of quasi-equivalence classes is a central problem in representation theory.

Representations of type I and II can be (essentially) classified, up to quasi-equivalence, by means of characters which turns them *quasi-invariant objects*. The set of characters, denoted by $\mathrm{Char}(G)$, forms a convex set; we denote by $\mathrm{Ex}(G)$ the set of its indecomposable (or extreme) elements, that is, characters that cannot be decomposed as a non-trivial linear convex combination of two characters. In the case where G is discrete, the set $\mathrm{Char}(G)$ is a a Choquet's simplex when equipped with the pointwise-convergence topology (another result due to Thoma ; see [88]), which means that, for every character $\varphi \in \mathrm{Char}(G)$, there is a *unique* probability measure μ, called the *Choquet measure* of φ, on $\mathrm{Ex}(G)$ such that

$$\varphi(g) = \int_{\mathrm{Ex}(G)} \xi(g)\, d\mu(\xi), \qquad g \in G.$$

In this fashion, the set $\mathrm{Ex}(G)$ can be understood as a *quasi-dual* object (for type I and II representations). We notice that, sometimes, the set of primitive ideals of the C^*-group algebra of G is considered as an alternative for a quasi-dual object; more details can be checked in [35].

In the case where G is a finite group (or more generally, a compact group), a character

is simply the normalized trace of a finite dimensional representation, and an indecomposable character is the normalized trace of an irreducible representation. For this reason, the character theory of infinite discrete groups can be understood as an extension of the classical character theory of finite groups.

The set of indecomposable characters has been characterized for some concrete examples of discrete groups, mostly belonging to two classes: direct limits of finite groups [36, 46, 49, 50, 93], and groups arising from discrete finitely dimensional nilpotent Lie algebras of zero characteristic [31, 52, 54, 86]. In the first case, the theory relies heavily on the knowledge of the character theory of the finite groups that form the direct limit; in fact, the so-called *Kerov-Vershik ergodic method* describes characters as pointwise limits of sequences of characters of the corresponding finite groups (we refer to [25, 90, 93] for more details on the subject). In the second case, the theory is usually carried out *via* an adaption of *Kirillov's orbit method* (Kirillov's original work is concerned with nilpotent real Lie groups [63, 66]) which describes characters in terms of coadjoint orbits on the dual space of the Lie algebra.

The class of discrete algebra groups includes both direct limits of finite groups, such as $U_\infty(\mathbb{F}_q)$ (here, $\mathbb{F}_q$ denotes a finite field with q elements), and nilpotent groups, such as $U_n(\mathbb{K})$ (for any arbitrary countable discrete field); however, the class is much bigger, having groups that are none of the above: $U_\infty(\mathbb{Q})$ is maybe the most blatant example. Consequently, one needs a suitable approach encompassing a broader family of countable discrete algebra groups that should, at least, include all countable discrete untriangular groups.

On the other hand, the set $\mathrm{Ex}(G)$ may be too large, or too difficult to describe, even if G is finite; in fact, the unitriangular group $U_n(\mathbb{F}_q)$ over a finite field is an example of such phenomenon (we refer to [47] where it is explained the difficulties in the classification of conjugacy classes of $U_n(\mathbb{F}_q)$). For this reason, it is of interest to consider a smaller and more manageable class of characters, which however should be rich enough to provide a viable way to deal with representation theoretical problems.

To address this questions, in the case of finite groups, Diaconis and Isaacs in [33] introduced the concept of a supercharacter theory[1] as an *approximation* of the usual irreducible character theory. A *supercharacter theory* of a finite group G is a pair $(\mathscr{K}, \mathscr{E})$ in which $\mathscr{K}$ is a partition of G, and $\mathscr{E}$ is a set of characters satisfying the following conditions:

[1] The terms supercharacter and super-representation often appear in the context of Lie superalgebras; however, we should mention that these are different and unrelated objects.

- $|\mathscr{K}| = |\mathscr{E}|$,
- every character $\chi \in \mathscr{E}$ takes a constant value on each member $K \in \mathscr{K}$, and
- each irreducible character is a constituent of one of the characters $\chi \in \mathscr{E}$.

We refer to the members of $\mathscr{K}$ as *superclasses* and to the characters in $\mathscr{E}$ as *supercharacters* of G. The main idea is that the relationship between the set $\mathrm{Cl}(G)$ of conjugacy classes of G and the set $\mathrm{Irr}(G)$ of irreducible characters should be *mimicked* by $\mathscr{K}$ and $\mathscr{E}$. In particular, given an arbitrary supercharacter theory, superclasses are always union of conjugacy classes, supercharacters are mutually orthogonal, and the regular character is decomposed uniquely as an non negative integer linear combination of *all* supercharacters.

Furthermore, [33] provides the construction of a particular supercharacter theory for finite algebra groups, usually called the *standard supercharacter theory*; in Diaconis and Isaacs own words, it "is a cruder version of the Kirillov orbit method", since in general (a version of) Kirillov's theory, presented in [65], does not provide all irreducible characters.

By now there is a quite extensive bibliography regarding supercharacter theories of finite groups (more on this can be found in the next section), and some connection with other areas of mathematics have been established, namely, with combinatorics [1, 11, 14] (to name a few examples), random walks on finite groups [13], and number theory [28, 41, 42]. Moreover, the finite unitriangular group $U_n(\mathbb{F}_q)$ may be equipped with an "almost-standard" supercharacter theory (which is simply a coarsening of the standard supercharacter theory) in a way that turns it *compatible* with the supercharacter theory for $U_{n-1}(\mathbb{F}_q)$; using this relationship, in [8], André, Gomes and the author were able to define and characterize a supercharacter theory of the locally finite unitriangular group $U_\infty(\mathbb{F}_q)$ (we note that, however, no formal definition of supercharacter theory was given there).

The aim of this work it to extend the cosntruction of [33] to infinite countable discrete algebra groups. However, the class of all countable discrete groups is still too large to fully generalize the standard supercharacter theory given in [33]. For this reason, we must add the additional hypothesis of *amenability*. *Amenable groups* enjoy nice properties (for the general theory of amenable groups we refer to [80]), and amenability is very often assumed in the context of dynamical systems induced by group actions (see [44, 95] for more details). A discrete amenable group is a group that "is not too big" in the sense that it admits a *Følner sequence*: a

Følner sequence for a group G is a family $\{F_n\}_{n\in\mathbb{N}}$ of finite subsets such that, for every $g \in G$,

$$\lim_{n\to\infty} \frac{|F_n \Delta(gF_n)|}{|F_n|} = 0,$$

where Δ denotes the symmetric difference of sets. The class of amenable countable discrete groups contains the finite, nilpotent and locally nilpotent algebra groups, and hence any unitriangular group is an amenable algebra group.

The construction of the supercharacter theory for amenable countable amenable discrete groups presented here is based on the construction given in [33], and can be understood as a generalization of this: its main properties have direct analogues in the finite group scenario. For this reason, we call it the *standard* supercharacter theory (for amenable discrete countable algebra groups).

Besides providing a general framework for the character theory analysis of a family of different algebra groups (depending on the base field $\mathbb{K}$ or on the $\mathbb{K}$-dimension of A), as far as we know, this work is the first general incursion on supercharacter theories of infinite discrete groups, and it extends both the character theory of discrete groups and supercharacter theory of finite groups. On the other hand, our construction, albeit being based on [33], presents an essential change of paradigm: our point of view is ergodic in its nature, which we believe could be fruitful in other scenarios since it allows great generality. We next briefly summarize our method.

Given an arbitrary (either finite, or infinite) amenable countable discrete algebra group $G = 1 + A$, the algebra A serves as a Lie algebra for G and the map $\vartheta : A \to G$, defined by the mapping $a \mapsto 1+a$, is a *surrogate* of the classical exponential map (notice that the exponential map may not be defined in positive characteristic). We consider the direct product $\mathbb{G} = G \times G$ acting on the left of A:

$$\mathbf{k} \cdot a = gah, \qquad \mathbf{k} = (g,h) \in \mathbb{G},\ a \in A.$$

Then, the superclass containing $1+a$, for $a \in A$, is defined to be the subset $1 + \mathbb{G} \cdot a$ of G (in the finite group case, these are precisely the standard superclasses appearing in [33]); a bounded function constant on superclasses is called a *superclass function*, and a character which is constant on the superclasses is called a *superclass character*. The $\mathbb{G}$-action on A induces the contragradient action on A°; throughout the thesis, A° denotes the set consisting of all characters of the abelian additive group A^+ (notice that A° is a compact space when equipped with the pointwise convergence topology). The $\mathbb{G}$-invariant finite complex measures on A° are in one-

to-one correspondence with the superclass functions according to the rule

$$\varphi(g) = \int_{A^\circ} \lambda(g-1)\, d\mu(\lambda), \qquad g \in G.$$

This correspondence between superclass functions and $\mathbb{G}$-invariant measures induces an affine homeomorphism between superclass characters (with the pointwise-convergence topology) and $\mathbb{G}$-invariant probability measures (with the weak*-convergence topology); in this way, the indecomposable superclass characters of G are in one-to-one correspondence with the indecomposable $\mathbb{G}$-invariant measures, which are the $\mathbb{G}$-ergodic measures on A°.

For every $\lambda \in A^\circ$, let $\mathscr{O}^\lambda$ denote the closure of the orbit $\mathbb{G}\cdot\lambda$. The assumption of G being amenable allows us to conclude that every $\mathbb{G}$-ergodic measure is supported on a single orbit closure, and that every orbit closure supports a unique $\mathbb{G}$-ergodic measure. If ω_λ denotes the unique ergodic measure supported on $\mathscr{O}^\lambda$, then the corresponding supercharacter, which we denote by χ^λ, admits an expression analogous to the usual *Kirillov character formula* (see [66] for the case of Lie groups and [31] for rational discrete nilpotent groups):

$$\chi^\lambda(g) = \int_{\mathscr{O}^\lambda} \lambda'(g-1)\, d\omega_\lambda, \qquad g \in G;$$

furthermore, when G is finite, this formula coincides with the normalized version of the supercharacter formula given in [33].

In terms of representations, for every $\mathbb{G}$-ergodic measure ω_λ on A°, there exists a cyclic representation $(\mathscr{T}^\lambda, L^2(A^\circ, \omega_\lambda))$ which affords the supercharacter χ^λ, and this gives us an explicit representation theoretical model of the supercharacters. In this sense, the description of $\mathbb{G}$-ergodic measures (and the corresponding orbit closures) yield a complete characterization of the supercharacter theory.

The main obstruction is that $\mathbb{G}$-ergodic measures might be difficult to understand. However, in the case where the group G is approximately finite (that is, if there is a family $\{G_n\}_{n\in\mathbb{N}}$ of finite algebra subgroups such that $G = \varinjlim_{n\in\mathbb{N}} G_n$), there exists a representation (π^λ, V^λ), which is induced by a one-dimensional representation of some algebra subgroup, that is quasi-equivalent to $(\mathscr{T}^\lambda, L^2(A^\circ, \omega_\lambda))$. Such induced representation provides a measure theoretical free model of supercharacters, making possible a supercharacter analysis without the full knowledge of $\mathbb{G}$-ergodic measures. Moreover, using the fact that supercharacters are induced characters, it is sometimes possible to decide if a supercharacter is associated with a representation of type *I* or *II* (this is accomplished in chapter 6 for the infinite unitriangular groups of positive characteristic).

Furthermore, there is an innermost relationship between the standard supercharacter theory of G and the standard supercharacter theories of all subgroups G_n ($n \in \mathbb{N}$). Using the amenability of $\mathbb{G}$ and the Lindenstrauss pointwise ergodic theorem [69], we can conclude that, for each supercharacter χ of G, there is a sequence $(\chi_n)_{n\in N}$, where χ_n, for $n \in \mathbb{N}$, is a standard (normalized) supercharacter of G_n, that pointwise-converges to χ. This fact yields a natural extension of the so-called *asymptotic representation theory* initiated by Kerov and Vershik in [90, 93]. Furthermore, the Kerov-Vershik ergodic method can be adapted to this setting and, as it will turn out, it is essentially equivalent to our approach. In this way, for approximately finite algebra groups, the standard supercharacter theory constructed in this thesis provides a link between the Kirillov orbit method and the Kerov-Vershik ergodic method.

There is also another supercharacter theory available for the discrete countable unitriangular groups, which is a coarsening of the standard supercharacter theory (and is obtained in a similar way by considering the action on A of a group $\mathbb{B}$ distinct from $\mathbb{G}$). In particular, such a supercharacter theory for the unitriangular group $U_\infty(\mathbb{F}_q)$ allows us to consider the Kingman's graph (originally defined in the context of population genetics; see [62]) in a supercharacter theory setting, providing a representation theoretical framework for some properties of that graph which had been studied in the past (as, for example, in [20, 77]).

It is worth to mention that the main ingredients of our theory is the amenability of the algebra group $G = 1 + A$ and the abelian structure of the discrete group A^+ (namely, the duality between the discreteness of A and the compactness of A°). Thus, the standard supercharacter theory may be considered *verbatim* in a slightly more general setting: one may consider groups of the form $1 + A$ where A is an associative nil algebra over a countable discrete ring, provided that $1 + A$ is amenable (by the way of example, the groups $U_n(\mathbb{Z})$ and $U_\infty(\mathbb{Z})$ which are generalizations of the discrete Heisenberg group) . However, for the sake of simplicity, we will only deal with algebra groups over a field.

Furthermore, it is possible to present a generalization of the definition of a supercharacter theory for an arbitrary countable discrete group G (see Definition 3.1.2), however, it is not clear if the standard supercharacter theory develloped here satisfies this definition.

While this chapter ends with an overview on the most relevant literature (Section 1.1 below), the rest of this thesis is structured as follows.

In Chapter 2, we briefly summarize the main definitions and results of both Measure Theory and Representation Theory which will be needed throughout the thesis.

Chapter 3 begins with the reasoning behind a possible general definition of a supercharacter theory (Definition 3.1.2); in Subsection 3.2 we provide a general method to obtain a supercharacter theory for an arbitrary discrete group G by means of an action (by automorphisms) of an amenable group $\mathscr{G}$.

We then define algebra groups in more detail (Definition 3.3.1) and flash out their main properties and in the Subsection 3.4 presents our construction; we relate superclass functions on G with $\mathbb{G}$-invariant measures on A° (Proposition 3.4.4), and Proposition 3.4.6 establish the correspondence between $\mathbb{G}$-ergodic measures and orbit closures which allows us to derive the orbit supercharacter formula in Proposition 3.4.7.

We end the chapter with Section 3.5, where we provide a brief study of the general properties of supercharacters as a whole topological object.

In Chapter 4, we turn our attention to approximately finite algebra groups of the form $G = \varinjlim_{n\in\mathbb{N}} G_n$. The finite approximation property is established in Proposition 4.1.4, providing a way to approximate supercharacters of G by supercharacters of the finite subgroups G_n ($n \in \mathbb{N}$). A summary of the Kerov-Vershik ergodic method is given in Subsection 4.2.1, and in Subsection 4.2.2 we apply this method in a supercharacter theory context and explain how it is equivalent to our approach. In Section 4.3.3, supercharacters are realized as induced characters using Mackey's *Imprimitivity Theorem*. This *induction property* allows us to factorize supercharacters as the product of some "simpler" supercharacters: in Corollary 4.3.5, we prove that the factorization is essentially unique when the algebra A has finite dimension; on the other hand, in the infinite dimensional case, Corollary 4.3.6 provides an asymptotic (in general non-unique) factorization.

The purpose of Chapter 5 is to explain how other supercharacter theories can be obtained with a similar method used for the standard supercharacter theory; namely, we explore some merits and limitations of Kirillov's orbit method for a general (amenable and countable) algebra group.

Finally, in Chapter 6 we classify two (related) supercharacter theories for the two families of infinite unitriangular groups of positive characteristic: $U_n(\mathbb{F})$ where $\mathbb{F}$ is the algebraic closure of a finite field, and $U_\infty(\mathbb{F}_q)$ where $\mathbb{F}_q$ is a finite field with q elements. The goal of this chapter is twofold: on one hand, it gives concrete examples of supercharacter theories, and on the other it shows, not only the difference between different supercharacter theories on the same group, but also it highlights the contrast of supercharacter theories behavior when groups have structural

differences. For both groups, and both supercharacter theories, we present exact formulas for the supercharacter values (depending on certain statistics defined through some matrix entries), a characterization of supercharacters by their type (using the induction property), and a brief discussion of the regular character.

1.1 A brief literature review

This section is meant to provide a short survey on the existing literature of both character theory of infinite discrete groups and supercharacter theories of finite groups. The main purpose is to give some historical context; while the existing literature is to vast to be all mentioned, in this section we try to our best to provide key references in the development of the theory.

Character theory for infinite discrete groups

The first major breakthrough in the representation theory of infinite countable discrete groups was the work of Thoma in the 1960's [87–89] where he extensively used *characters* as a representation theoretical object. In particular, a full description of the indecomposable characters of the infinite symmetric group S_∞ was achieved in [87], where each indecomposable character is parametrized by two decreasing sequences of real numbers $\alpha_1 \geq \alpha_2 \geq ... \geq 0$ and $\beta_1 \geq \beta_2 \geq ... \geq 0$ satisfying $\sum_{i\in\mathbb{N}}(\alpha_i + \beta_i) \leq 1$.

However, regarding Thoma's results on S_∞, in the words of Borodin and Olshansky ([25, Introduction]), they "looked too unusual and even exotic, and were largely away from the principal routes of representation theory that formed the mainstream in the 1960's and 1970's.". Indeed, the 1960's and 1970's were prolific in respect to the representation theory of Lie groups, namely with the works of Dixmier, Harish-Chandra and Kirillov[2] (to name a few). Nonetheless, with its character approach Thoma planted a seed that soon gave fruits.

Kirillov's orbit method for nilpotent Lie groups was introduced in 1962 by Kirillov in [63], and revealed to be an exceptional source of inspiration (for a wide array of real and p-adic Lie groups, with various degree of success, c.f [53, 66]); in 1977 Howe in [54] pioneered the first generalization of the orbit method for infinite algebraic nilpotent discrete groups over the field of rational numbers $\mathbb{Q}$, an approach also based on characters. However, there is a number of hypothesis imposed on the class of groups in question but, as he mentions in his paper

[2]Curiously enough, in 1965 Kirillov explored the characters of GL_n over a discrete field in [64].

"[...]this again is a fairly special extension of what eventually should be a very far-reaching theory[...]", he was not wrong. In [52] Howe explored the orbit method for nilpotent locally compact nilpotent groups, which can be applied, to some extent and limitations, to discrete nilpotent groups.

In 1994 Corwin and Johnston, based on Howe's ideas (presented in [54] and [52]), were able to drop some hypothesis and in [31] they described a fairly general orbit method for nilpotent groups over $\mathbb{Q}$. Later, in 1997, Bagget, Kaniuth and Moran connected the Kirillov orbit method for discrete nilpotent groups with both characters and primitive ideals on the group C^*-algebra, presenting an analysis of several representation theoretical topics. Such works turned Kirillov's ideas an essential methodology when dealing with discrete nilpotent groups.

Back to Thoma and his characterization of indecomposable characters of S_∞, in the early 80's Kerov and Vershik in [93] revisited the character theory of the infinite symmetric group using a different approach which relies on its direct limit structure. The group S_∞ can be understood as the union of all finite symmetric groups $\bigcup_{n\in\mathbb{N}} S_n$ and any character of S_∞ is fully determined by its restriction to all finite symmetric groups; using known properties of the restriction of irreducible characters from S_{n+1} to S_n they were able to realize the indecomposable characters of S_∞ as the pointwise limit of certain ascending families of normalized irreducible characters of the finite groups. To achieve this, a particular topological space is built from the above mentioned restrictions and, adapting the classical Birkhof ergodic theorem, the result follows (in Chapter 4, Section 4.2.1 we provide some more details). The success of such method, by now dubbed the *Kerov-Vershik ergodic method*, is twofold: in the first place it explains the Thoma parameters of indecomposable characters as the asymptotic limit of the growth of columns and rows of certain Young diagrams with n boxes as n goes to infinity (which has lead to a recent rich combinatorial theory as exemplified in [19, 20, 25, 45, 57, 74]). On the other hand, the Kerov-Vershik method can be applied to any group which can be realized as the direct limit of compact groups (we refer to [25, 90] for more details), giving rise to what is known as *asymptotic representation theory*. Moreover, the ergodic method allows to establish connections with other areas of mathematics, such as Markov processes [23, 24, 60, 91] and point processes [21, 22, 77]. The Kerov-Vershik ergodic method was applied to other groups and corresponding character theories; some examples can be found in [36, 46, 49–51].

It is worth mention that other methods have been applied when dealing with such groups, namely the *semigroup approach* due to Olshanski [75], which somehow generalizes the classical

Gelfand pairs into an infinite dimensional scenario; in this setting, characters can be understood as some sort of spherical functions. Such approach has been fruitful when dealing with the harmonic analysis of the infinite symmetric group as shown in [61, 76].

A recent book by Borodin and Olshanski [25], published in 2016, can be viewed as the main modern reference on the representation theory of the infinite symmetric group, reaching a broad list of topics connected with S_∞, including the link between symmetric functions on commutative variables and character theory, the Kerov-Vershik ergodic method and harmonic analysis.

Supercharacter Theory

Supercharacters first appeared, under the name of *basic characters*, in the early 1990's in André's attempt to deliver a manageable way to deal with the character theory of the finite unitriangular groups $U_n(\mathbb{F}_q)$, [3–5]. His construction worked under the hypothesis that $n < p$ (p being the characteristic of $\mathbb{F}_q$). Yan, in his Ph.D thesis [94] (2002), constructed a family of characters for $U_n(\mathbb{F}_q)$, for arbitrary n and p, using a more elementary approach, under the name of *transition characters*; when $n < p$, Yan's transition characters coincide with André's basic characters. Independently of Yan, André in 2002 was able to drop the assumption $n < p$ and in [6] he presented a general construction of his basic characters.

André's "basic characters" were named "supercharacters" by Carter (in various oral communications), and appeared in the literature for the first time in 2004 [13] where the theory saw its first achievement: supercharacters where used as substitutes of irreducible characters in an harmonic analysis context in order to bound the rate of convergence of a certain random walk on the finite unitriangular group $U_n(\mathbb{F}_q)$. Prompted by this success, in 2008 Diaconis and Isaacs provided in [33] a general definition of a *supercharacter theory* for an arbitrary finite group, and a general construction (based on Yan's work) of a *standard supercharacter theory* for an arbitrary finite algebra groups has been given.

From thereon, supercharacter theories soon began to grow. Supercharacter theories for concrete non-algebra groups were developed in works such as [10] (2008), [9, 34] (2009), [68] (2018) and [84] (2019). General supercharacter theory constructions have been attempted for example in [48] (2012), [12] (2016) and [2] (2017).

An important feature of the supercharacter theories for the unitriangular groups (and related subgroups) are its connections with combinatorics, namely with Hopf algebras [1, 11, 14, 15].

The paper [1] is of significant importance: for each unitriangular group $U_n(\mathbb{F}_q)$, there is a particular supercharacter theory (slightly coarser than the standard), known as the *uncolored supercharacter theory*, whose supercharacters are fully characterized by set partitions of $\{1,...,n\}$. Let $\mathbf{SC}_n$ be the vector space consisting of all complex functions on $U_n(\mathbb{F}_q)$ which are constant on the "uncolored" superclasses; in [1] the vector space $\mathbf{SC} = \bigoplus_{n\in\mathbb{N}} \mathbf{SC}_n$ is equipped with a Hopf algebra structure which is isomorphic to the Hopf algebra of symmetric functions in non-commutative variables. These facts establish a parallel with the irreducible representation theory of the finite symmetric group: the set $\mathrm{Irr}(S_n)$, consisting of irreducible characters of the finite symmetric group S_n, is parametrized by the integer partitions of n and the vector space $\bigoplus_{n\in\mathbb{N}} \langle \mathrm{Irr}(S_n)\rangle_{\mathbb{C}}$ admits a Hopf algebra structure which is isomorphic to the Hopf algebra of symmetric functions in commuting variables (fore more details see, for example, [43]). For this reason, the uncolored supercharacter theory of $U_n(\mathbb{F}_q)$ may not only be considered a *non-commuting* version of the irreducible character theory of S_n, but it also seems to be the "correct" way to address the (super)representation theory of $U_n(\mathbb{F}_q)$.

In the context of asymptotic representation theory, De Stavola [32] (2018) was the first to consider supercharacter-theoretic objects. Let $\mathscr{E}_n$ be the set consisting of all uncolored supercharacters of $U_n(\mathbb{F}_q)$, and recall that the regular character of $U_n(\mathbb{F}_q)$ can be uniquely written as a non-negative integer linear combination of all the supercharacters in $\mathscr{E}_n$. These coefficients define a measure $\mathbf{SPl}_n$ on $\mathscr{E}_n$, called the super-Plancherel measure (in direct analogy to the Plancherel measure on the set of irreducible characters of S_n); inspired in [92], De Stavola analyzed the asymptotic behavior of $\mathbf{SPl}_n$ as n grows to infinity, and presented a *limit shape* for the super-Plancherel measure. However, due to a lack of representation theoretical framework, his result is somehow lacking of meaning in terms of character/supercharacter theory (see Section 6.3 for more details).

On the other hand, also in 2018, André, Gomes and the author, in [8] considered and characterized indecomposable superclass characters of $U_\infty(\mathbb{F}_q)$; this was achieved by adapting in a suitable way the Kerov-Vershik ergodic method, and using strongly the combinatorial relationship between supercharacters all finite unitriangular groups (mostly from [1]). In this fashion, some parallels between the symmetric group (finite and infinite) and the unitriangular group (finite and infinite) were extended, providing a setting where further analogies may be found. Notwithstanding its merits, such approach lacks of generality, being too dependent on explicit properties of the supercharacter theory of $U_n(\mathbb{F}_q)$ so that it justifies the need of a different and

more general approach.

Chapter 2

Preliminaries

In this introductory chapter, we will do a brief incursion on Measure Theory, on the one hand, and on Group Representation Theory, on the other. It is primarily intended to fix the main concepts and terminology used throughout the thesis, and to provide the main background material; references will be provided for most of the relevant details.

In what concerns to Measure Theory, only a fairly amount of knowledge is needed to have a grasp on the approach used in this work; all results to be used are classical. For this reason, standard definitions and results are adapted to the setting of our interest in detriment of a more general setting. While we will try to retain the terminology used by most authors, here and there we will make some simplifications; by the way of example, by a *measure* we will always mean a finite complex-valued measure (because we will not deal with infinite measures and, generally speaking, we will need to consider complex measures). We will not delve too much into Lebesgue integration theory, and we will just explain the main idea.

The key results to have in mind are the Riesz-Markov-Kakutani representation theorem, Radon-Nikodym theorem and the ergodic decomposition of invariant measures (under the action of a countable discrete group), all heavily used in the main body of this work.

As for Group Representation Theory, all groups are assumed to be topological and we will consider only unitary representations. The class of *all* topological groups is too vast to have a reasonable representation theory; however, assuming mild topological conditions, the (classical) representation theory is fairly well developed. In more detail, unless otherwise stated, any topological group will be assumed to be Hausdorff, locally compact and second countable. It is worth to mention that any discrete group is Hausdorff and locally compact; furthermore, it is

second countable if and only if it has a countable number of elements.

In this setting, there is a huge interplay with the theory of C^*-algebras and von Neumman algebras and, although we will not make an explicit use of such connections, some familiarity with the classical concepts is recommended to better understand the quirks and difficulties of some representation theoretical issues arising when dealing with discrete groups. For this reason, since a summary of the theory of C^*-algebras and von Neumman algebras is out of the scope of this thesis, we will only state the main facts which we will be relevant for our purposes (providing references as we go along). The interested reader may consult the classical treaty by Dixmier [35], or the more modern reference by Blackadar [16].

2.1 Measure Theory

In this section, we recall the basic definitions and results about Measure Theory. Our main references are Bogachev's books [17, 18] (and the references therein) since they represent an (essentially) self-contained and extensive exposition of the general theory.

Given an arbitrary set X a family $\mathscr{A}$ of subsets of X is called a *σ-algebra* if it contains X and the empty set $\emptyset$, and if it is closed under (at most countable) unions, intersections and relative complements; a pair $(X,\mathscr{A})$ is called a *measurable space* if $\mathscr{A}$ is a σ-algebra of subsets of X, and we refer to the subsets in $\mathscr{A}$ as the *measurable subsets* (or, whenever necessary, as the *$\mathscr{A}$-measurable subsets* of X).

In what follows, we fix an arbitrary measurable space $(X,\mathscr{A})$.

Definition 2.1.1. A function $\mu : \mathscr{A} \to \mathbb{C}$ is called a *measure* if the following conditions are satisfied:

- $\mu(\emptyset) = 0$;
- For any family of disjoint measurable sets $A_1, A_2, \ldots \in \mathscr{A}$ and any positive integer N

$$\mu\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

(In a more general context, measures may have infinite values; however, such measures will not be considered in the present work. Furthermore, most of the authors assume a measure to have only positive values.)

The measurable space $(X, \mathscr{A})$ will be called a *measure space* if it is equipped with a measure μ; if there is no warn of ambiguity, we will omit the σ-algebra $\mathscr{A}$ from the notation and simply say that μ is a measure on X.

Given a measure μ on X, an important measure theoretical concept is the notion of μ-*almost-everywhere*: a property on X is said to be true μ-*almost-everywhere* (abbreviated μ-a.e.) if the subset where the property does not hold is measurable and has zero measure.

If X is a topological space, the smallest σ-algebra which contains all the open sets is called the *Borel σ-algebra* of X, and its measurable sets are called the *Borel subsets* of X; we denote by $\mathscr{B}(X)$ the set consisting of all Borel subsets of X. Notice that, by the definition of a σ-algebra, every closed set is measurable; indeed, the smallest σ-algebra containing all the closed sets coincides with the Borel σ-algebra. Furthermore, given a measure μ on X, its *support*, which we denote by supp(μ), is defined to be the set consisting of all points $x \in X$ such that every open neighbourhood of x has positive measure; equivalently, supp(μ) is the smallest closed set C such that $\mu(X \setminus C) = 0$.

A measure μ on X has different names and properties depending on its range:

- if $\mu(A) \in \mathbb{R}$ for all $A \in \mathscr{A}$, then μ is said to be a *signed measure*;
- if $\mu(A) \in \mathbb{R}_0^+$, then μ is said to be a *positive measure*;
- if μ is a positive measure and $\mu(X) = 1$, then μ is said to be a *probability measure*.

If μ is a complex-valued measure, then it is clear that μ admits a unique decomposition $\mu = \mu_1 + i\mu_2$ where i stands for the imaginary unit and μ_1, μ_2 are signed measures. On the other hand, if μ is a signed measure, then μ can be decomposed as a difference of two positive measures $\mu = \mu^+ - \mu^-$ where, for every measurable set $A \in \mathscr{A}$,

$$\mu^+(A) = \sup\{\mu(B) : \ B \in \mathscr{A},\ B \subseteq A\} \quad \text{and} \quad \mu^-(A) = \sup\{-\mu(B) : \ B \in \mathscr{A},\ B \subseteq A\};$$

furthermore, there are disjoint measurable sets $X^+, X^- \in \mathscr{A}$ such that $X = X^+ \cup X^-$ and $\mu^+(X^-) = \mu^-(X^+) = 0$ ([17, Theorem 3.11], its corollary and following remarks). The decomposition $\mu = \mu^+ - \mu^-$ is known as the *Hann-Jordan decomposition of* μ.

Definition 2.1.2. The *total variation* of a measure μ on X is a positive measure on X, which we denote by $|\mu|$ and define as follows: for every measurable set $A \in \mathscr{A}$,

$$|\mu|(A) = \sup_P \{ \sum_{B \in P} |\mu(B)| \},$$

where the supremum is taken over all measurable partitions P of A (that is, partitions of X whose parts are measurable sets). The value $\|\mu\| = |\mu|(X)$ is called the *total variation norm* of μ.

Notice that, if μ is a signed measure, then $\|\mu\| = \mu^+(X) + \mu^-(X)$ (see [17, Definition 3.1.4] and ensuing remarks); on the other hand, if μ is positive, then $|\mu| = \mu$.

Definition 2.1.3. If μ and ν are two positive measures on X, we say that:

- μ is *absolutely continuous* with respect to ν, and write $\mu \ll \nu$, if for any every measurable set A, $\nu(A) = 0 \Rightarrow \mu(A) = 0$;
- μ and ν are *equivalent* if $\mu \ll \nu \ll \mu$;
- μ and ν are *orthogonal*, or *singular* (with respect to one another), and write $\mu \perp \nu$, if there are disjoint measurable sets A and B such that $A \cup B = X$ and $\mu(A) = \nu(B) = 0$.

If $(Y, \mathscr{B})$ is another measurable space, then a function $f : X \to Y$ is said to be *measurable* if $f^{-1}(B) \in \mathscr{A}$ for all $B \in \mathscr{B}$. In particular, if X and Y are topological spaces, each equipped with its own Borel σ-algebra, then every continuous function $f : X \to Y$ is measurable.

For every measurable set $A \in \mathscr{A}$, we denote by $\mathbb{I}_A$ be the characteristic function of A; hence $\mathbb{I}_A(x) = 1$ if $x \in A$, and $\mathbb{I}_A(x) = 0$ if $x \in X \setminus A$. If $\mathbb{C}$ is considered with its Borel σ-algebra, then $\mathbb{I}_A$ is clearly a measurable function; we say that a function $f_0 : X \to \mathbb{C}$ is *simple* if there are a finite number of measurable sets $A_1, ..., A_n$ and complex numbers $\alpha_1, ..., \alpha_n$ such that

$$f_0 = \sum_{i=1}^{n} \alpha_i \mathbb{I}_{A_i},$$

and we define the *integral of f_0 with respect to μ*, which we denote by $\int_X f_0(x)\, d\mu(x)$, to be the sum

$$\int_X f_0(x)\, d\mu(x) = \sum_{i=1}^{n} \alpha_i \mu(A_i);$$

if there is no warn of ambiguity, we simplify the notation and write $\int_X f_0\, d\mu$ instead of $\int_X f_0(x)\, d\mu(x)$.

A sequence $(f_n)_{n \in \mathbb{N}}$ of simple functions is said to be *fundamental in the mean* if, for every $\varepsilon > 0$, there is an order n such that

$$\int_X |f_i - f_j|\, d\mu < \varepsilon, \qquad \text{for all } i, j \geq n.$$

For every measurable function $f : X \to \mathbb{C}$, there is a family $(f_n)_{n \in \mathbb{N}}$ of simple functions that converges pointwise to f ([17, Lemma 2.1.8]). We say that f is *μ-integrable* if the sequence

$(f_n)_{n\in\mathbb{N}}$ is fundamental in the mean; if this is the case, then we define the *integral* of f with respect to μ as

$$\int_X f\,d\mu = \lim_{n\to\infty}\int_X f_n\,d\mu.$$

As shown in [17, Lemma 2.4.2], this limit does not depend on the choice of the sequence fundamental in the mean, and [17, Lemma 2.3.4] ensures it is indeed finite.

As it turns out, absolutely continuous measures are related by a measurable function.

Theorem 2.1.4 (Radon-Nikodym theorem)**.** *If μ and ν are positive measures on X, then $\mu \ll \nu$ if and only if there is a unique (up to a set of measure zero) ν-integrable function $f : X \to \mathbb{R}$ such that, for every measurable set A,*

$$\mu(A) = \int_A f\,d\nu.$$

(For a proof, see [17, Theorem 3.2.2].) In the above notation, the function f is usually denoted by $\frac{d\mu}{d\nu}$, and is called the *Radon-Nikodym derivative*. The uniqueness of the Radon-Nikodym derivative ensures that, if μ and ν are two equivalent measures, then the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$ is invertible almost everywhere and that

$$\frac{d\nu}{d\mu} = \left(\frac{d\mu}{d\nu}\right)^{-1}.$$

An important scenario occurs when X is a topological compact Hausdorff space equipped with its Borel σ-algebra; we denote by $C(X)$ the set of all continuous functions equipped with the usual *uniform norm* $\|.\|_\infty$: for every $f \in C(X)$, we define

$$\|f\|_\infty = \sup_{x\in X}|f(x)|.$$

The uniform norm induces a topology on $C(X)$, and we denote by $C(X)^*$ the corresponding topological linear dual, that is, the set consisting of all continuous linear functionals on $C(X)$. Recall that a linear functional $\varphi \in C(X)^*$ is continuous if and only if it is bounded, that is, there is a constant $c > 0$ such that $|\varphi(f)| \le c\,\|f\|_\infty$, for all $f \in C(X)$; this fact allow us to consider the *operator norm* $\|.\|_{\mathrm{op}}$ on $C(X)^*$ which may be defined in two equivalent ways: for every $\varphi \in C(X)^*$

$$\|\varphi\|_{\mathrm{op}} = \inf\{c > 0\colon\ |\varphi(f)| \le c\,\|f\|_\infty\} = \sup\{|\varphi(f)|\colon\ \|f\|_\infty = 1\},$$

for the equivalence of the definitions we refer to [29]. The following result will be heavily used throughout the thesis (a proof can be found in [82, Theorem 6.19]); it relates linear functionals in $C(X)^*$ with *regular measures*: a measure μ is said to be *regular* if, for every Borel set $A \in \mathscr{A}$,

$$|\mu|(A) = \sup\{|\mu|(F) \colon F \subseteq A,\ F \text{ is closed}\} = \inf\{|\mu|(G) \colon A \subseteq G,\ G \text{ is open}\}.$$

Theorem 2.1.5 (Riesz-Markov-Kakutani representation theorem)**.** *If X is a topological compact Hausdorff space, then there is a bijection between regular measures on X and linear functionals in $C(X)^*$ given by:*

- *If μ is a regular Borel measure on X, then the mapping $f \mapsto \int_X f\, d\mu$ (for $f \in C(X)$) defines a linear function $\varphi(\mu) \in C(X)^*$;*
- *For every $\varphi \in C(X)^*$, there exists a unique regular Borel measure μ on X such that*

$$\varphi(f) = \int_X f\, d\mu, \qquad f \in C(X).$$

  (hence $\varphi = \varphi(\mu)$).

Furthermore, for every regular Borel measure μ on X,

$$\|\varphi(\mu)\|_{\mathrm{op}} = \|\mu\| = |\mu|(X).$$

We observe that, in the case where X is metric space, every Borel measure is regular; for a proof see [40, Theorem 7.17].

A paramount concept in what follows is that of *weak*-convergence* of measures: we say that a sequence of Borel measures $(\mu_n)_{n\in\mathbb{N}}$ on a topological space X *weak*-converges* to a Borel measure μ if

$$\lim_{n\to\infty} \int_X f\, d\mu_n = \int_X f\, d\mu, \qquad f \in C(X).$$

Notice that the convergence with respect to the total-variation norm is stronger than the weak*-convergence, in the sense that the former implies the latter. The following simple example shows that the topologies are in fact different: let $(\mu_n)_{n\in\mathbb{N}}$ be the sequence of measures on the unit interval $[0,1]$ defined by

$$\mu_n = \sum_{i=0}^{n-1} (-1)^i \delta_{i/n}, \qquad n \in \mathbb{N},$$

where $\delta_{i/n}$ denotes the Dirac measure on i/n, that is, $\delta_{i/n}(X) = 1$ if $i/n \in X$, and $\delta_{i/n}(X) = 0$ if $i/n \notin X$; then, $\mu_n \to 0$ in the weak*-topology while $\|\mu_n\| = 1$ for all $n \in \mathbb{N}$.

Whenever X is a topological space, we denote $\mathscr{M}(X)$ and $\mathscr{M}^+(X)$ the set of regular Borel measures and of regular probability Borel measures on X, respectively. The set $\mathscr{M}^+(X)$ lies inside the unit ball of the topological linear dual $C(X)^*$ of $C(X)$, and thus, when equipped with the weak*-topology, it is compact and metrizable (see [39, Proposition 3.101]).

Let X be a compact metric space, and let $\mathscr{G}$ a countable discrete group acting on the left of X; by a *left action* of $\mathscr{G}$ on X we mean a function $\mathscr{G} \times X \to X$, $(g,x) \mapsto g \cdot x$, satisfying

$$(gh) \cdot x = g \cdot (h \cdot x), \qquad g,h \in \mathscr{G},\ x \in X.$$

A Borel measure μ on X is said to be $\mathscr{G}$-invariant if

$$\mu(g \cdot A) = \mu(A), \qquad g \in \mathscr{G},\ A \in \mathscr{B}(X).$$

A $\mathscr{G}$-invariant probability measure μ on X is said to be *$\mathscr{G}$-ergodic* if, for every $\mathscr{G}$-invariant Borel set A, either $\mu(A) = 0$, or $\mu(A) = 1$; equivalently, a function defined on X is $\mathscr{G}$-invariant if and only if it it is constant μ-almost everywhere.

The proposition below allows us to decompose any $\mathscr{G}$-invariant measure in terms of $\mathscr{G}$-ergodic ones. We recall that in an arbitrary convex set V, an *indecomposable (or extreme) element* is one that cannot be written as a non-trivial linear convex combination of two other elements. If V is a compact metrizable convex subspace of a locally convex space E, then Choquet's theorem (see [79, Section 3] for a proof) states that, for every $v_0 \in V$, there is a probability measure μ on V supported on the indecomposable elements of V such that

$$v_0 = \int_V v\, d\mu;$$

we say that μ is the *representing measure* of v_0. The set V is said to be a *Choquet simplex* if for every element of V the corresponding representing measure is *unique*.

Proposition 2.1.6 (Ergodic decomposition)**.** *Let X be a compact metric space, and let $\mathscr{G}$ be countable group acting on X. Then, the set of $\mathscr{G}$-invariant measures is a Choquet simplex whose indecomposable elements are the ergodic measures. In other words, if* $\mathrm{Erg}_{\mathscr{G}}(X)$ *denotes the set consisting of all $\mathscr{G}$-ergodic measures on X, then for every $\mathscr{G}$-invariant probability measure μ there is a unique probability Borel measure μ^* on* $\mathrm{Erg}_{\mathscr{G}}(X)$ *such that*

$$\mu(B) = \int_{\mathrm{Erg}_{\mathscr{G}}(X)} \nu(B)\, d\mu^*(\nu) \qquad \text{and} \qquad \int_X f\, d\mu = \int_{\mathrm{Erg}_{\mathscr{G}}(X)} \left(\int_X f\, d\nu \right) d\mu^*(\nu)$$

for all $A \in \mathscr{B}(X)$ and all $f \in C(X)$.

Such an integral decomposition of any $\mathscr{G}$-invariant measure is usually referred to as the $\mathscr{G}$-*ergodic decomposition*. A proof can be found in [79, Section 12]; we should mention that the theorem there is stated in terms of Baire measures; however, on a metric space, Baire measures and Borel measures coincide (see [18, Corollary 6.3.5]).

2.2 Representation Theory

A *topological group* G is a group equipped with a topology for which the functions $G \times G \to G$, $(g,h) \mapsto gh$, and $G \to G$, $g \mapsto g^{-1}$, are continuous. In what follows, unless otherwise stated, any group is assumed to be locally compact and second countable; we also consider G equipped with its Borel σ-algebra. A (Borel) measure μ on G is said to be left *G-invariant* (or *invariant under left translations*) if

$$\mu(gB) = \mu(B), \qquad g \in G,\ B \in \mathscr{B}(G).$$

An important feature of topological groups is the existence of a G-invariant measure.

Theorem 2.2.1 (Haar)**.** *Every topological group G admits a left G-invariant measure η. Furthermore, the measure η is essentially unique, in the sense that, if η' is any other G-invariant measure, then there is a positive constant c such that*

$$\eta(B) = c\eta'(B), \qquad B \in \mathscr{B}(G).$$

If G is a compact topological group, then there is a unique left G-invariant probability measure.

For a proof we refer to [73, Chapter II, section 4, Theorem 1]; as it is customary, we refer to a left G-invariant measure on G as a *left Haar measure* on G. We now fix a left Haar measure η on G, and for every $g \in G$ we define the measure η_g by

$$\eta_g(B) = \eta(Bg), \qquad B \in \mathscr{B}(G).$$

The measure η_g (for $g \in G$) is also a left Haar measure, and thus there exists a constant $\Delta(g)$ such that $\Delta(g)\eta = \eta_g$; the mapping $g \mapsto \Delta(G)$ defines a function $\Delta\colon\ G \to \mathbb{R}^+$ which is called the *modular function* of G. The group G is said to be *unimodular* if $\Delta(g) = 1$ for all $g \in G$. We observe that, when the group G is countable and discrete, the *counting* measure d on G (that is, $d(g) = 1$ for all $g \in G$) is a left and right Haar measure, thus, countable discrete groups are unimodular.

If G is a topological group, then by a *unitary representation* of G we mean a pair $(\pi, \mathscr{H})$ where $\mathscr{H}$ is a separable Hilbert space and $\pi : G \to \mathrm{U}(\mathscr{H})$ is a group homomorphism from G to the group $\mathrm{U}(\mathscr{H})$ consisting of all bounded linear unitary operators on $\mathscr{H}$, such that for every $v \in \mathscr{H}$ the mapping $g \mapsto \pi(g)v$ defines a norm continuous map $G \to \mathscr{H}$. Except when explicitly stated, all representations are assumed to be unitary; moreover, when the context is clear, we sometimes omit the space $\mathscr{H}$ and refer simply to the representation π.

Example 2.2.2. Let G be a topological group, fix a left Haar measure η, and consider the Hilbert space $L^2(G,\eta)$. For every $f \in L^2(G,\eta)$ and every $g \in G$, let $\pi(g) : L^2(G,\eta) \to L^2(G,\eta)$ be the map defined by

$$\pi(g)f(h) = f(g^{-1}h), \qquad h \in G.$$

Then, the pair $(\pi, L^2(G,\eta))$ is a unitary representation to which we refer to as the *(left) regular representation* of G.

If $(\pi, \mathscr{H})$ is a representation of G and $\mathscr{H}'$ is a $\pi(G)$-invariant closed subspace of $\mathscr{H}$, then π defines naturally a group homomorphism $\pi : G \to \mathrm{U}(\mathscr{H}')$, so that the pair $(\pi, \mathscr{H}')$ becomes a representation of G; in this situation, we refer to $(\pi, \mathscr{H}')$ as a *subrepresentation* of $(\pi, \mathscr{H})$. As it is usual, a representation $(\pi, \mathscr{H})$ is said to be *irreducible* if it does not admit non-trivial subrepresentations; in other words, if 0 and $\mathscr{H}$ are the only closed $\pi(G)$-invariant subspaces of $\mathscr{H}$.

Given an arbitrary representation $(\pi, \mathscr{H})$ of G, we may consider the von Neumman algebra W^π generated by the set $\{\pi(g) : \; g \in G\}$. The von Neumman algebra W^π decomposes uniquely as a direct sum of von Neumman algebras $W^\pi_I \oplus W^\pi_{II} \oplus W^\pi_{III}$, where W^π_I, W^π_{II} and W^π_{III} are, possibly trivial, von Neumman algebras of types *I*, *II* or *III* (we follow the classical Murray-von Neumman terminology; for more details see [16, III.1.4.7]); accordingly, we say that the representation $(\pi, \mathscr{H})$ is of type *I*, *II*, *III*, or mixture of types, depending on the decomposition of W^π.

The group G is said to be *tame* if every representation is type *I*, and *wild* otherwise. This terminology is justified by the fact that representations of type *II* and *III* exhibit a pathological behavior: if π is a type *II* or *III* representation, then every subrepresentation is a multiple of a proper subrepresentation (see [16, III.5.1.9]), and therefore they do not admit irreducible subrepresentations.

There are various classes which are known to be tame: for example, abelian groups, compact

groups, connected semisimple or nilpotent Lie groups and linear algebraic groups are tame (see [67, Chapter A Short Historical Sketch and a Guide to the Literature]). On the other hand, infinite discrete groups tend to be wild; in fact, it was shown by Thoma in [89] that a countable infinite discrete group is tame if and only if it has a commutative subgroup of finite index (in this fashion, a large family of interesting infinite discrete groups are excluded, as it is the case of the infinite symmetric group $S_\infty = \bigcup_{n\in\mathbb{N}} S_n$ consisting on all finite permutations of the natural numbers $\mathbb{N}$).

An *intertwining operator* between two representations $(\pi_1,\mathscr{H}_1)$ and $(\pi_2,\mathscr{H}_2)$ is a continuous unitary linear operator $\Psi:\mathscr{H}_1\to\mathscr{H}_2$ which commutes with the group action, that is,

$$\Psi\circ\pi_1(g)=\pi_2(g)\circ\Psi,\qquad g\in G.$$

We denote $\mathrm{Hom}_G(\pi_1,\pi_2)$ the vector space consisting of all intertwining operators between π_1 and π_2, and say that π_1 and π_2 are *disjoint* if $\mathrm{Hom}_G(\pi_1,\pi_2)=\emptyset$ and *equivalent*, denoted by $\pi_1\simeq\pi_2$, if there is an invertible element in $\mathrm{Hom}_G(\pi_1,\pi_2)$. On the other hand, π_1 and π_2 are said to be *quasi-equivalent*, and write $\pi_1\approx\pi_2$, if no subrepresentation of π_1 is disjoint from π_2, and conversely no subrepresentation of π_2 is disjoint from π_1; equivalently, $\pi_1\approx\pi_2$ if and only if the von Neumman algebras W^{π_1} and W^{π_2} are isomorphic (for a proof of the two equivalences we refer to [35, proposition 5.3.1]).

We note that, quasi-equivalence preserve the representation type (if $\pi_1\approx\pi_2$, then π_1 is of type *I*, *II*, *III*, or a mixture of types, if and only of π_2 is type *I*, *II*, *III*, or a mixture of types); furthermore, two irreducible unitary representations are quasi-equivalent if and only if they are equivalent. (Despite its name, the quasi-equivalent relation is indeed a true equivalence relation.)

Given two representations $(\pi_1,\mathscr{H}_1)$ and $(\pi_2,\mathscr{H}_2)$ of G, their *sum* $(\pi_1,\mathscr{H}_1)\oplus(\pi_2,\mathscr{H}_2)$ is defined in the natural way as the operator $\pi_1\oplus\pi_2$ acting on the direct sum $\mathscr{H}_1\oplus\mathscr{H}_2$ of Hilbert spaces. The tensor product of unitary representations can also be defined. Given two Hilbert spaces H_1 and $\mathscr{H}_2$ equipped with norms p_1 and p_2, respectively, we consider the usual tensor product $\mathscr{H}_1\otimes\mathscr{H}_2$ of vector spaces, and define a norm p on $\mathscr{H}_1\otimes\mathscr{H}_2$ *via*

$$p(v_1\otimes v_2)=p_1(v_1)p_2(v_2),\qquad v_1\in\mathscr{H}_1,\ v_2\in\mathscr{H}_2;$$

the completion of $\mathscr{H}_1\otimes\mathscr{H}_2$ with respect to p, denoted by $\mathscr{H}_1\mathbin{\overline{\otimes}}\mathscr{H}_2$, is a Hilbert space. Now, if $(\pi_1,\mathscr{H}_1)$ and $(\pi_2,\mathscr{H}_2)$ are two unitary representations of G, then for every $g\in G$ the map

$\pi_1(g) \otimes \pi_2(g)$ defined on pure tensors as

$$\pi_1(g) \otimes \pi_2(g)\big(v_1 \otimes v_2\big) = \pi_1(g)v_1 \otimes \pi_2(g)v_2, \qquad v_1 \in \mathscr{H}_1,\ v_2 \in \mathscr{H}_2,$$

extends to a unitary operator of $\mathscr{H}_1 \mathbin{\overline{\otimes}} \mathscr{H}_2$, and the pair $(\pi \otimes \pi_2, \mathscr{H}_1 \mathbin{\overline{\otimes}} \mathscr{H}_2)$ is a unitary representation of G.

Both the sum and product of any finite number of unitary representations is well defined; furthermore, the notion of sum of representations can be extended in order to define a *continuous sum* of an infinite (possibly uncountable) number of representations. Let $(X, \mathscr{A}, \mu)$ be an arbitrary measure space, and let $\{(\pi_x, \mathscr{H}_x);\ x \in X\}$ be a family of unitary representations of G. We recall the definition of the *direct integral of Hilbert spaces*. Let Γ be a set consisting of vector functions of $f : X \to \prod_{x \in X} \mathscr{H}_x$ (that is, a function such that $f(x) \in \mathscr{H}_x$ for all $x \in X$) satisfying:

- For every $f_1, f_2 \in \Gamma$, the mapping $x \mapsto \langle f_1(x) \mid f_2(x)\rangle$ defines a measurable function;
- For every $x \in X$, the $\mathbb{C}$-linear span of the set $\{f(x) \colon f \in \Gamma\}$ is dense in $\mathscr{H}_x$;

A vector function $f : X \to \prod_{x \in X} \mathscr{H}_x$ is said to be *measurable* if for every $f' \in \Gamma$ the mapping $x \mapsto \langle f(x) | f'(x)\rangle$ defines a measurable function. If we denote by $\mathscr{H}^0$ the $\mathbb{C}$-linear span of such functions, then the *direct integral*

$$\mathbf{H}^{\mu} = \int_X^{\oplus} \mathscr{H}_x \, d\mu$$

is defined to be the Hilbert space generated by $\mathscr{H}^0$ endowed with the Hermitian product defined by

$$\langle f_1 | f_2 \rangle = \int_X \langle f_1(x) | f_2(x) \rangle \, d\mu(x), \qquad f_1, f_2 \in \mathbf{H}^{\mu}.$$

Accordingly, for every $g \in G$ we define the map $\boldsymbol{\pi}^{\mu}(g) : \mathbf{H}^{\mu} \to \mathbf{H}^{\mu}$ by

$$(\boldsymbol{\pi}^{\mu}(g) f)(y) = \pi_y(g) f(y), \qquad f \in \mathbf{H}^{\mu},\ y \in X;$$

we sometimes write $\boldsymbol{\pi}^{\mu} = \int_{x \in X}^{\otimes} \pi_x \, d\mu$. Then, $(\boldsymbol{\pi}, \mathbf{H}^{\mu})$ is a unitary representation of G to which we refer as the *direct integral of the representations* $(\pi_x, \mathscr{H}_x)$, $x \in X$; for more details, we refer to [35, 67].

Every unitary representation of G admits a decomposition as a direct integral of irreducible unitary representations (see [67, Chapter 8, Corollary to Theorem 2]), and we may think of the set of equivalence classes of representations as a dual space (by a dual space we mean an object whose elements are enough to describe all representations); however, uniqueness (up to

equivalent representations) is in general lacking (see [71, Theorem 11] where it is presented an example of a Type *II* unitary representation which does not admit uniqueness). In this fashion, irreducible representations are not reasonable invariants (as they lead to a poor decomposition theory), and thus we need to consider a different dual space capable of a unique decomposition.

In [38], a general decomposition theory is achieved using *factor representations*: a factor representation $(\pi, \mathscr{H})$ is one whose corresponding von Neumman algebra W^π is a factor, equivalently, any subrepresentation is quasi-equivalent to $(\pi, \mathscr{H})$ ([35, Proposition 5.2.5]); however, from the representation theoretical point of view, type *III* representations are, in a sense, too large and ill behaved. In particular, representations of type *III* do not admit any kind of *trace* (nevertheless, with the so called Tomita-Takesaki theory, one can understand the nature of type *III* von Neumann algebras). On the other hand, representations of type *I* and *II* admit a unique trace (up to quasi-equivalence), and this allows us to recover much of the character theory of finite (or, more generally, compact) groups.

Definition 2.2.3. A continuous function $\varphi : G \to \mathbb{C}$ is called a *character* of G if it satisfies the following properties:

- φ is normalized, that is, $\varphi(1) = 1$;
- φ is central, that is, $\varphi(h^{-1}gh) = \varphi(g)$ for all $g, h \in G$;
- φ is positive definite, that is, for all $n \in \mathbb{N}$, all $z_1, \dots, z_n \in \mathbb{C}$ and all $g_1, \dots, g_n \in G$

$$\sum_{i,j=1}^{n} z_i \overline{z_j} \varphi(g_i g_j^{-1}) \geq 0.$$

We denote by $\mathrm{Char}(G)$ the set consisting of all characters of G, and equip it with the topology of uniform convergence on compact sets, so that $\mathrm{Char}(G)$ becomes a compact topological space (see [35, Proposition 17.3.5]). It is clear that $\mathrm{Char}(G)$ is a convex set, and thus we may consider the subset $\mathrm{Ex}(G)$ consisting of all *indecomposable* of G; we recall that, in any convex set, an element is called indecomposable (or, extreme) if it cannot be written as a non-trivial linear convex combination of any other two elements. (Some authors refer to characters as *Thoma characters* while others (such as Dixmier) reserve the term character to denominate an indecomposable character. However, we prefer to use the definition above in order to maintain the similarity with the classical character theory of finite groups.)

Let φ be a character of a group G. Since φ is positive definite, for every $g \in G$ the matrix

$$\begin{bmatrix} \varphi(1) & \varphi(g) \\ \varphi(g^{-1}) & \varphi(1) \end{bmatrix}$$

must be positive and hermitian, and this implies that

$$|\varphi(g)| \leq \varphi(1) = 1 \qquad \text{and} \qquad \varphi(g^{-1}) = \overline{\varphi(g)}.$$

Let $(\pi, \mathscr{H})$ be a unitary representation of G, and let $v \in \mathscr{H}$ a normalized vector. Then, the function $\chi : G \to \mathbb{C}$ defined as

$$\chi(g) = \langle \pi(g)v|v \rangle, \qquad g \in G,$$

is a character. A representation $(\pi, \mathscr{H})$ of G is said to be a *cyclic representation* if there is a normalized vector $v \in \mathscr{H}$ such that the $\mathbb{C}$-linear span of $\{\pi(g)v : \ v \in \mathscr{H}\}$ is dense in $\mathscr{H}$; if this is the case, then we say that the character associated with v (as above) is the *character of G afforded* by $(\pi, \mathscr{H})$.

As it turns out, every character is obtained in this fashion; in order to understand how, we need to consider the C^*-algebra of the group. We fix a left Haar measure η for G, and consider the *involutive algebra* $L^1(G, \eta)$. By a *representation of* $L^1(G, \eta)$ we mean a pair $(\pi, \mathscr{H})$ where $\mathscr{H}$ is a Hilbert space and π is a homomorphism from $L^1(G)$ to the involutive algebra consisting of all bounded linear operators on $\mathscr{H}$. A representation $(\pi, \mathscr{H})$ of $L^1(G, \eta)$ is said to be *non-degenerate* if for every $v \in \mathscr{H}$ there is an element $f \in L^1(G, \eta)$ such that $\pi(f)v \neq 0$.

Now, let $(\pi, \mathscr{H})$ be a unitary representation of G. For every $f \in L^1(G, \eta)$, the linear operator $\pi(f)$ defined by the mapping

$$v \mapsto \int_G f(g)\pi(g)v \, d\eta(g), \qquad v \in \mathscr{H},$$

determines a unique non-degenerate representation of $L^1(G, \eta)$; furthermore, this correspondence defines a bijection between unitary representations of G and non-degenerate representations of $L^1(G, \eta)$ (we refer to [35, Proposition 13.3.1] for a proof).

On the other hand, consider the norm of $L^1(G, \eta)$ given by

$$\|f\| = \sup_{\pi} \|\pi(f)\|, \qquad f \in L^1(G, \eta),$$

where the supremum is taken over all non-degenerate representations of $L^1(G, \eta)$. The completion of $L^1(G, \eta)$ with respect to such norm yields a C^*-algebra, denoted by $C^*(G)$ and called the

group C^*-algebra of G. As it turns out, any non-degenerate representation of $L^1(G,\eta)$ admits a unique extension to a non-degenerate representation of $C^*(G)$. (We notice that, if G is finite, then $C^*(G)$ is the usual group algebra $\mathbb{C}[G]$.)

Every character φ of G admits a unique extension to a positive linear functional on $C^*(G)$ which we will also denote by φ, and the *Gelfand-Naimark-Segal construction* allows us to construct a cyclic representation $(\pi_\varphi, \mathscr{H}_\varphi)$ of G which affords the character φ (for all details we refer to [16, II.6.4]). Let

$$N_\varphi = \{x \in C^*(G) \colon \varphi(xx^*) = 0\}.$$

N_φ is a closed left ideal of $C^*(G)$, and so we may define a inner product $\langle \cdot | \cdot \rangle_\varphi$ on the quotient $C^*(G)/N_\varphi$ by

$$\langle x|y\rangle_\varphi = \varphi(xy^*), \qquad x, y \in C^*(G)/N_\varphi.$$

The completion of $C^*(G)/N_\varphi$ with respect to $\langle \cdot | \cdot \rangle_\varphi$ yields a Hilbert space, denoted by $\mathscr{H}_\varphi$. Finally, we use continuous extension to define the representation $(\pi_\varphi, \mathscr{H}_\varphi)$ of G such that

$$\pi_\varphi(x)(y+N_\varphi) = xy + N_\varphi \qquad x \in C^*(G),\ y + N_\varphi \in C^*(G)/N_\varphi.$$

Furthermore, we note that there is a cyclic vector $v_\varphi \in \mathscr{H}_\varphi$ such that

$$\varphi(x) = \langle \pi_\varphi(x) v_\varphi | v_\varphi \rangle_\varphi, \qquad x \in C^*(G);$$

if G is discrete, then v_φ is the image in $\mathscr{H}_\varphi$ of the group identity $1 \in G$.

Using the Gelfand-Naimark-Segal construction, it is fairly easy to check that two cyclic representations having the same character are unitarily equivalent. In this fashion, there is a bijective correspondence between characters of G and cyclic representations of G. Furthermore, there is a natural bijection between $\mathrm{Char}(G)$ and the set of *quasi-equivalence* classes of representations of type *I* and *II* (see [35, Proposition 17.3.4]), and hence the classification of representations of types *I* and *II* (up to quasi-equivalence) may be only concerned with the cyclic representations.

Example 2.2.4. Let G be a discrete group, so that the *discrete measure* (or, counting measure) d on G (that is, $d(g) = 1$ for all $g \in G$) serves as a (left) Haar measure. For every $g \in G$, let δ_g be the *Dirac function* on g (that is, for every $h \in G$ $\delta_g(h) = 1$ if $g = h$, and $\delta_g(h) = 0$ if $g \neq h$); notice that the $\mathbb{C}$-linear span of $\{\delta_g :\ g \in G\}$ is dense in $L^2(G,d)$.

Consider the (left) regular representation $(\pi, L^2(G,d))$, and note that $\pi_g \delta_h = \delta_{gh}$ for all $g, h \in G$; consequently, δ_1 is a cyclic vector and $(\pi, L^2(G,d))$ affords a character ρ such that for

all $g \in G$

$$\rho(g) = \langle \pi_g \delta_1 | \delta_1 \rangle = \sum_{h \in G} \delta_g(h)\delta_1(h) = \begin{cases} 1, & \text{if } g = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The character ρ is called the *(left) regular character of* G.

In the case where G is a discrete countable group, Thoma showed in [88] that $\mathrm{Char}(G)$ is a *Choquet simplex*. This means that $\mathrm{Char}(G)$ is fully determined by its indecomposable elements: for every $\varphi \in \mathrm{Char}(G)$, there exists a unique Borel, regular probability measure μ on $\mathrm{Ex}(G)$ such that

$$\varphi(g) = \int_{\mathrm{Ex}(G)} \xi(g)\, d\mu(\xi), \qquad g \in G.$$

We will refer to such a decomposition as the *Choquet integral decomposition* (sometimes also called the *spectral decomposition*) of φ, and we will refer to μ as the *Choquet measure* associated with φ.

By the way of example, let G be a finite group. Then, G is a tame group, and hence every finite-dimensional representation admits a character: indeed, the set $\mathrm{Char}(G)$ consists of all normalized traces of finite-dimensional representations, and the indecomposable characters are nothing more than the normalized traces of the irreducible representations (note that an irreducible representation of a finite group is always finite-dimensional). If we denote by $\mathrm{Irr}(G)$ the set consisting of all irreducible characters of G, then $\mathrm{Irr}(G)$ is a finite set, and every character $\varphi \in \mathrm{Char}(G)$ is uniquely written as a non-negative integer linear combination of irreducible characters, that is,

$$\varphi = \sum_{\xi \in \mathrm{Irr}(G)} m(\xi, \varphi)\xi$$

where, for every $\xi \in \mathrm{Irr}(G)$, the coefficient $m(\xi, \varphi)$ is a non-negative integer, to which we refer as the *multiplicity* of ξ in φ (notice that we are allowing zero multiplicities). In order to interpret the Choquet decomposition of characters in this case, we consider the set

$$\mathrm{Ex}(G) = \{\widehat{\xi} = \xi / \xi(1) \colon \xi \in \mathrm{Irr}(G)\}$$

consisting of all normalized irreducible characters of G. For an arbitrary character $\varphi \in \mathrm{Char}(G)$, we consider its normalized version $\widehat{\varphi} = \varphi / \varphi(1)$, so that

$$\widehat{\varphi} = \sum_{\widehat{\xi} \in \mathrm{Ex}(G)} \frac{\xi(1)}{\varphi(1)} m(\xi, \varphi)\widehat{\xi},$$

and we define the *normalized multiplicity* of $\widehat{\xi}$ in $\widehat{\varphi}$ to be

$$\widehat{m}(\widehat{\xi},\widehat{\varphi}) = \frac{\xi(1)}{\varphi(1)} m(\xi,\varphi).$$

Since $\varphi(1) = \sum_{\xi \in \mathrm{Irr}(G)} m(\xi,\varphi)\xi(1)$, we conclude that

$$0 \leq \widehat{m}(\widehat{\xi},\widehat{\varphi}) \leq 1 \quad \text{and} \quad \sum_{\widehat{\xi} \in \mathrm{Ex}(G)} \widehat{m}(\widehat{\xi},\widehat{\varphi}) = 1$$

and thus, the values $\{\widehat{m}(\widehat{\xi},\widehat{\varphi})\}_{\widehat{\xi} \in \mathrm{Ex}(G)}$ determine a unique probability measure μ on $\mathrm{Ex}(G)$ such that

$$\widehat{\varphi} = \int_{\mathrm{Ex}(G)} \widehat{\xi}\, d\mu = \sum_{\widehat{\xi} \in \mathrm{Ex}(G)} \widehat{m}(\widehat{\xi},\widehat{\varphi})\widehat{\xi}.$$

Consequently, the Choquet integral decomposition is equivalent to the ordinary decomposition of characters as an integer linear combination of irreducible characters, and it is in this sense that the character theory of infinite discrete groups generalize the usual character theory of finite groups. For this reason, the set $\mathrm{Ex}(G)$ is considered a *quasi-dual space*.

Chapter 3

Supercharacter Theories and Algebra Groups

For an arbitrary group G, the set $\mathrm{Ex}(G)$ of indecomposable characters may be too large or even too complicated to describe. Nonetheless, it may be possible to consider a smaller and more tractable family of characters which could be used as an "approximation" of $\mathrm{Ex}(G)$. Just as in the finite group case, this may be accomplished by considering *supercharacter theories* of the given group; a prototype example is the finite unitriangular group which is known to have an intractable character theory.

The following section describes a possible way to generalize the definition of a supercharacter theory for an arbitrary countable discrete algebra group, which we present in Definition 3.1.2. Also we provide a general way to construct a supercharacter theory in the sense of Definition 3.1.2.

The Section 3.3 introduces the class of algebra groups and in 3.4 we generalize the supercharacter theory given in [33] for finite algebra groups. We mention that it is not clear if such a construction obeys the axioms of Definition 3.1.2, however it coincides with the standard supercharacter theory of [33] when the algebra group is finite, for this reason we shall refer it as *the standard supercharacter theory*.

3.1 On the definition of a supercharacter theory

Let G be an arbitrary finite group. For the moment, by a *(complex) character* of G we mean the usual (non-normalized) trace of a finite-dimensional complex representation of G. The notion

of a supercharacter theory of G was introduced by Diaconis and Isaacs in [33] to generalize an approach used by André (e.g. [4–6]) and Yan [94] to study the irreducible characters of the finite unitriangular groups. The basic idea is to coarsen the usual irreducible character theory of a group by replacing irreducible characters with integer linear combinations of irreducible characters that are constant on a set of clumped conjugacy classes. By a *supercharacter theory* of G we mean a pair $(\mathscr{K},\mathscr{E})$ where $\mathscr{K}$ is set partition of G and $\mathscr{E}$ an orthogonal set of characters of G (not necessarily irreducible), satisfying the following properties:

- $|\mathscr{K}| = |\mathscr{E}|$,
- every character $\chi \in \mathscr{E}$ takes a constant value on each member $K \in \mathscr{K}$, and
- each irreducible character is a constituent of one of the characters $\chi \in \mathscr{E}$.

We refer to the members of $\mathscr{K}$ as *superclasses* and to the characters in $\mathscr{E}$ as *supercharacters* of G. We note that, as shown in [33, Theorem 2.2], the superclasses of G are always unions of conjugacy classes; moreover, 1 forms a superclass and the principal character 1_G is always a supercharacter of G.

As a "trivial" example, the set $\mathrm{Cl}(G)$ of conjugacy classes G, together with the set $\mathrm{Irr}(G)$ of irreducible characters, form a supercharacter theory $(\mathrm{Cl}(G),\mathrm{Irr}(G))$ of G. In general, given an arbitrary supercharacter theory $(\mathscr{K},\mathscr{E})$ of G, the set $\mathscr{E}$ induces a partition

$$\mathscr{X}_{\mathscr{E}} = \{X_\chi : \chi \in \mathscr{E}\}$$

of $\mathrm{Irr}(G)$ where, for every $\chi \in \mathscr{E}$, X_χ is the set of irreducible constituents of χ; in this fashion, $(\mathscr{K}, \mathscr{X}_{\mathscr{E}})$ can be understood as a quotient of $(\mathrm{Cl}(G),\mathrm{Irr}(G))$, and thus we may think of a supercharacter theory as an approximation of the (usual) irreducible character theory. As it turns out (see [33, Lemma 2.1]), the partitions $\mathscr{K}$ of G and $\mathscr{X}_{\mathscr{E}}$ of $\mathrm{Irr}(G)$ uniquely determine each other. Let $\mathscr{X}$ be a partition of $\mathrm{Irr}(G)$, and define the character

$$\sigma_X = \sum_{\xi\in\mathrm{Irr}(G)} \xi(1)\xi, \qquad X \in \mathscr{X}.$$

If there is a partition $\mathscr{K}$ of G such that $\{1\} \in \mathscr{K}$ and $|\mathscr{K}| = |\mathscr{E}|$, and if each σ_X is constant on the members of $\mathscr{K}$, then the pair $(\mathscr{K},\{\sigma_X : X \in \mathscr{X}\})$ forms a supercharacter theory of G; indeed, the following is true.

Lemma 3.1.1 ([33, Lemma 2.1]). *Let $\mathscr{K}$ be a partition of a finite group G, let $\mathscr{X}$ be a partition of* $\mathrm{Irr}(G)$*, and assume that $\mathscr{E} = \{\chi_X : X \in \mathscr{X}\}$ is a family of characters which are constant on*

elements of $\mathscr{K}$ and such that, for every $X \in \mathscr{X}$, the irreducible characters of χ_X lie in X. If $|\mathscr{K}| = |\mathscr{E}|$, then the following are equivalent:

- $\{1\} \in \mathscr{K}$;
- *Each irreducible character is a constituent of a unique element of $\mathscr{E}$;*
- *For every $X \in \mathscr{X}$, the character χ_X is a constant multiple of σ_X.*

If we consider G as a topological group (equipped with the discrete topology) and normalize characters in order to agree with the general definition of character given in Definition 2.2.3, then every (non-normalized) supercharacter theory $(\mathscr{K}, \mathscr{E})$ yields a normalized supercharacter theory which is uniquely determined by the corresponding partitions $\mathscr{K}$ and $\mathscr{X}_{\mathscr{E}}$: if χ_X is the supercharacter associated with $X \in \mathscr{X}_{\mathscr{E}}$ and $\chi_X = \alpha_X \sigma_X$ for some positive integer α_X, then

$$\frac{\chi_X}{\chi_X(1)} = \frac{\alpha_X \sigma_X}{\alpha_X \sigma_X(1)} = \frac{\sigma_X}{\sigma_X(1)}.$$

In this context, a supercharacter theory for a finite group admits a slightly different characterization. Let

$$\mathrm{Ex}(G) = \{\zeta/\zeta(1) \colon \zeta \in \mathrm{Irr}(G)\},$$

and let φ be an arbitrary (normalized) character of G. Then, we say that $\xi \in \mathrm{Ex}(G)$ is a *constituent* of φ if the normalized multiplicity $\widehat{m}(\xi, \varphi)$ is non-zero. If $(\mathscr{K}, \mathscr{E})$ is a (normalized) supercharacter theory for G, then a character which is constant on the superclasses will be referred to as a *superclass character*; more generally, by a *superclass function* of G we mean a complex-valued function defined on G which is constant on the superclasses of G (hence, a superclass character is a superclass function which is also a character of G). We will denote by $\mathrm{SCl}^+_{\mathscr{K}}(G)$ the set consisting of all superclass characters of G. Then, $\mathrm{SCl}^+_{\mathscr{K}}(G)$ is a convex set, and $\mathscr{E}$ is precisely the set of indecomposable elements of $\mathrm{SCl}^+_{\mathscr{K}}(G)$ (because $|\mathscr{K}| = |\mathscr{E}|$ and because every indecomposable character is a constituent of a unique supercharacter). Consequently, a *normalized supercharacter theory* of G may be defined as a pair $(\mathscr{K}, \mathscr{E})$, where $\mathscr{K}$ is partition of G and $\mathscr{E}$ is the set consisting of all indecomposable elements of $\mathrm{SCl}^+_{\mathscr{K}}(G)$, such that $|\mathscr{K}| = |\mathscr{E}|$ and each indecomposable character $\xi \in \mathrm{Ex}(G)$ is a constituent of a unique element in $\mathscr{E}$;

Now, let G be an *approximately finite group*, that is, there is a chain $\{G_n\}_{n \in \mathbb{N}}$ of finite

subgroups of G such that $G_n \subseteq G_{n+1}$ and

$$G = \varinjlim_{n\in\mathbb{N}} G_n = \bigcup_{n\in\mathbb{N}} G_n.$$

Furthermore, suppose that for each $n \in \mathbb{N}$ the subgroup G_n is endowed with a normalized supercharacter theory $(\mathscr{K}_n, \mathscr{E}_n)$. Then, if we assume mild compatibility conditions on the supercharacter theories $(\mathscr{K}_n, \mathscr{E}_n)$ for $n \in \mathbb{N}$, there is a natural way to extend the notion of supercharacter theory to the group G.

On the one hand, suppose that, for every $n \in \mathbb{N}$ and every superclass $K_n \in \mathscr{K}_n$, there is a unique superclass $K_{n+1} \in \mathscr{K}_n$ such that $K_n \subseteq K_{n+1}$. Then, the set $\mathscr{K}$ of superclasses of G is defined to be the set consisting of all unions

$$K = \bigcup_{n\in\mathbb{N}} K_n$$

where $K_n \in \mathscr{K}_n$ and $K_n \subseteq K_{n+1}$. On the other hand, for every $n \in \mathbb{N}$ and every $K_{n+1} \in \mathscr{K}_{n+1}$, the intersection $K_{n+1} \cap G_n$ must be a union of superclasses of G_n, and this implies that the restriction of any supercharacter in $\mathscr{E}_{n+1}$ to G_n must be a convex linear combination of supercharacters in $\mathscr{E}_n$. Consequently, for every $n \in \mathbb{N}$, the restriction to G_n of every superclass character of G is a superclass character of G_n, and this fact allows us to adapt the Kerov-Vershik ergodic method (see [90] and also 4.2) in order to conclude that, for every indecomposable superclass character χ of G, there is at least one sequence $(\chi_n)_{n\in\mathbb{N}}$, where $\chi_n \in \mathscr{E}_n$ for all $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \chi_n(g) = \chi(g), \qquad g \in G.$$

Such a finite approximation property extends the notion of a supercharacter in an asymptotic fashion, and thus it is natural to define the set of supercharacters of G to be the set $\mathscr{E}$ consisting of all indecomposable superclass characters of G. This definition of a supercharacter is used implicitly in [8] where the indecomposable supercharacters[1] are described for the infinite unitriangular group

$$U_\infty(\mathbb{F}_q) = \varinjlim_{n\in\mathbb{N}} U_n(\mathbb{F}_q);$$

the resulting supercharacter theory is in fact a natural extension of the supercharacter theory of $U_n(\mathbb{F}_q)$ (in particular, it allows us to understand the paper [32] by De Stavola in a purely

[1] We notice that, in [8] superclass characters are called supercharacters and the indecomposable superclass characters are called indecomposable supercharacters; we choose not to follow this terminology in order to be coherent with the finite group case.

representation theoretical setting; this will be explained in detail in Section 7.3). These facts suggest that the set $\mathscr{E}$ of indecomposable superclass characters are the right substitute for the set of supercharacters, and that the pair $(\mathscr{K}, \mathscr{E})$ should be considered a supercharacter theory for any approximately finite group G.

In the more general situation of an arbitrary countable discrete group G, there is no evidence of which object should be considered as a supercharacter theory. A supercharacter theory for G should be a pair $(\mathscr{K}, \mathscr{E})$, understood as an approximation of $(\mathrm{Cl}(G), \mathrm{Ex}(G))$, which generalizes the finite case definition in the same way that indecomposable characters generalize the irreducible characters, that is, we should have "natural analogues" to fulfill the following schematic diagram:

$$\begin{array}{ccc} \textbf{Finite groups} & & \textbf{Infinite groups} \\ \text{Irreducible characters} & \longrightarrow & \text{Indecomposable characters} \\ \big\downarrow & & \big\downarrow \\ \text{Supercharacters} & \longrightarrow & \text{Supercharacters} \end{array}$$

Since, in general, the number of indecomposable characters of an infinite countable discrete group is "bigger" than the number of conjugacy classes (for example, the indecomposable characters of the abelian group of integer numbers is the unit circle), it does not seem reasonable to request the number of superclasses to be equal to the number of supercharacters. Nevertheless, any extension of the definition of a supercharacter theory $(\mathscr{K}, \mathscr{E})$ should coincide with the finite group definition if, additionally, $|\mathscr{K}| = |\mathscr{E}|$.

The equality, in the case of finite groups, between the number of superclasses and supercharacters has important consequences that can be omitted in the definition of a supercharacter theory; however, these consequences need to be part of the definition when one considers an infinite countable group. The purpose of a supercharacter theory is to provide some sort of approximation of the indecomposable character theory, where superclasses play the role of conjugacy classes and supercharacters are a substitute for indecomposable characters. Hence, in order for a partition $\mathscr{K}$ of G to be a candidate for the set consisting on superclasses, we should require any member $K \in \mathscr{K}$ to be a union of conjugacy classes and that $\{1\} \in \mathscr{K}$; if this is case, then we shall say that $\mathscr{K}$ is a *family of superclasses* of G.

A naive attempt to define a supercharacter theory of G would be to choose a pair $(\mathscr{K}, \mathscr{E})$ where $\mathscr{K}$ is a family of superclasses, and where $\mathscr{E}$ is the subset consisting of all indecomposable

elements of the convex set $\mathrm{SCl}^+_{\mathscr{K}}(G)$ of all superclass characters (that is, characters which are constant on each element of $\mathscr{K}$). However, this definition has an obvious obstruction: for example, it does not agree with the (normalized) definition of [33] in the particular case where G is a finite group (because the elements of $\mathscr{E}$ are not necessarily orthogonal nor induce a partition of $\mathrm{Irr}(G)$). For this reason, the definition of a supercharacter theory must be refined.

Although there are various similarities between the character theory of infinite discrete groups and of finite groups, the analogy breaks down in two crucial relayed points: finiteness and decomposition. For a finite group, there is only a finite number of irreducible characters, and this implies that every character is uniquely decomposed as a sum of irreducible characters; on the other hand, for an infinite discrete group, such a decomposition is carried out by means of a measure (in general, non-discrete). In the former case, it makes sense to consider the irreducible characters which appear in the decomposition of a given character, but in the latter things are not that precise due to the fact that one has to deal with sets of zero measure, which causes a significant hindrance in generalizing the notion of a supercharacter theory to infinite groups.

Our way to avoid these (and other) difficulties is to look at the axioms of a supercharacter theory of finite groups from a measure theoretical point of view. Let $(\mathscr{K},\mathscr{E})$ be a normalized supercharacter theory of a finite group G. For every supercharacter $\chi \in \mathscr{E}$, let M_χ be the Choquet measure on $\mathrm{Ex}(G)$ associated with χ that is,

$$M_\chi(\xi) = \widehat{m}(\xi,\chi), \qquad \xi \in \mathrm{Ex}(G),$$

and let $\mathrm{supp}(M_\chi)$ be its support. Then, the fact that every irreducible character is a constituent of a unique supercharacter is equivalent to the following two conditions:

- for all $\chi,\chi' \in \mathscr{E}$, $\chi \neq \chi'$, the measures M_χ and $M_{\chi'}$ are mutually singular, and
- $\bigcup\limits_{\chi\in\mathscr{E}} \mathrm{supp}(M_\chi) = \mathrm{Ex}(G)$.

(We recall that two measures M and M' on $\mathrm{Ex}(G)$ are *mutually singular* if there are two disjoint Borel subsets E and E' such that $E \cup E' = \mathrm{Ex}(G)$ and $M(E') = M'(E) = 0$.) Having this in mind, we define a supercharacter theory for an arbitrary countable discrete group as follows.

Definition 3.1.2. Let G be an arbitrary countable discrete group. By a *supercharacter theory* of G we mean a pair $(\mathscr{K},\mathscr{E})$ where $\mathscr{K}$ is a partition of G and $\mathscr{E}$ is a set of characters of G such that

- $\mathscr{K}$ is a *superclass family*, that is, $\{1\} \in \mathscr{K}$ and every $K \in \mathscr{K}$ is a union of conjugacy classes;
- the characters of G which are constant on the elements of $\mathscr{K}$ form a Choquet simplex whose indecomposable elements is $\mathscr{E}$;
- the Choquet measures on $\mathrm{Ex}(G)$ associated with the elements of $\mathscr{E}$ are mutually singular, and $\mathrm{Ex}(G)$ equals the union of all the corresponding supports.

For simplicity, and in order to maintain some analogy with the finite group scenario, we will say that two characters of G are *orthogonal* if the corresponding Choquet measures on $\mathrm{Ex}(G)$ are mutually singular (we note however that this is not a standard terminology).

Every countable discrete group G admits at least one "trivial" supercharacter theory, namely the pair $(\mathrm{Cl}(G), \mathrm{Ex}(G))$ where, as before, $\mathrm{Cl}(G)$ denotes the set of all conjugacy classes of G. In this sense, a supercharacter theory generalizes the (usual) character theory; furthermore, if $(\mathscr{K}, \mathscr{E})$ is a supercharacter theory (in the above sense) of a finite group G, then it is a supercharacter theory in the sense of [33] provided that $|\mathscr{K}| = |\mathscr{E}|$.

At this point it is worth to mention that the aforementioned definition is not the only *natural* extension of the concept of a supercharacter theory to an infinite countable discrete group. In [48], Hendrickson showed that a supercharacter theory of a finite group is fully determined by the set of superclasses, suggesting an alternative definition. Let G be a finite group, and let $\mathbb{C}[G]$ denote the complex group algebra of G. Given a partition $\mathscr{K}$ of G, we associate with every $K \in \mathscr{K}$ the element

$$\widehat{K} = \sum_{g \in G} g \in \mathbb{C}[G],$$

and denote by $S_{\mathscr{K}}$ the $\mathbb{C}$-linear span of the set $\{\widehat{K} \colon K \in \mathscr{K}\}$; this set $S_{\mathscr{K}}$ is called a *Schur ring* (associated with $\mathscr{K}$) if it is a subalgebra of $\mathbb{C}[G]$ and $\mathscr{K}$ is such that $\{1\} \in \mathscr{K}$ and

$$K^{-1} = \{g^{-1} \colon g \in K\} \in \mathscr{K}, \qquad K \in \mathscr{K}.$$

If a Schur ring $S_{\mathscr{K}}$ is a subalgebra of the centre $\mathbf{Z}(\mathbb{C}[G])$ of $\mathbb{C}[G]$ (that is, if $S_{\mathscr{K}}$ is a *central Schur ring*), and $\mathscr{E}$ denotes the set of indecomposable elements of $\mathrm{SCl}^{+}_{\mathscr{K}}(G)$, then the pair $(\mathscr{K}, \mathscr{E})$ is a supercharacter theory; conversely, every supercharacter theory $(\mathscr{K}, \mathscr{E})$ determines a unique Schur ring $S_{\mathscr{K}}$ which is a subalgebra of $\mathbf{Z}(\mathbb{C}[G])$. In this fashion, we may define a supercharacter theory of a finite group G to be a pair $(\mathscr{K}, \mathscr{E})$ where $S_{\mathscr{K}}$ is a central Schur ring

and $\mathscr{E}$ is the set of indecomposable elements of $\mathrm{SCl}^{+}_{\mathscr{K}}(G)$. The advantage of this definition is that it does not directly rely on the equality between the number of superclasses and the number of supercharacters; furthermore, properties of supercharacters are not straightforwardly required.

Notice that, given any partition $\mathscr{K}$ of G, the product $\widehat{K}\widehat{K'}$ of two elements $K, K' \in \mathscr{K}$ is an element in $S_{\mathscr{K}}$ if and only if the set

$$KK' = \{gh\colon g \in K,\ h \in K'\}$$

is a union of members in $\mathscr{K}$; moreover, $S_{\mathscr{K}}$ is a subalgebra of $\mathbf{Z}(\mathbb{C}[G])$ if and only if every $K \in \mathscr{K}$ is a union of conjugacy classes. Consequently, given an arbitrary countable discrete group G, we might be tempted to define a supercharacter theory of G to be a pair $(\mathscr{K}, \mathscr{E})$ where $\mathscr{K}$ is a superclass family such that $K^{-1} \in \mathscr{K}$ for all $K \in \mathscr{K}$, and KK' is a (possibly infinite) union of superclasses for all $K, K' \in \mathscr{K}$, and such that $\mathrm{SCl}^{+}_{\mathscr{K}}(G)$ is a Choquet Simplex whose indecomposable elements is $\mathscr{E}$. The main issue is that, in general, it is not clear that this definition would be equivalent to Definition 3.1.2.

We choose Definition 3.1.2 as the definition of supercharacter theory, mainly for two reasons. On one hand, it (loosely) replicates the main feature of a supercharacter theory of a finite group in the sense that we still have a notion of orthogonality between supercharacters; on the other hand, we feel that the definition conveys (in a more or less straightforward sense) what we mean by an approximation of indecomposable characters.

3.2 Supercharacter Theories defined by group actions

Let G be an arbitrary countable discrete group since, we shall explain how any countable amenable group $\mathscr{G}$ acting on G *via* automorphism defines a supercharacter theory.

There are several equivalent definitions of discrete amenable groups (we refer to [80] for more details on amenable groups); the most typical one is the presence of a *Følner sequence*: a discrete countable group $\mathscr{G}$ is amenable if it admits a Følner sequence, that is, a family of finite subsets $\{F_n\}_{n\in\mathbb{N}}$ such that for all $\mathbf{k} \in \mathscr{G}$

$$\lim_{n\to\infty} \frac{|F_n \Delta(\mathbf{k}F_n)|}{|F_n|} = 0,$$

where Δ denotes the symmetric difference of sets. However, for our purposes, the most useful characterization of amenable groups is the *fixed point property*. Let K be a convex subset of a

locally convex vector space E, we say that a locally compact Hausdorff group $\mathscr{G}$ acts *affinely and in a separately continuous way* on K if for all $t \in [0,1]$ and all $x, y \in K$

$$\mathbf{k}\cdot(tx+(1-t)y) = t(\mathbf{k}\cdot x)+(1-t)(\mathbf{k}\cdot y), \qquad \mathbf{k} \in \mathscr{G},$$

and the map $\mathscr{G} \times K \to K$, defined by the mapping $(\mathbf{k}, x) \mapsto \mathbf{k}\cdot x$, is separately continuous; the group $\mathscr{G}$ is said to have the *fixed point property* over K if it acts affinely and in a separately continuous way on K and the this action has a fixed point. The following theorem provides an alternative definition of amenability (a proof can be found in [80, Theorem 5.4]):

Theorem 3.2.1 (Day's fixed point theorem)**.** *A locally compact Hausdorff group $\mathscr{G}$ is amenable if and only if $\mathscr{G}$ has the fixed point property over any convex subset K of a locally convex vector space E.*

Notice that since $\mathscr{G}$ acts on G *via* automorphisms, any element $\mathbf{k} \in \mathscr{G}$ induces a permutation of conjugacy classes, thus, the set $\mathscr{K}_{\mathscr{G}} = \{\mathscr{G}\cdot g \colon g \in G\}$ is a superclass family. We denote by $\mathrm{SCl}^+_{\mathscr{G}}(G)$ the set of characters constant on elements in $\mathscr{K}_{\mathscr{G}}$.

Now consider $\mathrm{Char}(G)$, the set consisting of all characters of G, since $\mathscr{G}$ acts *via* automorphism on G there is a natural corresponding action of $\mathscr{G}$ on $\mathrm{Char}(G)$, which is affinely and in a separately continuous: for all $\mathbf{k} \in \mathscr{G}$ and all $\varphi \in \mathrm{Char}(G)$

$$\mathbf{k}\cdot\varphi(g) = \varphi(\mathbf{k}^{-1}\cdot g), \qquad g \in G.$$

Moreover, $\mathscr{G}$ acts on $\mathrm{Ex}(G)$, indeed if $\xi\,\mathrm{Ex}(G)$ and $\mathbf{k} \in \mathscr{G}$, then, if $t \in]0,1[$ and $\varphi, \psi \in \mathrm{Char}(G)$ are such that $\mathbf{k}\cdot\xi = t\varphi+(1-t)\psi$, then

$$\xi = t\mathbf{k}^{-1}\cdot\psi+(1-t)\mathbf{k}^{-1}\psi; \Rightarrow\ \varphi = \psi.$$

Since $\mathrm{Char}(G)$ is a Choquet simplex, any character $\varphi \in \mathrm{Char}(G)$ is fully determined by a unique Borel probability measure on $\mathrm{Ex}(G)$, for this reason we identify $\mathrm{Char}(G)$ with $\mathscr{M}^+(\mathrm{Ex}(G))$ (the set consisting of all Borel probability measures on $\mathrm{Ex}(G)$) equipped with the weak*-convergence topology. A measure $M \in \mathscr{M}^+(\mathrm{Ex}(G))$ is said to be $\mathscr{G}$-invariant if for any $\mathbf{k} \in \mathscr{G}$

$$\mathbf{k}\cdot M(X) = M(\mathbf{k}^{-1}\cdot X) = M(X), \qquad X \in \mathscr{B}(\mathrm{Ex}(G)),$$

and we denote by $\mathscr{M}^+_{\mathscr{G}}(\mathrm{Ex}(G))$ the set consisting of all $\mathscr{G}$-invariant measures on $\mathrm{Ex}(G)$.

It is straight forward to check $\mathscr{M}^+_{\mathscr{G}}(\mathrm{Ex}(G))$ is afinely homeomorphic to $\mathrm{SCl}^+_{\mathscr{G}}(G)$, furthermore $\mathscr{M}^+_{\mathscr{G}}(\mathrm{Ex}(G))$ is a Choquet simplex whose indecomposable elements are precisely the

$\mathscr{G}$-ergodic measures on $\mathrm{Ex}(G)$ (in virtue of proposition 2.1.6); we denote by $\mathscr{E}_{\mathscr{G}}$ the characters in $\mathrm{SCl}^{+}_{\mathscr{G}}(G)$ determined by the $\mathscr{G}$-ergodic measures on $\mathrm{Ex}(G)$.

Proposition 3.2.2. *The pair $(\mathscr{K}_{\mathscr{G}}, \mathscr{E}_{\mathscr{G}})$ forms a supercharacter theory for G.*

Proof. Notice that for the pair $(\mathscr{K}_{\mathscr{G}}, \mathscr{E}_{\mathscr{G}})$ to be a supercharacter theory (in the sense of Definition 3.1.2) it only remains to show that elements in $\mathscr{E}_{\mathscr{G}}$ are orthogonal and that the union of the support of all $\mathscr{G}$-ergodic measures on $\mathrm{Ex}(G)$ is equal to $\mathrm{Ex}(G)$.

let M_1 and M_2 be $\mathscr{G}$-ergodic probability measures on $\mathrm{Ex}(G)$. According to the Lebesgue's decomposition theorem (see, for example [17, Theorem 3.2.3]), the measure M_1 decomposes as a sum

$$M_1 = M_1^c + M_1^d$$

where M_1^c and M_1^d are two signed measures such that M_1^c is absolutely continuous with respect to M_2 and M_1^d is singular (or orthogonal) with respect to M_2. We observe that M_1^c and M_1^d are also singular with respect to each other, and thus there are two disjoint Borel sets $B_1, B_2 \in \mathscr{B}(\mathrm{Ex}(G))$ such that

$$\mathrm{Ex}(G) = B_1 \cup B_2 \qquad \text{and} \qquad M_1^d(B_1) = M_2(B_2) = 0.$$

This means that, for every Borel set $X \in \mathscr{B}(\mathrm{Ex}(G))$ with $X \subseteq B_1$, we have

$$0 \leq M_1^c(X) = M_1(X) \leq 1,$$

and hence M_1^c is a $\mathscr{G}$-invariant positive measure; in its turn, this implies that M_1^d is also a positive $\mathbb{G}$-invariant measure (because M_1 is positive, $M_1 = M_1^c + M_1^d$, and M_1^c and M_1^d are singular measures).

If both M_1^c and M_1^d are non-zero measures, then after normalizing them we can write M_1 as a convex sum of two $\mathscr{G}$-invariant probability measures, which contradicts the ergodicity of M_1. Consequently, either $M_1 = M_1^c$ (which implies that $M_1 = M_2$), or $M_1 = M_1^d$, and thus the corresponding superclass characters are orthogonal.

For every $\xi \in \mathrm{Ex}(G)$, let $\mathscr{O}^{\xi}$ denote the closure of the $\mathscr{G}$-orbit $\mathscr{G} \cdot \xi \subseteq \mathrm{Ex}(G)$. Since $\mathscr{O}^{\xi}$ is a closed subset of a compact Hausdorff space, it is also compact and Hausdorff. The set $\mathscr{M}^{+}(\mathscr{O}^{\xi})$, consisting of all probability measures on $\mathscr{O}^{\xi}$, is non-empty (because it can be identified with the topological linear dual of $C(\mathscr{O}^{\xi})$) and the $\mathbb{G}$-action on $\mathscr{O}^{\xi}$ induces a natural $\mathscr{G}$-action on $\mathscr{M}^{+}(\mathscr{O}^{\xi})$: for every $M \in \mathscr{M}^{+}(\mathscr{O}^{\xi})$ and every $\mathbf{k} \in \mathscr{G}$, we define $\mathbf{k} \cdot M \in \mathscr{M}^{+}(\mathscr{O}^{\xi})$ by

$$(\mathbf{k} \cdot M)(X) = M(\mathbf{k}^{-1} \cdot X), \qquad X \in \mathscr{B}(\mathscr{O}^{\xi}).$$

Since $\mathbb{G}$ is amenable, there is at least one fixed point in $\mathscr{M}^+(\mathscr{O}^\xi)$ (in virtue of Day's Theorem 3.2.1), that is, there is at least one $\mathbb{G}$-invariant measure on $\mathscr{O}^\xi$.

Every $\mathscr{G}$-invariant measure on $\mathscr{O}^\xi$ admits a unique decomposition in terms of $\mathscr{G}$-invariant ergodic measures, and hence there must exist at least one $\mathscr{G}$-invariant ergodic measure M_0 on $\mathscr{O}^\xi$. The fact that M_0 is $\mathscr{G}$-invariant implies that $\operatorname{supp}(M_0) = \mathscr{O}^\xi$; furthermore, M_0 admits an extension to a measure M on $\operatorname{Ex}(G)$: for every Borel set $B \in \mathscr{B}(\operatorname{Ex}(G))$ we set

$$M(B) = M_0(B \cap \mathscr{O}^\xi).$$

Finally, note that, if $B \in \mathscr{B}(\operatorname{Ex}(G))$ is $\mathscr{G}$-invariant set, then so is $B \cap \mathscr{O}^\xi$, and thus either $M(B) = 0$ or $M(B) = 1$; consequently, we conclude that M is a $\mathbb{G}$-ergodic measure on $\operatorname{Ex}(G)$ with $\operatorname{supp}(M) = \mathscr{O}^\xi$.

Since

$$\operatorname{Ex}(G) = \bigcup_{\xi \in \operatorname{Ex}(G)} \mathscr{G} \cdot \xi,$$

it follows that the union of the supports of all $\mathbb{G}$-ergodic measures equals $\operatorname{Ex}(G)$, and therefore the pair $(\mathscr{K}_{\mathscr{G}}, \mathscr{E}_{\mathscr{G}})$ is in fact a supercharacter theory of G.

□

3.3 Algebra Groups and Supercharacter Theories

Let $\mathbb{K}$ be a field and let A be an associative nil algebra over $\mathbb{K}$ (that is, an associative algebra over $\mathbb{K}$ where every element is nilpotent). Throughout the thesis, we enlarge the ring A with an identity element 1 (to be more rigorous, we naturally embed A as a subring of the direct sum $\mathscr{A} = \mathbb{K} \oplus A$), and consider the set $1 + A$ consisting of all formal sums $1 + a$ with $a \in A$. Then, the set $1 + A$ can be equipped with the product

$$(1+a)(1+b) = 1 + a + b + ab, \qquad a, b \in A.$$

Since A is nil, for every $a \in A$ there is a positive integer $n \in \mathbb{N}$ such that $a^n = 0$, and so

$$(1+a)\left(\sum_{t=0}^{n-1} (-1)^t a^t\right) = 1 + a - a + \sum_{t=2}^{n-1} (-1)^t a^t + \sum_{t=2}^{n-1} (-1)^{t-1} a^t = 1.$$

Consequently, every element in $1 + A$ has an inverse; moreover, since A is associative, the product of $1 + A$ is also associative, and thus $1 + A$ is a group (with identity 1).

Definition 3.3.1. A group G is said to be an *algebra group* over a field $\mathbb{K}$ (or simply a $\mathbb{K}$-*algebra group*) if there is an associative nil algebra A over $\mathbb{K}$ such that $G = 1 + A$. We refer to $\mathbb{K}$ as the base field of G, and define the *characteristic* and *dimension* of G to be, respectively, the characteristic of $\mathbb{K}$ and the dimension of A (as $\mathbb{K}$-vector space). Following [55], we define an *algebra subgroup* of G to be a subgroup H for which there is a subalgebra B of A such that $H = 1 + b$; similarly, an *ideal subgroup* of G is a subgroup N for which there is a two-sided ideal I of A such that $N = 1 + I$ (notice that and ideal subgroup is a normal subgroup of G).

(We observe that if $G = 1 + A$ is an algebra group then G is the set of unipotent elements of $\mathscr{A} = \mathbb{K} \oplus A$.)

Now, suppose that $\mathbb{K}$ is a topological field. Then, being a vector space over $\mathbb{K}$, A may be equipped with the product topology induced by the topology of K (which in general is not discrete), and we may use the obvious bijection $\vartheta : A \to 1 + A$ (given by the mapping $a \mapsto 1 + a$) to induce a topology on the group $G = 1 + A$: a subset $U \subseteq G$ is open if and only if $\vartheta^{-1}(U) = U - 1$ is an open subset of A. Furthermore, it is well-known that (with respect to the product topology), A is a topological vector space; in particular, if in addition the multiplication of A is continuous, then the multiplication of G is also continuous, and hence G becomes a topological group. Notice that A can be considered as a Lie algebra of G, and that the bijection $\vartheta : A \to G$ may be viewed as a crude version of an "exponential map"; indeed, in the particular case where $G = 1 + A$ is a finite dimensional algebra group over the real field $\mathbb{R}$ or over the complex field $\mathbb{C}$, then G is indeed a Lie group whose Lie algebra is A, and ϑ is the usual exponential map truncated at the second term.

Our main focus is on discrete, locally compact and second countable algebra groups; for this reason, we shall consider only *countable* discrete algebra groups which, in particular, are second countable locally compact Hausdorff and unimodular groups (the assumption on the cardinality of the group is to ensure second countability). We observe that, any countable discrete algebra group $G = 1 + A$ must be defined over a countable discrete field $\mathbb{K}$, and A must have countable dimension (as a vector space over $\mathbb{K}$).

Furthermore, we will often require algebra groups to be amenable, which is a somewhat reasonable hypothesis since it does not seem to restrict too much the class of groups considered. In fact, most interesting examples are either nilpotent or locally nilpotent groups, which are amenable: on one hand, nilpotent groups are solvable, and solvable groups are amenable (for a proof we refer to [80, Corollary 13.5]); on the other, an algebra group $G = 1 + A$ is locally

nilpotent if and only if the algebra A is locally nilpotent, and since locally nilpotent algebras can be realized as the direct limit of nilpotent algebras, the group G is a direct limit of nilpotent groups, and thus amenable (for a proof that a direct limit of amenable groups is amenable, we refer to [80, Proposition 13.6]).

An important subclass of countable discrete algebra groups that concern us consists of all *approximately finite* (AF for short) algebra groups: an algebra group $G = 1 + A$ (over a field $\mathbb{K}$) is said to be *approximately finite* if there is a chain of subgroups $\{G_n\}_{n\in\mathbb{N}}$ where, for every $n \in \mathbb{N}$, $G_n = 1 + A_n$ is a finite algebra group (over a finite field $\mathbb{K}_n$) such that $G_n \subseteq G_{n+1}$ and

$$G = \varinjlim_{n\in\mathbb{N}} G_n,$$

the direct limit being taken with respect to the inclusion maps; equivalently,

$$G = \bigcup_{n\in\mathbb{N}} G_n = 1 + \bigcup_{n\in\mathbb{N}} A_n.$$

Here, G is equipped with the direct limit topology which turns out to be the discrete topology (because finite groups are assumed to be discrete); therefore, every AF-algebra group is indeed a discrete group. Furthermore, finite groups are (trivially) amenable, and thus AF-algebra groups are in fact amenable groups.

Since G_n is a finite algebra group, the base field $\mathbb{K}_n$ of G_n must be finite for all $n \in \mathbb{N}$; moreover, since $G_n \subseteq G_{n+1}$, it is also clear that $\mathbb{K}_n$ must be a subfield of $\mathbb{K}_{n+1}$ for all $n \in \mathbb{N}$. It follows that the base field of G is the union

$$\mathbb{K} = \bigcup_{n\in\mathbb{N}} \mathbb{K}_n,$$

which implies that the base field of an AF-algebra group G is of positive characteristic.

Example 3.3.2 (Unitriangular groups)**.** If $\mathbb{K}$ is a countable discrete field, then a *unitriangular group* over $\mathbb{K}$ is one of the following:

- for every $n \in \mathbb{N}$, the unitriangular group $U_n(\mathbb{K}) = 1_n + \mathfrak{u}_n(\mathbb{K})$ consisting of all $n \times n$ upper triangular matrices over $\mathbb{K}$ with all diagonal entries equal to $1 \in \mathbb{K}$; here, 1_n is the identity matrix, and $\mathfrak{u}_n(\mathbb{K})$ denotes the algebra over $\mathbb{K}$ consisting of all strictly upper triangular matrices over $\mathbb{K}$;

- The locally finite unitriangular group $U_\infty(\mathbb{K}) = 1 + \mathfrak{u}_\infty(\mathbb{K})$ consisting of all infinite upper-triangular square matrices over $\mathbb{K}$ with diagonal entries equal to $1 \in \mathbb{K}$ and such that every

element has only a finite number of non-zero entries above the main diagonal: for every $n \in \mathbb{N}$, the (finite dimensional) $\mathbb{K}$-algebra $\mathfrak{u}_n(\mathbb{K})$ may be identified with the subalgebra of $\mathfrak{u}_{n+1}(\mathbb{K})$ having the last column filled with zeros, so that we get a natural an inclusion $U_n(\mathbb{K}) \hookrightarrow U_{n+1}(\mathbb{K})$, and the group $U_\infty(\mathbb{K})$ can be identified with the direct limit

$$U_\infty(\mathbb{K}) = \varinjlim_{n\in\mathbb{N}} U_n(\mathbb{K}) = \bigcup_{n\in\mathbb{N}} U_n(\mathbb{K}).$$

Notice that $U_n(\mathbb{K})$ is a nilpotent group and that $U_\infty(\mathbb{K})$ is locally nilpotent; therefore. they are both amenable groups.

Example 3.3.3 (McLain groups (or pattern groups))**.** Let $(\mathscr{P}, \preceq)$ be a countable poset (that is, a partially ordered set). With all $\alpha, \beta \in P$ with $\alpha \prec \beta$, we associate a formal element $e_{\alpha,\beta}$, and for all $\alpha, \beta, \gamma, \tau \in P$ with $\alpha \prec \beta$ and $\gamma \prec \tau$, we define the product

$$e_{\alpha,\beta} e_{\gamma,\tau} = \begin{cases} e_{\alpha,\tau}, & \text{if } \beta = \gamma, \\ 0, & \text{otherwise.} \end{cases}$$

If $\mathbb{K}$ is a countable discrete field (as in the previous example), then the $\mathbb{K}$-algebra $A(\mathscr{P}, \preceq)$ linearly spanned over $\mathbb{K}$ by the set $\{e_{\alpha,\beta} \colon \alpha, \beta \in \mathscr{P},\ \alpha \prec \beta\}$ is a nil associative $\mathbb{K}$-algebra and

$$G(\mathscr{P}, \preceq) = 1 + A(\mathscr{P}, \preceq)$$

is a discrete algebra group over $\mathbb{K}$. By the way of example, if $\mathscr{P} = \{1, ..., n\}$ and $\preceq$ is the usual linear order $\leq$, then the corresponding McLain group is the unitriangular group $U_n(\mathbb{K})$.

Notice that if $(\mathscr{P}, \preceq)$ is a finite poset, the algebra $A(\mathscr{P}, \preceq)$ is a nilpotent algebra, which implies that $G(\mathscr{P}, \preceq)$ is a nilpotent group, hence amenable. On the other hand, if $(\mathscr{P}, \preceq)$ is an infinite countable poset and we choose any numbering $\mathscr{P} = \{\alpha_1, ..., \alpha_n, ...\}$, then the partial order $\preceq$ induces a partial order on $\mathscr{P}_n = \{\alpha_1, ..., \alpha_n\}$, for all $n \in \mathbb{N}$, and $G(\mathscr{P}, \preceq)$ can be realized as the direct limit

$$G(\mathscr{P}, \preceq) = \varinjlim_{n\in\mathbb{N}} G(\mathscr{P}_n, \preceq),$$

and thus it is an amenable group.

In the case where $\mathbb{K}$ is a finite field $(\mathscr{P}, \preceq)$ a finite poset, a supercharacter theory of the group $G(\mathscr{P}, \preceq)$ has been constructed in [34]; in the infinite case, in [85] the representation theory of $G(\mathscr{P}, \preceq)$ was studied *via* some sort of generalization of André's work on the finite unitriangular groups.

Example 3.3.4 (Algebra groups of prime characteristic associated with augmentation ideals)**.** Let p be a prime number, let $q = p^e$, $e \in \mathbb{N}$, and denote by $\mathbb{F}_q$ the finite field with q elements; given an arbitrary p-group P, we consider the group algebra $\mathbb{F}_q[P]$ and its augmentation ideal A (that is, the $\mathbb{F}_q$-vector space linearly spanned by the set $\{a-1\colon\, a \in P\}$). Then, A is a nil algebra and we can form the algebra group $G = 1+A$.

3.4 Supercharacters and ergodic measures on A°

In what follows, we fix an arbitrary amenable countable discrete algebra group $G = 1+A$ over a field $\mathbb{K}$, and we present a family of superclasses $\mathscr{K}$ of G such that the characters of G which are constant on the elements of $\mathscr{K}$ (to which we refer as supercharacters) form a Choquet simplex.

This construction is strongly based on [33] and generalizes the standard supercharacter theory for finite algebra groups given there; moreover, it is *loosely* based on Kirillov's orbit method (see [63] for the case of nilpotent real Lie groups) as we can think of A as a Lie algebra for G where the map $\vartheta : A \to G$ plays the role of the exponential map. Our method yields a generalization of the construction presented in [33], allowing us to exhibit a supercharacter formula, quite reminiscent of Kirillov's character formula, where supercharacters values are obtained as an integral over the closure of certain orbits on the dual group of A (see also [31] for the analogue version of Kirillov's orbit method for discrete rational nilpotent groups). Similarly to Proposition 3.2.2, supercharacters are in one-to-one correspondence with $\mathbb{G}$-ergodic measures on the dual group A° of the additive group A^+.

Consider the additive group A^+ of A, and its Pontryagin dual group A° of A; by definition, A° consists on all continuous group homomorphisms from A to the complex unit circle $\mathbb{S}^1$, and is equipped with the topology induced by convergence on compact sets (since A is discrete, this is nothing but the pointwise-convergence topology). We observe that, since abelian groups are tame, A° is in fact the set of indecomposable characters of A. As mentioned, A° admits a structure of an abelian topological group (that we write additively) which is determined by the pointwise product of functions: for every $\lambda, \lambda' \in A^\circ$, the element $\lambda+\lambda' \in A^\circ$ is defined by

$$(\lambda+\lambda')(a) = \lambda(a)\lambda'(a), \qquad a \in A,$$

and the identity element of A° is the trivial character $\mathbf{1}_A$ of A^+ (we sometimes write $0 = \mathbf{1}_A$).

Since A° is the set consisting of all indecomposable characters of A, it is a compact Hausdorff space (see for example [83, Proposition 1.2.5 (a)] for a proof not involving C^*-algebras). We consider A° equipped with its Borel σ-algebra of measurable sets, and the set consisting of all complex measures on A° will be denoted by $\mathscr{M}(A^\circ)$ (we assume that $\mathscr{M}(A^\circ)$ is equipped with the weak*-convergence topology). The set $C(A^\circ)$ consisting of all complex continuous functions is equipped with its usual uniform norm, and we identify its topological dual space with $\mathscr{M}(A^\circ)$ according to the Riesz-Markov-Kakutani theorem; consequently, any measure on A° is fully determined by the integration of complex continuous functions.

For every $g \in G$, consider the function $T_g : A^\circ \to \mathbb{C}$ defined by

$$T_g(\lambda) = \lambda(\vartheta^{-1}(g)) = \lambda(g-1), \qquad g \in G,$$

and denote by T_G the $\mathbb{C}$-linear span of the set $\{T_g \colon g \in G\}$.

Proposition 3.4.1. *For all $g \in G$, the function T_g is continuous. Furthermore, $\{T_g : g \in G\}$ is a linearly independent set, and T_G is a dense subalgebra of $C(A^\circ)$.*

Proof. According to Pontryagin duality theorem, the dual group $(A^\circ)^\circ$ of A° is canonically isomorphic to A: for all $a \in A$, the function $\widehat{a} : A^\circ \to \mathbb{C}$ defined by

$$\widehat{a}(\lambda) = \lambda(a), \qquad \lambda \in A^\circ,$$

is a character of A°, and every character of A° is of this form (see [83, Theorem 1.7.2] for a proof). On the other hand, $\vartheta^{-1} : G \to A$ is an homeomorphism and

$$T_g = \widehat{\vartheta^{-1}(g)}, \qquad g \in G,$$

which allow us to conclude that T_g is continuous for all $g \in G$.

We now identify $\{T_g : g \in G\}$ with $\{\widehat{a} : a \in A\}$. This set is linearly independent if and only if, for every mutually distinct elements $a_1, \ldots, a_n \in A$, a complex linear combination

$$\alpha_1 \widehat{a_1}(\lambda) + \cdots + \alpha_n \widehat{a_n}(\lambda) = 0$$

holds for all $\lambda \in A^\circ$ only if $\alpha_1 = ... = \alpha_n = 0$. We argue that this is the case by induction on n; the case $n = 1$ is trivial, hence we assume that $a_1, \ldots, a_{n+1} \in A$ are mutually distinct and that $\alpha_1, \ldots, \alpha_{n+1} \in \mathbb{C}$ are such that

$$\alpha_1 \widehat{a_1}(\lambda) + \cdots + \alpha_{n+1} \widehat{a_{n+1}}(\lambda) = 0, \qquad \lambda \in A^\circ.$$

Let $\lambda' \in A^\circ$ be such that $\widehat{a_1}(\lambda') \neq \widehat{a_{n+1}}(\lambda')$. Then, for every $\lambda \in A^\circ$ we have

$$\begin{aligned} 0 &= \alpha_1 \widehat{a_1}(\lambda'+\lambda) + \cdots + \alpha_{n+1}\widehat{a_{n+1}}(\lambda'+\lambda) \\ &= \alpha_1 \widehat{a_1}(\lambda')\widehat{a_1}(\lambda) + \ldots + \alpha_{n+1}\widehat{a_{n+1}}(\lambda')\widehat{a}_{n+1}(\lambda); \end{aligned}$$

on the other hand, we have

$$\alpha_1 \widehat{a}_{n+1}(\lambda')\widehat{a_1}(\lambda) + \cdots + \alpha_{n+1}\widehat{a_{n+1}}(\lambda')\widehat{a}_{n+1}(\lambda) = 0,$$

and thus

$$\alpha_1\big(\widehat{a_1}(\lambda') - \widehat{a_{n+1}}(\lambda')\big)\widehat{a_1}(\lambda) + \cdots + \alpha_n\big(\widehat{a_n}(\lambda') - \widehat{a_{n+1}}(\lambda')\big)\widehat{a_n}(\lambda) = 0.$$

By induction, it follows that

$$\alpha_i(\widehat{a_i}(\lambda') - \widehat{a_{n+1}}(\lambda')) = 0, \qquad 1 \leq i \leq n;$$

in particular, we conclude that $\alpha_1 = 0$, and this implies that $\alpha_2 = \ldots = \alpha_{n+1} = 0$ (again by induction). Therefore, $\{\widehat{a} :\ a \in A\}$, and hence $\{T_g :\ g \in G\}$, is a linearly independent set.

Now, for every $a, b \in A$, we have

$$(T_{1+a}T_{1+b})(\lambda) = \lambda(a)\lambda(b) = \lambda(a+b) = T_{1+a+b}(\lambda), \qquad \lambda \in A^\circ,$$

which means that $T_gT_h \in T_G$ for all $g, h \in G$; on the other hand, for every $a \in A$, we see that

$$\overline{T_{1+a}(\lambda)} = \overline{\lambda(a)} = \lambda(-a) = T_{1-a}(\lambda), \qquad \lambda \in A^\circ,$$

and hence $\overline{T_g} \in T_G$ for all $g \in G$. It follows that T_G is a subalgebra of $C(A^\circ)$ (notice that T_1 is the identity).

The set T_G separates elements of A°, because for every $\lambda, \lambda' \in A^\circ$, with $\lambda \neq \lambda'$, there is $a \in A$ such that

$$T_{1+a}(\lambda) = \lambda(a) \neq \lambda'(a) = T_{1+a}(\lambda').$$

In virtue of the Stone-Weierstrass theorem, we conclude that the subalgebra T_G is dense in $C(A^\circ)$, as stated. □

The density of T_G ensures that any measure on A° (equivalently, any linear continuous functional on $C(A^\circ)$) is fully determined by the integration values of the functions T_g for $g \in G$.

Indeed, let μ be a continuous linear functional defined on T_G, and let $(f_n)_{n\in\mathbb{N}}$ be a convergent sequence of continuous functions with limit point $f \in C(A^\circ)$. The continuity of μ ensures that

$$|\mu(f_n) - \mu(f_m)| \leq \|\mu\|_{\mathrm{op}} \, \|f_n - f_m\|_\infty, \qquad n,m \in \mathbb{N};$$

since $C(A^\circ)$ is a complete space, it follows that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Therefore, $(\mu(f_n))_{n\in\mathbb{N}}$ is also a Cauchy sequence, and thus it is convergent. It follows that μ extends naturally to a linear functional on $C(A^\circ)$, and it is straightforward to show that this extension is unique.

The group G induces a natural *adjoint* action (by automorphisms) on A, for all $g \in G$

$$g \cdot a = g^{-1} a g, \qquad a \in A.$$

Such action fully determines the conjugation action of G onto itself since

$$g^{-1}(1+a)g = 1 + (g^{-1}ag), \qquad g \in G,\ a \in A.$$

The adjoint action induces a *coadjoint* action on $A^\circ = \mathrm{Ex}(A)$, for all $\lambda \in A^\circ$

$$(g \cdot \lambda)(a) = \lambda(gag^{-1}), \qquad g \in G, a \in A.$$

Let $\varphi \in \mathrm{Char}(G)$, we define a continuous linear functional μ on $C(A^\circ)$ (equivalently, a measure on A°) such that

$$\mu(T_g) = \int_{A^\circ} T_g \, d\mu = \varphi(g),$$

furthermore, it is straightforward to check that the measure μ must be G-invariant (beacause φ is constant on conjugacy classes).

The first step towards the definition of a supercharacter theory of G is to exhibit a family of superclasses. We consider the direct product $\mathbb{G} = G \times G$, and the natural action of $\mathbb{G}$ on the left of A *via* left/right multiplication:

$$\mathbf{k} \cdot a = g^{-1} a h, \qquad \mathbf{k} = (g,h) \in \mathbb{G},\ a \in A;$$

If we identify G with the diagonal group $\Delta(G) = \{(g,g) : g \in G\}$, then we get the conjugation by restriction of the action of $\mathbb{G}$ to $\Delta(G)$. Moreover, every $\mathbb{G}$-orbit on A is a disjoint union of conjugacy G-orbits; indeed, we have

$$\mathbb{G} \cdot a = \bigcup_{g \in G} \Delta(G) \cdot (ag), \qquad a \in A.$$

Definition 3.4.2. If $G = 1 + A$ is an arbitrary countable discrete algebra group and $a \in A$, then we define the *superclass* of $g = 1 + a \in G$ to be the set

$$K_g = 1 + \mathbb{G}\cdot a = 1 + GaG,$$

and we denote by $\mathscr{K} = \{K_g : \ g \in G\}$ the set consisting of all superclasses of G.

Since $K_1 = \{1\} \in \mathscr{K}$ and every member $K \in \mathscr{K}$ is a union of conjugacy classes, the set $\mathscr{K}$ is in fact a superclass family. By a *superclass function* we mean a *bounded* complex function on G which takes a constant value on each superclass in $K \in \mathscr{K}$, and as before a character of G which is also a superclass function will be referred to as a *superclass character*. We denote by $\mathrm{SCl}_{\mathscr{K}}(G)$ and $\mathrm{SCl}^+_{\mathscr{K}}(G)$ the sets consisting of all superclass functions and of all superclass characters, respectively. The set $\mathrm{SCl}^+_{\mathscr{K}}(G)$ is clearly a convex set, and we denote by $\mathscr{E}_{\mathscr{K}}(G)$ (or, if there is no risk of confusion, simply by $\mathscr{E}$) the subset of $\mathrm{SCl}^+_{\mathscr{K}}(G)$ consisting of all indecomposable elements.

The $\mathbb{G}$-action on A yields the natural continuous contragradient action on the left of the dual group A°: for every $\mathbf{k} \in \mathbb{G}$ and every $\lambda \in A^\circ$, we define $\mathbf{k}\cdot\lambda \in A^\circ$ by

$$(\mathbf{k}\cdot\lambda)(a) = \lambda(\mathbf{k}^{-1}\cdot a), \qquad a \in A.$$

For simplicity, for every $g \in G$, we write

$$g\lambda = (g,1)\cdot\lambda \qquad \text{and} \qquad \lambda g = (1,g)\cdot\lambda = \lambda g,$$

and thus it makes sense to talk about the *left/right* action of G on A° and left/right G-orbits on A°.

On the other hand, we consider the sets $\mathscr{M}_{\mathbb{G}}(A^\circ)$ and $\mathscr{M}^+_{\mathbb{G}}(A^\circ)$ consisting of all $\mathbb{G}$-invariant measures and $\mathbb{G}$-invariant probability measures on A°, respectively, and equip both with the weak*-convergence topology. Notice that both $\mathscr{M}_{\mathbb{G}}(A^\circ)$ and $\mathscr{M}^+_{\mathbb{G}}(A^\circ)$ are non-empty (because the Dirac measure $\delta_{\mathbf{1}_A}$ supported on the trivial character of A^+ is clearly a $\mathbb{G}$-invariant probability measure); furthermore, $\mathscr{M}^+_{\mathbb{G}}(A^\circ)$ is a Choquet Simplex (due to Proposition 2.1.6).

Lemma 3.4.3. *For every $\mathbb{G}$-invariant measure μ on A°, we have*

$$\int_{A^\circ} T_g\overline{T_h}\, d\mu = \int_{A^\circ} T_{h^{-1}g}\, d\mu, \qquad g,h \in G.$$

Proof. Let $a, b \in A$ be arbitrary, and set $g = 1+a$ and $h = 1+b$. Firstly, we claim that

$$T_h(h\lambda) = \overline{T_{h^{-1}}(\lambda)}.$$

To see this, let $b' \in A$ be such that $h^{-1} = 1+b'$, and notice that $b+b'b = -b'$ (because $h^{-1}h = 1+b+b'+b'b = 1$). Then,

$$\begin{aligned} T_h(h\lambda) &= (h\lambda)(h-1) = \lambda(h^{-1}(h-1)) = \lambda(1-h^{-1}) \\ &= \lambda(-b') = \overline{\lambda(b')} = \overline{T_{1+b'}(\lambda)} = \overline{T_{h^{-1}}(\lambda)}. \end{aligned}$$

Consequently,

$$\int_{A^\circ} T_g(\lambda)\overline{T_h(\lambda)}\, d\mu = \int_{A^\circ} T_g(\lambda) T_{h^{-1}}(h^{-1}\lambda)\, d\mu = \int_{A^\circ} T_g(h\lambda) T_{h^{-1}}(\lambda)\, d\mu$$

(the last equality because μ is $\mathbb{G}$-invariant). Since

$$T_g(h\lambda)T_{h^{-1}}(\lambda) = \lambda(h^{-1}a)\lambda(a') = \lambda(h^{-1}a + b') = \lambda(a + b'a + b')$$

and $h^{-1}g = 1+b'+a+b'a$, we conclude that

$$\int_{A^\circ} T_g(\lambda)\overline{T_h(\lambda)}\, d\mu = \int_{A^\circ} T_{h^{-1}g}(\lambda)\, d\mu,$$

and this completes the proof. □

For every superclass function φ on G, we define a linear functional μ^φ on $C(A^\circ)$ (equivalently, a measure on A°) (*via* linearity and continuous extension) by

$$\mu^\varphi(T_g) = \int_{A^\circ} T_g\, d\mu^\varphi = \varphi(g), \qquad g \in G.$$

Notice that, if $\varphi \neq \varphi'$ are superclass functions on G, then there is $g \in G$ such that

$$\mu^\varphi(T_g) = \varphi(g) \neq \varphi'(g) = \mu^{\varphi'}(T_g),$$

which means that the mapping $\varphi \mapsto \mu^\varphi$ defines an injective map. On the other hand, the fact that φ is a superclass function ensures that μ^φ is a $\mathbb{G}$-invariant measure: indeed, for every $\mathbf{k} \in \mathbb{G}$ and every $a \in A$, we evaluate

$$\begin{aligned} \int_{A^\circ} T_{1+a}(\mathbf{k}^{-1}\cdot\lambda)\, d\mu^\varphi &= \int_{A^\circ} T_{1+\mathbf{k}\cdot a}(\lambda)\, d\mu^\varphi = \varphi(1+\mathbf{k}\cdot a) \\ &= \varphi(1+a) = \int_{A^\circ} T_{1+a}(\lambda)\, d\mu^\varphi. \end{aligned}$$

Conversely, for every $\mathbb{G}$-invariant measure $\mu \in \mathscr{M}_{\mathbb{G}}(A^\circ)$, we define the function $\varphi^\mu : G \to \mathbb{C}$ by

$$\varphi^\mu(g) = \int_{A^\circ} T_g \, d\mu, \qquad g \in G.$$

If μ and μ' are two distinct measures in $\mathscr{M}_{\mathbb{G}}(A^\circ)$, then there is an element $f \in C(A^\circ)$ such that

$$\int_{A^\circ} f \, d\mu \neq \int_{A^\circ} f \, d\mu'.$$

Since T_G is dense in $C(A^\circ)$, there must exist at least one element $g \in G$ such that

$$\int_{A^\circ} T_g \, d\mu \neq \int_{A^\circ} T_g \, d\mu'.$$

Therefore, the mapping $\mu \mapsto \varphi^\mu$ defines an injective map. Moreover, the fact that μ is $\mathbb{G}$-invariant implies that φ^μ is a superclass function on G; in fact, for every $g \in G$ and every $\mathbf{k} \in \mathbb{G}$, we have

$$(\mathbf{k}\cdot T_g)(\lambda) = T_g(\mathbf{k}^{-1}\cdot\lambda) = \lambda(\mathbf{k}\cdot(g-1)) = T_{1+\mathbf{k}\cdot(g-1)}(\lambda),$$

and thus

$$\varphi^\mu(1+\mathbf{k}\cdot(g-1)) = \int_{A^\circ} T_{1+\mathbf{k}\cdot(g-1)} \, d\mu = \int_{A^\circ} (\mathbf{k}\cdot T_g) \, d\mu = \int_{A^\circ} T_g \, d\mu = \varphi^\mu(g)$$

where we use the fact that μ is $\mathbb{G}$-invariant (in the third equality).

Keeping the notation as above, we now prove the following.

Proposition 3.4.4. *The mapping $\varphi \mapsto \mu^\varphi$ defines an affine homeomorphism between $\mathrm{SCl}_{\mathscr{K}}(G)$ and $\mathscr{M}_{\mathbb{G}}(A^\circ)$ with inverse given by the mapping $\mu \mapsto \mu^\varphi$.*

Furthermore, the measures in $\mathscr{M}_G^+(A^\circ)$ are in one-to-one correspondence with the elements of $\mathrm{SCl}^+_{\mathscr{K}}(G)$ (and hence elements in $\mathscr{E}$ correspond to $\mathbb{G}$-ergodic measures on A°).

Proof. The fact that the above correspondence is an affine homeomorphism is a matter of straightforward calculations.

We now claim that the measure μ associated with a superclass character φ is a probability measure. Let $B \in \mathscr{B}(A^\circ)$ be arbitrary, and let $\mathbb{I}_B \in L^2(A^\circ, \mu)$ be the corresponding indicator function. Since T_G is dense in $C(A^\circ)$, so is its image in $L^2(A^\circ, \mu)$, and hence

$$\mathbb{I}_B = \lim_{n\to\infty} \sum_{i=1}^n \alpha_i T_{g_i}$$

for some family $\{\alpha_i T_{g_i} \colon \alpha_i \in \mathbb{C},\ g_i \in G\}_{i\in\mathbb{N}}$. Since

$$\overline{\mathbb{I}_B} = \lim_{n\to\infty} \sum_{i=1}^n \overline{\alpha_i}\overline{T_{g_i}} \qquad \text{and} \qquad \mathbb{I}_B(\lambda) = \mathbb{I}_B(\lambda)\overline{\mathbb{I}_B(\lambda)}, \quad \lambda \in A^\circ,$$

we see that

$$\mu(B) = \lim_{n\to\infty} \sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \int_{A^\circ} T_{g_i} \overline{T_{g_j}}\, d\mu.$$

According to Lemma 3.4.3, we conclude that

$$\mu(B) = \lim_{n\to\infty} \sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \int_{A^\circ} T_{g_j^{-1} g_i}\, d\mu = \lim_{n\to\infty} \sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \varphi(g_j^{-1} g_i).$$

Since φ is a positive definite function, we have

$$\sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \varphi(g_j^{-1} g_i) \geq 0, \qquad n \in \mathbb{N},$$

and thus $\mu(B) \geq 0$. On the other hand,

$$\mu(A^\circ) = \int_{A^\circ} T_1\, d\mu = \varphi(1) = 1,$$

and so μ is a probability measure.

Conversely, in order to prove that every probability measure $\mu \in \mathscr{M}_{\mathbb{G}}^{+}(A^\circ)$ determines a superclass character, we exhibit a cyclic representation which affords the character φ^μ. Let $\mathscr{H}^\mu$ be the Hilbert space $L^2(A^\circ, \mu)$, and for every $f \in \mathscr{H}^\mu$ and every $g \in G$, define the operator $\mathscr{T}^\mu(g) : \mathscr{H}^\mu \to \mathscr{H}^\mu$ by

$$(\mathscr{T}^\mu(g)f)(\lambda) = T_g(\lambda) f(g^{-1}\lambda), \qquad \lambda \in A^\circ.$$

We claim that $\mathscr{T}^\mu(g)$ is a unitary operator for all $g \in G$. Firstly, for every $g \in G$, we compute the adjoint operator of $\mathscr{T}^\mu(g)$: for every $f_1, f_2 \in \mathscr{H}^\mu$, we evaluate

$$\begin{aligned}
\langle \mathscr{T}^\mu(g) f_1 | f_2 \rangle &= \int_{A^\circ} T_g(\lambda) f_1(g^{-1}\lambda) \overline{f_2(\lambda)}\, d\mu = \int_{A^\circ} T_g(g\lambda) f_1(\lambda) \overline{f_2(g\lambda)}\, d\mu \\
&= \int_{A^\circ} f_1(\lambda) \overline{T_{g^{-1}}(\lambda) f_2(g\lambda)}\, d\mu = \langle f_1 | \mathscr{T}^\mu(g^{-1}) f_2 \rangle
\end{aligned}$$

(in the second equality, we took into account that μ is $\mathbb{G}$-invariant), and thus the adjoint operator of $\mathscr{T}^\mu(g)$ is $\mathscr{T}^\mu(g^{-1})$. On the other hand, we have

$$\mathscr{T}^\mu(g) \mathscr{T}^\mu(g^{-1}) = \mathscr{T}^\mu(g^{-1}) \mathscr{T}^\mu(g) = \mathrm{Id}, \qquad g \in G,$$

where $\mathrm{Id} \colon \mathscr{H}^\mu \to \mathscr{H}^\mu$ is the identity operator, and hence $\mathscr{T}_g$ is unitary for all $g \in G$. It is easy to check that the map $\mathscr{T}^\mu \colon G \to \mathrm{U}(\mathscr{H}^\mu)$ (defined by the mapping $g \mapsto \mathscr{T}^\mu(g)$) is a group homomorphism, and so it is a representation of G.

Since T_G is a dense subalgebra in $C(A^\circ)$, its image in $\mathscr{H}^\mu$ is also dense; moreover, we have

$$\mathscr{T}^\mu(g) T_1 = T_g, \qquad g \in G,$$

and so T_1 is a cyclic vector and the representation $(\mathscr{T}^\mu, \mathscr{H}^\mu)$ affords a character given by the formula

$$\langle \mathscr{T}^\mu(g) T_1 | T_1 \rangle = \int_{A^\circ} T_g \, d\mu = \varphi^\mu(g), \qquad g \in G,$$

which implies that φ^μ is a superclass character of G, as required.

The rest of the statements follows from the fact that $\mathbb{G}$-ergodic measures are the indecomposable elements of $\mathscr{M}^+_{\mathbb{G}}(A^\circ)$. □

For an arbitrary $\mathbb{G}$-invariant probability measure μ on A°, the representation $(\mathscr{T}^\mu, \mathscr{H}^\mu)$ will be referred to as the *standard super-representation* associated with μ, since it provides a canonical model for representing the superclass character φ^μ.

On the other hand, elements in $\mathscr{E}$ are enough to describe any superclass function in the following sense.

Proposition 3.4.5. *For every $\varphi \in \mathrm{SCl}_{\mathscr{K}}(G)$, there is a measure μ^* (in general, complex) on $\mathscr{E}$ such that*

$$\varphi(g) = \int_{\mathscr{E}} \chi(g) \, d\mu^*, \qquad g \in G.$$

Proof. Let $\varphi \in \mathrm{SCl}_{\mathscr{K}}(G)$ with associated $\mathbb{G}$-invariant measure $\mu = \mu_1 + i\mu_2$ on A°, where i denotes the imaginary unit and μ_1 and μ_2 are real signed measures. Since μ is $\mathbb{G}$-invariant, it is clear that both μ_1 and μ_2 are also $\mathbb{G}$-invariant. Let $j = 1,2$, and consider the Hann-Jordan decomposition $\mu_j = \mu_j^+ - \mu_j^-$ of μ_j; recall that for every $X \in \mathscr{B}(A^\circ)$

$$\mu_j^+(X) = \sup_{Y \subseteq X} \mu_j(Y) \qquad \text{and} \qquad \mu_j^- = -\inf_{Y \subseteq X} \mu_j(Y),$$

where the supremum and infimum are taken over all Borel sets $Y \subseteq X$, and hence both μ_j^+ and μ_j^- are $\mathbb{G}$-invariant measures. We set $\alpha_j^\pm = \mu_j^\pm(A^\circ)$, and notice that

$$\nu_j^\pm = \frac{1}{\alpha_j^\pm} \mu_j^\pm$$

are $\mathbb{G}$-invariant probability measures. Let $\mathrm{Erg}_{\mathbb{G}}(A^\circ)$ stand for the set consisting of all $\mathbb{G}$-invariant ergodic measures on A° (which are the indecomposable elements of the Choquet Simplex $\mathscr{M}^+_{\mathbb{G}}(A^\circ)$). Then, there are unique probability measures $(\nu_j^\pm)^* \in \mathscr{M}^+(\mathrm{Erg}_{\mathbb{G}}(A^\circ))$ such that

$$\nu_j^\pm = \int_{\mathrm{Erg}_{\mathbb{G}}(A^\circ)} \omega \, d(\nu_j^\pm)^*.$$

Consequently, if we define $(\mu_j^{\pm})^* = \alpha_j^{\pm}(\nu_j^{\pm})^*$ and $\mu_j^* = (\mu_j^+)^* - (\mu_j^-)^*$, then we conclude that

$$\mu = \int_{\mathrm{Erg}_{\mathbb{G}}(A^\circ)} \omega \, d(\mu_1^* + i\mu_2^*),$$

the result follows (by setting $\mu^* = \mu_1^* + i\mu_2^*$). □

In this fashion, we can formulate superclass functions theoretical problems in terms of $\mathbb{G}$-invariant measures on A°; since supercharacters allow us to describe all superclass functions, the description of ergodic measures yields a description of the standard supercharacter theory and the corresponding superclass functions.

If G is a finite algebra group, then the dual group A° is finite and equipped with the discrete topology (that is, any subset is a Borel set). Therefore, it is clear that every $\mathbb{G}$-orbit on A° supports a unique ergodic measure, and that any ergodic measure is of this form; in fact, for any $\mathbb{G}$-orbit $\mathbb{G}\boldsymbol{\cdot}\lambda \subseteq A^\circ$ the corresponding ergodic measure, which we will denote by ω_λ, is defined as follows: for every $\lambda' \in A^\circ$

$$\omega_\lambda(\lambda') = \begin{cases} \frac{1}{|\mathbb{G}\boldsymbol{\cdot}\lambda|}, & \text{if } \lambda' \in \mathbb{G}\boldsymbol{\cdot}\lambda, \\ 0, & \text{otherwise.} \end{cases}$$

As it is expected, we recover the (normalized) supercharacter formula given in [33]: for every $\lambda \in A^\circ$, let $\chi^{\mathbb{G}\boldsymbol{\cdot}\lambda}$ denote the supercharacter associated with the $\mathbb{G}$-orbit $\mathbb{G}\boldsymbol{\cdot}\lambda \subseteq A^\circ$; then,

$$\chi^{\mathbb{G}\boldsymbol{\cdot}\lambda}(g) = \int_{A^\circ} T_g \, d\omega_\lambda = \frac{1}{|\mathbb{G}\boldsymbol{\cdot}\lambda|} \sum_{\lambda' \in \mathbb{G}\boldsymbol{\cdot}\lambda} \lambda'(g-1), \qquad g \in G.$$

However, in [33] this formula is achieved using purely representation theoretical arguments, while in our proof, we rely heavily on ergodic theory (which allows a different point of view on supercharacters in the finite group case).

For an arbitrary infinite countable algebra group, any $\mathbb{G}$-orbit $\mathbb{G}\boldsymbol{\cdot}\lambda \subseteq A^\circ$ is a discrete subset of an infinite compact topological space, and thus it is closed if and only if it is finite. Consequently, no infinite $\mathbb{G}$-orbit can support a $\mathbb{G}$-invariant measure. The way to bypass this issue, is to consider *orbit closures*: for every $\lambda \in A^\circ$, we will denote by $\mathscr{O}^\lambda$ the closure in A° of $\mathbb{G}\boldsymbol{\cdot}\lambda$. Our next goal is to associate supercharacters with orbit closures.

Recall that, for every probability measure ω on A°, its support $\mathrm{supp}(\omega)$ consists of all characters $\lambda \in A^\circ$ for which for every open neighborhood has positive measure; equivalently, it is the smallest closed subset C of A° such that $\omega(A^\circ \setminus C) = 0$.

Proposition 3.4.6. *For every $\mathbb{G}$-invariant ergodic measure ω on A° there is at least one $\lambda \in A^\circ$ such that* $\operatorname{supp}(\omega) = \mathscr{O}^\lambda$*. Conversely, every orbit closure supports a unique $\mathbb{G}$-invariant ergodic measure.*

Proof. Let ω be a $\mathbb{G}$-ergodic measure on A°, and consider its support $\operatorname{supp}(\omega)$ equipped with the subspace topology. The group $\mathbb{G}$ acts on $\operatorname{supp}(\omega)$, and ω can be though naturally as a $\mathbb{G}$-ergodic measure on $\operatorname{supp}(\omega)$ having full support. Let us choose a topological basis $\{U_n\}_{n\in\mathbb{N}}$ for $\operatorname{supp}(\omega)$; thus, $\omega(U_n) > 0$ for all $n \in \mathbb{N}$. Since $\mathbb{G}\cdot U_n$ is a $\mathbb{G}$-invariant set of positive measure and ω is ergodic, we must have $\omega(\mathbb{G}\cdot U_n) = 1$. Moreover, the family $\{\mathbb{G}\cdot U_n\}_{n\in\mathbb{N}}$ is also a topological basis for $\operatorname{supp}(\omega)$. Let

$$V = \bigcap_{n\in\mathbb{N}} \mathbb{G}\cdot U_n,$$

and note that, since V is an intersection of sets with measure 1, we also have $\omega(V) = 1$ and this clearly implies that V is non-empty. Furthermore, V is $\mathbb{G}$-invariant, and thus for every $\lambda \in V$, the $\mathbb{G}$-orbit $\mathbb{G}\cdot\lambda$ intersects every element of a topological basis of $\operatorname{supp}(\omega)$. Consequently, $\mathbb{G}\cdot\lambda$ is dense in $\operatorname{supp}(\omega)$, which means that its closure $\mathscr{O}^\lambda = \overline{\mathbb{G}\cdot\lambda}$ equals $\operatorname{supp}(\omega)$.

On the other hand, running a similar argument to the used in the proof of Proposition **??**, we conclude that the amenability of $\mathbb{G}$ implies that any orbit closure $\mathscr{O}^\lambda$ supports a unique $\mathbb{G}$-ergodic measure. □

Let $\Omega = \{\mathscr{O}^\lambda : \lambda \in A^\circ\}$ denote the *orbit-closure* space, and notice that the previous proposition ensures that there is a one-to-one correspondence between Ω and the $\mathbb{G}$-ergodic measures on A° (and hence between Ω and the supercharacters of G). Henceforth, we fix the following notation. For every $\mathscr{O} = \mathscr{O}^\lambda \in \Omega$, we denote by either $\omega_{\mathscr{O}}$ or ω_λ the $\mathbb{G}$-ergodic measure supported on $\mathscr{O}$, and by $\chi^{\mathscr{O}}$ or χ^λ the corresponding supercharacter of G ; hence, $\mathscr{E} = \{\chi^{\mathscr{O}} : \mathscr{O} \in \Omega\}$; furthermore, we denote either by $(\mathscr{T}^{\mathscr{O}}, \mathscr{H}^{\mathscr{O}})$ or $(\mathscr{T}^\lambda, \mathscr{H}^\lambda)$ the standard super-representation associated with $\omega_{\mathscr{O}} = \omega_\lambda$.

The correspondence between supercharacters and orbit closures, together with Proposition 3.4.4, allow us to establish the following supercharacter formula.

Proposition 3.4.7 (Orbit supercharacter formula)**.** *For every orbit closure $\mathscr{O} \in \Omega$ with associated $\mathbb{G}$-ergodic measure $\omega_{\mathscr{O}}$ and supercharacter $\chi^{\mathscr{O}}$, we have*

$$\chi^{\mathscr{O}}(g) = \int_{\mathscr{O}} T_g(\lambda)\, d\omega_{\mathscr{O}} = \int_{\mathscr{O}} \lambda(g-1)\, d\omega_{\mathscr{O}}, \qquad g \in G.$$

In this fashion, the representation theory of countable discrete algebra groups is brought into an ergodic theoretical setting, where the description of the standard supercharacter theory is equivalent to the description of $\mathbb{G}$-orbit closures and the corresponding $\mathbb{G}$-ergodic measures. Although the correspondence between supercharacters and orbit closures establishes a parallel with the finite group algebra scenario, infinite countable discrete algebra groups can exhibit a different behavior depending on the nature of the $\mathbb{G}$-action on A°. Namely, there are examples, such as $U_\infty(\mathbb{F}_q)$, where there exist a dense $\mathbb{G}$-orbit; as we explain in the following section this is the case if and only if the regular character of G is a supercharacter. (We observe that, being the closure of a $\mathbb{G}$-orbit, $\mathscr{O}^\lambda$ is $\mathbb{G}$-invariant, and hence it is a union of $\mathbb{G}$-orbits; therefore, it is not so odd to have distinct $\mathbb{G}$-orbits associated with the same supercharacter, contrary to the case of finite algebra groups.)

3.4.1 The regular representation

The group G acts on itself *via* left multiplication, and this induces a left action of G on $L^2(G,d)$, where d is the counting measure on G (which serves as a Haar measure for G): for every $g \in G$ and every $f \in L^2(G,d)$, we define $\pi(g)f \in L^2(G,d)$ by

$$\pi(g)f(x) = f(g^{-1}x), \qquad x \in G.$$

It is a matter of straightforward calculations to show that the pair $(\pi, L^2(G,d))$ is a unitary representation of G, to which we refer as the (left) *regular representation*. (Since the regular representation is defined *via* the group multiplication, this is the most natural representation that one can consider.)

For every $g \in G$, let $\delta_g \in L^2(G,d)$ be the Dirac function supported on g, that is, $\delta_g(g) = 1$, and $\delta_g(h) = 0$ for all $h \in G$, $h \neq g$. The $\mathbb{C}$-linear span of the set $\{\delta_g \colon g \in G\}$ is dense in $L^2(G,d)$, and since $\pi(g)\delta_1 = \delta_g$ for all $g \in G$, we see that the function δ_1 is a cyclic vector for $(\pi, L^2(G,d))$. Consequently, the regular representation affords a character, which we will denote by ρ and refer to as the *regular character* of G; hence, for every $g \in G$

$$\rho(g) = \langle \pi(g)\delta_1 | \delta_1 \rangle = \begin{cases} 1, & \text{if } g = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The regular character is clearly a superclass character of G, and thus it is uniquely determined by a unique $\mathbb{G}$-invariant measure on A°. In what follows, we characterize the regular

character in the context of the standard supercharacter theory by understanding the measure which is associated with ρ; in particular, we provide a criterium in terms of the $\mathbb{G}$-action on A°, for the regular character to be a supercharacter.

Let η denote the unique probability Haar measure on A° (recall that, being an abelian group, A° is unimodular and the compactness of A° ensures uniqueness). For every $\mathbf{k} \in \mathbb{G}$, we define the measure $\mathbf{k}\cdot\eta$ on A° by

$$(\mathbf{k}\cdot\eta)(B) = \eta(\mathbf{k}^{-1}\cdot B), \qquad B \in \mathscr{B}(A^\circ);$$

notice that $(\mathbf{k}\cdot\eta)(A^\circ) = \eta(A^\circ) = 1$, and hence $\mathbf{k}\cdot\eta$ is a probability measure.

Lemma 3.4.8. *The measure η is $\mathbb{G}$-invariant.*

Proof. Let $\mathbf{k} \in \mathbb{G}$ and $\lambda \in A^\circ$ be arbitrary. Then, since η is A°-invariant, we deduce that

$$(\mathbf{k}\cdot\eta)(\lambda + B) = \eta(\mathbf{k}^{-1}\cdot\lambda + \mathbf{k}^{1}\cdot B) = \eta(\mathbf{k}^{-1}\cdot B) = (\mathbf{k}\cdot\eta)(B), \qquad B \in \mathscr{B}(A^\circ),$$

that is, the measure $\mathbf{k}\cdot\eta$ is A°-invariant. Being a probability measure, the uniqueness of the Haar measure ensures that $\mathbf{k}\cdot\eta = \eta$, and thus η is $\mathbb{G}$-invariant. $\square$

Consequently, we can consider the corresponding standard super-representation $(\mathscr{T}^\eta, \mathscr{H}^\eta)$ where $\mathscr{H}^\eta = L^2(A^\circ, \eta)$ and

$$(\mathscr{T}^\eta(g)f)(\lambda) = T_g(\lambda)f(g^{-1}\lambda), \qquad g \in G,\ f \in \mathscr{H}^\eta, \lambda \in A^\circ.$$

Our next goal is to show that $(\mathscr{T}^\eta, \mathscr{H}^\eta)$ is equivalent to the regular representation. In order to do so, we recall some facts of the harmonic analysis of abelian groups (the details can be found in [83]).

Since the group A is countable and discrete, we may consider the counting measure d_0 as its Haar measure. For every function $F : A \to \mathbb{C}$ with compact support (equivalently, with finite support), the corresponding *Fourier transform* is the function $\mathscr{F}(F) : A^\circ \to \mathbb{C}$ defined by

$$\mathscr{F}(F)(\lambda) = \sum_{a \in A} F(a)\lambda(a), \qquad \lambda \in A^\circ.$$

Since F has finite support, its Fourier transform $\mathscr{F}(F)$ is continuous, and therefore measurable; moreover, if $\operatorname{supp}(F) = \{a_1, \ldots, a_n\}$, then

$$\int_{A^\circ} |\mathscr{F}(F)|^2 \, d\eta \le \sum_{i=1}^{n} |F(a_i)|^2 \int_{A^\circ} |\lambda(a_i)|^2 \, d\eta < \infty,$$

and hence $\mathscr{F}(F) \in L^2(A^\circ, \eta)$. Due to the fact that functions with compact support form a dense subset of $L^2(A, d_0)$, the Fourier transform admits a unique extension to an operator $\mathscr{F}: L^2(A, d_0) \to L^2(A^\circ, \eta)$. Such an operator admits an inverse: the *inverse Fourier transform* of a function $f \in L^2(A^\circ, \eta)$ is the function $\mathscr{F}^{-1}(f) \in L^2(A, d_0)$ defined by

$$\mathscr{F}^{-1}(f)(a) = \int_{A^\circ} f(\lambda)\overline{\lambda(a)}\, d\eta, \qquad a \in A.$$

Furthermore, both $\mathscr{F}$ and $\mathscr{F}^{-1}$ are continuous unitary operators.

Proposition 3.4.9. *Given a countable discrete algebra group $G = 1 + A$ (not necessarily amenable). the linear operator $\mathscr{L}: \mathscr{H}^\eta \to L^2(G, d)$ given by*

$$\mathscr{L}(f)(g) = \int_{A^\circ} f(\lambda)\overline{\lambda(g-1)}\, d\eta, \qquad g \in G,$$

defines an invertible intertwining operator between the representations $(\mathscr{T}^\eta, \mathscr{H}^\eta)$ and $(\pi, L^2(G, d))$ of G, whose inverse is defined on the functions with finite support $F \in C_c(G)$ by

$$\mathscr{L}^{-1}(F)(\lambda) = \sum_{g \in G} F(g)\lambda(g-1), \qquad \lambda \in A^\circ.$$

Proof. Firstly, we observe that the map $\vartheta^{-1}: G \to A$ (defined by the mapping $g \mapsto g-1$) induces a unitary isomorphism of Hilbert spaces $\theta^*: L^2(G, d) \to L^2(A, d_0)$. Furthermore, we have $\mathscr{L} = \mathscr{F}^{-1} \circ \theta^*$, and hence it only remains to show that $\mathscr{L}$ is an intertwining operator. To see this, let $g \in G$ and $f \in \mathscr{H}^\eta$ be arbitrary; then, for every $h \in G$ we evaluate

$$\begin{aligned}
\mathscr{L}\big(\mathscr{T}^\eta(g)f\big)(h) &= \int_{A^\circ} \big(\mathscr{T}^\eta(g)f\big)(\lambda)\,\overline{\lambda(h-1)}\, d\eta \\
&= \int_{A^\circ} T_g(\lambda)f(g^{-1}\lambda)\overline{\lambda(h-1)}\, d\eta \\
&= \int_{A^\circ} \lambda(g-1)f(g^{-1}\lambda)\overline{\lambda(h-1)}\, d\eta \\
&= \int_{A^\circ} (g\lambda)(g-1)f(\lambda)(g\lambda)(-h+1)\, d\eta \\
&= \int_{A^\circ} f(\lambda)\lambda(1-g^{-1})\lambda(-g^{-1}h+g^{-1})\, d\eta \\
&= \int_{A^\circ} f(\lambda)\overline{\lambda(g^{-1}h-1)}\, d \\
&= \big(\pi(g)\mathscr{L}(f)\big)(h),
\end{aligned}$$

as required. $\square$

Consequently, the probability Haar measure η on A° fully determines the regular representation of G. Since $\operatorname{supp}(\eta) = A^\circ$, Proposition 3.4.6 implies the following immediate corollary.

Corollary 3.4.10. *Let G be an amenable discrete countable algebra group. Then, the regular character of G is a supercharacter if and only if there is a dense $\mathbb{G}$-orbit in A°.*

Hence, the nature of the regular character, as a superclass character, is not intrinsic to the class of amenable discrete countable algebra groups, but rather to the nature of the $\mathbb{G}$-action. As explained in Chapter 6, the regular character of the infinite unitriangular group $U_n(\mathbb{F})$ is not a supercharacter; however, the regular character of the locally finite unitriangular group $U_\infty(\mathbb{F}_q)$ is a supercharacter. Nevertheless, the operators $\mathscr{L}$ and $\mathscr{L}^{-1}$ always establish an isometry between $L^2(G,d)$ and $L^2(A^\circ,\eta)$, which is to be understood as a Fourier transform for countable discrete algebra groups (not necessarily amenable).

3.5 The super-dual space topology

The correspondence between supercharacters of G and $\mathbb{G}$-ergodic measures on A° provides a fairly good understanding of supercharacters at the individual level. In this section, we study supercharacters as a single topological object.

For an arbitrary topological group G, its *dual object* is by definition the set consisting of all equivalence classes of irreducible representations; however, for non-type I groups, due to its poor decomposition theory in terms of irreducible representations, the set consisting of all quasi-equivalence classes of factor representations, called the *quasi-dual object*, is considered as dual space. In the context of supercharacters, when G is an amenable countable discrete algebra group, equipped with its standard supercharacter theory $(\mathscr{K},\mathscr{E})$, it is only reasonable to consider $\mathscr{E}$ as a *super-dual object*, since every superclass character decomposes uniquely as an integral over $\mathscr{E}$. Since the decomposition of superclass characters is obtained *via* a Borel measure on $\mathscr{E}$, it seems to be relevant to understand the topology on $\mathscr{E}$. However, the set $\mathscr{E}$ reveals to have some limitations as a dual space; indeed, it may be used to decompose superclass characters, but this decomposition does not translate into a decomposition of the corresponding super-representations.

Let $C(A^\circ)^*$ denote the topological linear dual of $C(A^\circ)$ equipped with the usual operator norm. Besides establishing a bijection between $C(A^\circ)^*$ and $\mathscr{M}(A^\circ)$, the Riesz-Markov-Kakutani representation theorem also states that for every $\psi \in C(A^\circ)^*$, with corresponding measure $\mu \in \mathscr{M}(A^\circ)$, the operator norm of ψ equals the total variation norm of μ. On the other

hand, since A° is separable, the unit ball

$$\mathscr{B} = \{\mu \in \mathscr{M}(A^\circ) \colon \|\mu\| \leq 1\}$$

is Hausdorff and compact for the weak*-topology; thus, based on [39, Lemma 3.101], it is possible to present an explicit metrization of $\mathrm{SCl}^+_{\mathscr{K}}(G)$.

Proposition 3.5.1. *Fix a numbering $g_1, g_2, \ldots, g_n, \ldots$ of G, and define the function*

$$d \colon \mathrm{SCl}^+_{\mathscr{K}}(G) \times \mathrm{SCl}^+_{\mathscr{K}}(G) \to \mathbb{R}^+_0$$

by

$$d(\varphi, \varphi') = \sum_{n=1}^{\infty} \frac{1}{2^n} |\varphi(g_n) - \varphi'(g_n)|, \qquad \varphi, \varphi' \in \mathrm{SCl}^+_{\mathscr{K}}(G).$$

Then, d is a metric on $\mathrm{SCl}^+_{\mathscr{K}}(G)$ which is compatible with the weak-topology.*

Proof. Let $\varphi, \varphi' \in \mathrm{SCl}^+_{\mathscr{K}}(G)$ be arbitrary. It is clear that $d(\varphi, \varphi') = 0$ if and only if $\varphi = \varphi'$. On the other hand, for every $\psi \in \mathrm{SCl}^+_{\mathscr{K}}(G)$, we have

$$\begin{aligned} d(\varphi, \psi) + d(\psi, \varphi') &= \sum_{n=1}^{\infty} \frac{1}{2^n} \big(|\varphi(g_n) - \psi(g_n)| + |\psi(g_n) - \varphi'(g_n)| \big) \\ &\geq \sum_{n=1}^{\infty} \frac{1}{2^n} \big(|\varphi(g_n) - \psi(g_n) + \psi(g_n) - \varphi'(g_n)| \big) = d(\varphi, \varphi'), \end{aligned}$$

which implies that d is in fact a metric on $\mathrm{SCl}^+_{\mathscr{K}}(G)$.

Now, let $(\varphi_i)_{i \in \mathbb{N}}$ be a convergent sequence of superclass characters with limit point φ; notice that

$$|\varphi_i(g) - \varphi(g)| \leq |\varphi_i(g)| + |\varphi(g)| \leq 1 + 1 = 2, \qquad g \in G,\ i \in \mathbb{N}.$$

Furthermore, for every $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty} \frac{1}{2^{n-1}} < \frac{\varepsilon}{2}.$$

On the other hand, there is $i_0 \in \mathbb{N}$ such that

$$\sum_{n=1}^{n_0} \frac{1}{2^n} |\varphi_i(g_n) - \varphi(g_n)| < \frac{\varepsilon}{2}, \qquad i \geq i_0.$$

It follows that

$$d(\varphi_i, \varphi) < \varepsilon, \qquad i \geq i_0,$$

and this means that the sequence $(d(\varphi_i, \varphi))_{i \in \mathbb{N}}$ converges to zero.

Conversely, assume that $(\varphi_i)_{i \in \mathbb{N}}$ is a sequence in $\mathrm{SCl}^+_{\mathscr{K}}(G)$ such that the sequence $(d(\varphi_i, \varphi))_{i \in \mathbb{N}}$ converges to zero. Then, for every $g \in G$, the sequence $(|\varphi_i(g) - \varphi(g)|)_{i \in \mathbb{N}}$ must converge to zero, and thus $(\varphi_i)_{i \in \mathbb{N}}$ converges pointwise to φ. The proof is complete. □

Hence, the following topological properties hold.

- $\mathrm{SCl}^+_{\mathscr{K}}(G)$ is compact, Hausdorff and metrizable;
- $\mathrm{SCl}^+_{\mathscr{K}}(G)$ is second countable (because it is a Hausdorff metric space);
- $\mathrm{SCl}^+_{\mathscr{K}}(G)$ is separable (because it is second countable, and hence it contains a countable dense subset);
- The space $\mathscr{E}$ is compact, metrizable, Hausdorff, second countable and separable (because metrizability, second countability and compactness are hereditary properties).

Furthermore, the geometry of the measure space $\mathscr{M}_{\mathbb{G}}(A^\circ)$ allows us to use supercharacters to approximate elements in $\mathrm{SCl}^+_{\mathscr{K}}(G)$.

Proposition 3.5.2. *The convex hull of $\mathscr{E}$ is dense in $\mathrm{SCl}^+_{\mathscr{K}}(G)$. In particular, if $\mathscr{E}_0$ is any countable dense subspace of $\mathscr{E}$, then the $\mathbb{C}$-linear span of $\mathscr{E}_0$ is dense in $\mathrm{SCl}^+_{\mathscr{K}}(G)$.*

Proof. Recall that a topological vector space is said to be *locally convex* if it has a topological basis consisting of open convex sets. According to [27, Proposition 3.12], for every $\mu_0 \in \mathscr{M}_{\mathbb{G}}(A^\circ)$, every finite family $\{f_1,\dots,f_n\} \subseteq C(A^\circ)$ and $\varepsilon > 0$, the set

$$V_{\mu_0}(f_1,...,f_n;\varepsilon) = \left\{\mu \in \mathscr{M}_{\mathbb{G}}(A^\circ) \colon \left|\int_{A^\circ} f_i\, d(\mu-\mu_0)\right| < \varepsilon,\ 1 \le i \le n\right\}$$

is an open neighborhood of μ_0, furthermore, the family consisting of all this sets is a topological basis for $\mathscr{M}_G(A^\circ)$.

Now, we fix $\mu_0 \in \mathscr{M}_{\mathbb{G}}(A^\circ)$ and set $V = V_{\mu_0}(f_1,...,f_n;\varepsilon)$. Let $\mu_1,\mu_2 \in V$ be arbitrary, let $t \in [0,1]$, and consider $\mu = t\mu_1 + (1-t)\mu_2$. For every $1 \le i \le n$, we have

$$\left|\int_{A^\circ} f_i\, d(\mu_1-\mu_0)\right| < \varepsilon \qquad \text{and} \qquad \left|\int_{A^\circ} f_i\, d(\mu_2-\mu_0)\right| < \varepsilon.$$

Since the set $\{z \in \mathbb{C} \colon |z| < \varepsilon\}$ is convex, it follows that

$$\left|t\left(\int_{A^\circ} f_i\, d(\mu_1-\mu_0)\right) + (1-t)\left(\int_{A^\circ} f_i\, d(\mu_2-\mu_0)\right)\right| < \varepsilon$$

$$\iff \left|\int_{A^\circ} f_i\, d\big(t\mu_1 - (1-t)\mu_2 + t\mu_0 + (1-t)\mu_0\big)\right| < \varepsilon.$$

Consequently, $\mu \in V$, and so V is convex. Therefore, we conclude that $\mathscr{M}_{\mathbb{G}}(A^\circ)$ is locally convex.

Since $\mathrm{SCl}^{+}_{\mathscr{K}}(G) \simeq \mathscr{M}^{+}_{\mathbb{G}}(A^{\circ})$ is a compact subset of $\mathscr{M}_{\mathbb{G}}(A^{\circ})$, it follows from the Krein-Milman theorem (see [39, Theorem 3.65] for a proof) that $\mathscr{M}^{+}_{\mathbb{G}}(A^{\circ})$ is the closed convex hull of $\mathscr{E}$ and that $\mathscr{E}$ is dense in $\mathrm{SCl}^{+}_{\mathscr{K}}(G)$.

On the other hand, if $\mu \in \mathscr{M}_{\mathbb{G}}(A^{\circ})$, then μ can be decomposed as

$$\mu = (\mu_1^+ - \mu_1^-) + i(\mu_2^+ - \mu_2^-)$$

where $\mu_j^{\pm}$, $j = 1,2$, are finite positive measures. After normalizing them, the density of the convex hull of $\mathscr{E}$ implies that the $\mathbb{C}$-span of $\mathscr{E}$ is dense, and the result follows because $\mathscr{E}_0$ is dense in $\mathscr{E}$. $\square$

For every superclass character φ of G, with associated $\mathbb{G}$-invariant probability measure μ on A°, there is a unique probability measure μ^* on the compact space $\mathscr{E} \simeq \Omega$ that fully describes the decomposition of φ , so that

$$\mu = \int_{\Omega} \omega_{\lambda}\, d\mu^* \qquad \text{and} \qquad \varphi = \int_{\mathscr{E}} \chi^{\lambda}\, d\mu^*;$$

recall that $\Omega = \{\mathscr{O}^{\lambda} \colon \lambda \in A^{\circ}\}$. The measure μ^* may be used to define the direct integral of representations: we set

$$\boldsymbol{\pi}^{\mu^*} = \int_{\mathscr{E}}^{\oplus} \mathscr{T}^{\lambda}\, d\mu^* \qquad \text{and} \qquad \mathbf{H}^{\mu^*} = \int_{\mathscr{E}}^{\oplus} \mathscr{H}^{\lambda}\, d\mu^*,$$

and consider the representation $(\boldsymbol{\pi}^{\mu^*}, \mathbf{H}^{\mu^*})$ of G.

Lemma 3.5.3. *Let ν and μ to be two $\mathbb{G}$-invariant probability measures on A°. If ν is absolutely continuous with respect to μ, then ν^* is absolutely continuous with respect to μ^*.*

Proof. Let ν and μ to be two $\mathbb{G}$-invariant probability measures on A°. Assume that ν is absolutely continuous with respect to μ, and let $f : A^{\circ} \to \mathbb{R}_0^+$ denote the corresponding Radon-Nikodym derivative; notice that, since ν and μ are $\mathbb{G}$-invariant, the uniqueness of f ensures that f is a $\mathbb{G}$-invariant function.

Since f is $\mathbb{G}$-invariant, the restriction $f_{|\mathscr{O}^{\lambda}}$ of f to the orbit closure $\mathscr{O}^{\lambda} = \overline{\mathbb{G}\cdot\lambda}$, for any $\lambda \in A^{\circ}$, must be constant ω_{λ}-almost everywhere for every ergodic measure ω_{λ} on $\mathscr{O}^{\lambda}$; let $\widehat{f}(\mathscr{O}^{\lambda})$ denote the ω_{λ}-almost everywhere constant value of $f_{|\mathscr{O}^{\lambda}}$. We claim that the function $\widehat{f} : \Omega \to \mathbb{R}_0^+$ defined by the mapping $\mathscr{O}^{\lambda} \mapsto \widehat{f}(\mathscr{O}^{\lambda})$, is measurable . Indeed, for every $B \in \mathscr{B}(A^{\circ})$,

let $\mathbb{I}_B$ denote the corresponding indicator function, and evaluate

$$\begin{aligned}
\nu(B) &= \int_{A^\circ} \mathbb{I}_B(x) f(x)\, d\mu \\
&= \int_{A^\circ} \mathbb{I}_B(x) f(x)\, d\Big(\int_\Omega \omega_\lambda\, d\mu^*\Big) \\
&= \int_\Omega \Big(\int_{A^\circ} \mathbb{I}_B(x) f(x)\, d\omega_\lambda\Big)\, d\mu^* \\
&= \int_\Omega \widehat{f}(\mathscr{O}^\lambda)\Big(\int_{A^\circ} \mathbb{I}_B(x)\, d\omega_\lambda\Big)\, d\mu^* \\
&= \int_\Omega \widehat{f}(\mathscr{O}^\lambda)\ \omega_\lambda(B)\, d\mu^*,
\end{aligned}$$

and hence, not only $\widehat{f}$ is measurable, but also this implies that ν^* is absolutely continuous with respect to μ^* with $\widehat{f}$ as the corresponding Radon-Nikodym derivative. □

Proposition 3.5.4. *The representation $(\boldsymbol{\pi}^{\mu^*}, \mathbf{H}^{\mu^*})$ contains a subrepresentation which is quasi-equivalent to any super-representation associated with a measure ν which is equivalent to μ.*

Proof. Let φ be a superclass character of G with associated $\mathbb{G}$-invariant measure μ on A°. For every $g \in G$ we define the vector function $\mathbf{T}_g : \Omega \to \prod_{\mathscr{O}^\lambda \in \Omega} \mathscr{H}^\lambda$ by

$$\mathbf{T}_g(\mathscr{O}^\lambda) = T_g \in \mathscr{H}^\lambda, \qquad \lambda \in A^\circ.$$

Let $\mathbf{T}_G$ denote the $\mathbb{C}$-linear span of the set $\{\mathbf{T}_g : g \in G\}$.

Notice that the mapping $\mathscr{O}^\lambda \mapsto \langle \mathbf{T}_g(\mathscr{O}^\lambda) | \mathbf{T}_h(\mathscr{O}^\lambda)\rangle_\lambda$, for $\lambda \in A^\circ$, defines a measurable function; indeed, a continuous function because

$$\langle \mathbf{T}_g(\mathscr{O}^\lambda) | \mathbf{T}_h(\mathscr{O}^\lambda)\rangle_\lambda = \int_{A^\circ} T_{h^{-1}g}\, d\omega_\lambda = \chi^\lambda(h^{-1}g), \qquad g,h \in G.$$

Furthermore, the set $\{\mathbf{T}(\mathscr{O}^\lambda) : \mathbf{T} \in \mathbf{T}_G\}$ is dense in $\mathscr{H}^\lambda$, and thus $\mathbf{H}^{\mu^*}$ consists on all vector functions $F : \Omega \to \prod_{\mathscr{O}^\lambda \in \Omega} \mathscr{H}^\lambda$ such that $\mathscr{O}^\lambda \mapsto \langle F(\mathscr{O}^\lambda) | \mathbf{T}(\mathscr{O}^\lambda)\rangle_\lambda$ defines a measurable function for all $\mathbf{T} \in \mathbf{T}_G$, where two functions are identified up to a set of zero μ^*-measure.

Let $\mathbf{H}_0^{\mu^*}$ be the Hilbert subspace of $\mathbf{H}^{\mu^*}$ generated by $\mathbf{T}_G$. Since $\boldsymbol{\pi}^{\mu^*}(g)\mathbf{T}_1 = \mathbf{T}_g$ for all $g \in G$, we see that $\mathbf{T}_1$ is a cyclic vector of $\mathbf{H}_0^{\mu^*}$, and thus $\mathbf{H}_0^{\mu^*}$ affords a character such that for all $g \in G$

$$\langle \boldsymbol{\pi}^{\mu^*}(g)\mathbf{T}_1 | \mathbf{T}_1\rangle = \int_\Omega \langle \mathscr{T}^\lambda(g) T_1 | T_1\rangle_\lambda\, d\mu^* = \int_\Omega \chi^\lambda(g)\, d\mu^* = \varphi(g).$$

It follows that $\mathbf{H}^{\mu^*}$ contains a subrepresentation quasi-equivalent to $(\mathscr{T}^\mu, \mathscr{H}^\mu)$.

On the other hand, if ν is a $\mathbb{G}$-invariant measure equivalent to μ, then it is straightforward to check that the mapping

$$F \mapsto \left(\frac{d\nu^*}{d\mu^*}\right)^{1/2} F, \qquad F \in \mathbf{H}^{\nu^*}$$

defines an invertible intertwining operator $\mathbf{H}^{\nu^*} \to \mathbf{H}^{\mu^*}$, and so the representations $(\boldsymbol{\pi}^{\nu^*}, \mathbf{H}^{\nu^*})$ and $(\boldsymbol{\pi}^{\mu^*}, \mathbf{H}^{\mu^*})$ are equivalent. The result follows. $\square$

Such a phenomena justifies the claim that $\mathscr{E}$ cannot be used as a dual space for super-representations. Furthermore, it shows how complex the direct integral decomposition of representations is (because it does not depend on a measure, but rather on the its class of absolutely continuous measures), and how the relationship between characters and representations can breakdown when it comes to decomposition.

The uniform topology

Since superclass functions are assumed to be bounded, we can consider $\mathrm{SCl}_{\mathscr{K}}(G)$ equipped with the *uniform norm*: for all $\varphi \in \mathrm{SCl}_{\mathscr{K}}(G)$

$$\|\varphi\|_\infty = \sup_{g\in G} |\varphi(g)|, \qquad \varphi \in \mathrm{SCl}_{\mathscr{K}}(G)$$

This norm induces a topology on $\mathrm{SCl}_{\mathscr{K}}(G)$, which we naturally call the *uniform topology*: notice that this topology is a stronger then the pointwise topology in the sense that convergence with respect to the uniform norm implies pointwise convergence.

Proposition 3.5.5. *By identifying $\mathrm{SCl}_{\mathscr{K}}(G)$ with a subspace of the topological dual $C(A^\circ)^*$ (via $\mathbb{G}$-invariant measures and using the Riesz-Markov-Kakutani theorem), then the uniform norm coincides with the usual operator-norm of linear operators.*

Proof. Let $\varphi \in \mathrm{SCl}_{\mathscr{K}}(G)$ be associated with the $\mathbb{G}$-invariant measure μ on A°, and recall that as a linear functional the operator-norm of μ is given by

$$\|\mu\|_{\mathrm{op}} = \sup_{\|f\|_\infty = 1} \left| \int_{A^\circ} f \, d\mu \right| = |\mu|(A^\circ).$$

Since $\|T_g\| = 1$ for all $g \in G$, it follows that

$$\|\varphi\|_\infty \leq \|\mu\|_{\mathrm{op}}.$$

On the other hand, the operator norm is equivalently given by

$$\|\mu\|_{op} = \inf\{c > 0 \colon \left| \int_{A^\circ} f \, d\mu \right| \leq c \|f\|_\infty : \text{ for all } f \in C(A^\circ)\} = |\mu|(A^\circ).$$

Taking into account that $\|T_g\| = 1$ for all $g \in G$, it follows that

$$\|\mu\|_{\mathrm{op}} \leq \inf\{c > 0 \colon |\mu(T_g)| \leq c\} = \|\varphi\|_\infty ,$$

and so $\|\mu\|_{\mathrm{op}} = \|\varphi\|_\infty$ as stated.

□

If we restrict our attention to a superclass function associated with a signed measure μ with Hann-Jordan decomposition $\mu^+ - \mu^-$, then we get

$$\|\varphi\|_\infty = \mu^+(A^\circ) + \mu^-(A^\circ)$$

(see [17, 3.14] for the details), and this allows us to have a fairly decent understanding of the distance between two superclass characters of G.

Let φ_1 and φ_2 be two superclass characters with associated $\mathbb{G}$-invariant measures μ_1 and μ_2 on A°, respectively; hence μ_1 and μ_2 are probability measures, and $\nu = \mu_1 - \mu_2$ is a signed measure on A°. Let $\nu = \nu^+ - \nu^-$ be the Hann-Jordan decomposition of ν, and let X^+ and X^- be two disjoint subsets of A° such that

$$X^+ \cup X^- = A^\circ \qquad \text{and} \qquad \nu^+(X^-) = \nu^-(X^+) = 0.$$

Since $\nu(A^\circ) = 0$, we see that

$$\nu^+(X^+) = \nu^+(A^\circ) = \nu^-(A^\circ) = \nu^-(X^-),$$

and so

$$2|\nu(X^+)| = 2\nu^+(X^+) = \nu(X^+) + \nu(X^-) = \nu^+(A^\circ) + \nu^-(A^\circ) = |\nu|(A^\circ).$$

On the other hand, for every $B \subseteq \mathscr{B}(A^\circ)$ we have

$$2|\nu(B)| = 2|\nu^+(B) - \nu^-(B)| \leq 2\max\{\nu^+(A^\circ), \nu^-(A^\circ)\} = \nu^+(A^\circ) + \nu^-(A^\circ) = |\nu|(A^\circ)$$

and thus

$$|\mu_1 - \mu_2|(A^\circ) = \sup_{B \in \mathscr{B}(A^\circ)} \{\mu_1(B) - \mu_2(B)\}.$$

Proposition 3.5.6. *If $\lambda, \lambda' \in A^\circ$ are such that the supercharacters χ^λ and $\chi^{\lambda'}$ are distinct, then*

$$\left\|\chi^\lambda - \chi^{\lambda'}\right\|_\infty = 2$$

Proof. Let $\lambda, \lambda' \in A^\circ$ be such that the orbit closures $\mathscr{O}^\lambda$ and $\mathscr{O}^{\lambda'}$ are distinct. Since ω_λ and $\omega_{\lambda'}$ are mutually singular

$$2|\omega_\lambda(B) - \omega_{\lambda'}(B)| \leq 2\max\{\omega_\lambda(B), \omega_{\lambda'}(B)\} = 2, \qquad B \in \mathscr{B}(A^\circ).$$

On the other hand,

$$2|\omega_\lambda(\mathscr{O}^\lambda) - \omega_{\lambda'}(\mathscr{O}^\lambda)| = 2\omega_\lambda(\mathscr{O}^\lambda) = 2$$

and this completes the proof.

□

In this fashion, with respect to the uniform norm, supercharacters are always far away from each other, and for this reason the topology on $\mathscr{E}$ induced by the uniform norm is not very interesting. In what concerns to viewing $\mathscr{E}$ as a topological space, we may use the bijection between $\mathscr{E}$ and the space Ω (which is essentially the quotient space $\mathbb{G}\,A^\circ$, and hence is equipped with the quotient topology) to introduce a topology on $\mathscr{E}$ so that the aforementioned bijection becomes an homeomorphism. However this would, in general, yield a ill-behaved topology since the quotient topology may lack "good" topological features: for example, suppose that there exists $\lambda \in A^\circ$ such that the $\mathbb{G}$-orbit $\mathbb{G}\cdot\lambda$ is dense in A°; then, the only open set (for the quotient topology) which contains $\mathscr{O}^\lambda$ is Ω itself, and thus not only Ω does not separate points, but also every sequence $(\mathscr{O}^{\lambda_n})_{n\in\mathbb{N}}$ in Ω converges to $\mathscr{O}^\lambda$.

Chapter 4

AF-algebra groups and supercharacters

The class of AF-algebra groups is of particular interest as they represent the foremost generalization of finite algebra groups.

In what follows we fix a prime number p, and an AF-algebra group $G = 1 + A$ over a field $\mathbb{K}$ of characteristic p; hence, by definition,

$$G = \varinjlim_{n \in \mathbb{N}} G_n = \bigcup_{n \in \mathbb{N}} G_n$$

where $\{G_n\}_{n \in \mathbb{N}}$ is a chain of subgroups such that, for every $n \in \mathbb{N}$, $G_n = 1 + A_n$ is a finite algebra group (over a finite field $\mathbb{K}_n$) with $G_n \subseteq G_{n+1}$ (and where the direct limit is taken with respect to the inclusion maps). We recall that every AF-algebra group is an amenable countable discrete group, and thus we may consider the standard supercharacter theory $(\mathscr{K}, \mathscr{E})$ of G. Furthermore, we also consider each finite group G_n equipped with its standard supercharacter theory $(\mathscr{K}_n, \mathscr{E}_n)$, and for every $n \in \mathbb{N}$ we fix the following notation (which will will use throughout the chapter without always recalling its meaning):

- $\mathbb{G}_n$ will denote the direct product $G_n \times G_n$;
- SCl_n and SCl_n^+ will stand for the sets consisting of all superclass functions and all superclass characters of G_n, respectively;
- $\Omega_n = \{\mathbb{G}_n \cdot \gamma \colon \gamma \in (A_n)^\circ\}$ will denote the space of $\mathbb{G}_n$-orbits, and for every $\gamma \in (A_n)^\circ$ we will write $\mathscr{O}^\gamma = \mathbb{G} \cdot \gamma$;
- For every $g \in G_n$, we will set $K_g^{(n)} = 1 + \mathbb{G}_n \cdot g$ to denote the superclass $K_g^{(n)} \in \mathscr{K}_n$ which contains g, and for every $\mathscr{O} \in \Omega_n$ we will denote by $\chi^{\mathscr{O}}$ the supercharacter which corresponds to $\mathscr{O}$; whenever convenient, we will also use the notation χ^γ for $\chi^{\mathscr{O}}$ when $\gamma \in \mathscr{O}$.

The standard supercharacter theory of G is intimately connected with the corresponding standard supercharacters theories of the subgroups G_n for all $n \in \mathbb{N}$. Indeed, for every $a \in A$, we may choose the smallest $n_0 \in \mathbb{N}$ such that $a \in A_{n_0}$, and then the superclass of G which contains the element $g = 1 + a$ is

$$K_g = 1 + \bigcup_{n \in \mathbb{N}} \mathbb{G}_n \cdot a = \bigcup_{n \geq n_0} K_g^{(n)}.$$

Therefore, for every $n \in \mathbb{N}$ the restriction to the subgroup G_n of an arbitrary superclass function $\varphi \in \mathrm{SCl}_{\mathscr{K}}(G)$, which we will denote by $\varphi_{|n}$, is a superclass function of G_n. Moreover, since $\mathscr{E}_n$ is a $\mathbb{C}$-basis of SCl_n, there are uniquely determined complex numbers $m(\chi^{\mathscr{O}}, \varphi_{|n}) \in \mathbb{C}$, for $\mathscr{O} \in \Omega_n$, such that

$$\varphi_{|n} = \sum_{\mathscr{O} \in \Omega_n} m(\chi^{\mathscr{O}}, \varphi_{|n}) \chi^{\mathscr{O}}, \qquad n \in \mathbb{N};$$

if it is non-zero, then the coefficient $m(\chi^{\mathscr{O}}, \varphi_{|n})$ is called the *multiplicity* of $\chi^{\mathscr{O}}$ in $\varphi_{|n}$ (to be accurate, these are the *normalized* supercharacter multiplicities and they should be denoted by $\widehat{m}(\cdot,\cdot)$; however, we choose to avoid this heavy notation, mainly because in what follows we will deal only with normalized multiplicities, and thus there is no risk of confusion). Notice that, if φ is a superclass character, then so is $\varphi_{|n}$ and this implies that all its multiplicities are non-negative rational numbers, for all $n \in \mathbb{N}$. On the other hand, due to the fact that $G = \bigcup_{n \in \mathbb{N}} G_n$, a superclass function φ is uniquely determined by the family $\{\varphi_{|n}\}_{n \in \mathbb{N}}$; for this reason it is important to analyze the behavior of such restrictions, as we will do in the next section.

We will establish a connection between multiplicities and $\mathbb{G}$-invariant measures (on A°) by understanding the topological nature of the dual group A°. Furthermore, using an ergodic theorem on amenable groups, it will be possible to derive a finite approximation of supercharacters in $\mathscr{E}$ by finite supercharacters in $\mathscr{E}_n$, for $n \in \mathbb{N}$; this *finite approximation property* will allow us not only to comprehend the nature of the multiplicities of supercharacters, but also to establish an asymptotic formula for them.

On the other hand, for every AF-algebra group there is a graded graph associated with its standard supercharacter theory, the so-called *superbranching graph*, which allows us to use the theory of graded graphs developed by Kerov and Vershik (see for example [90]). The Kerov-Vershik ergodic method relates superclass functions with measures on the set of paths of the superbranching graph, making it possible to establish an asymptotic formula for supercharacters *via* the Kerov-Vershik ergodic theorem. As it turns out, the Kerov-Vershik ergodic method is equivalent to the ergodic approach that we developed in the previous chapter. For this reason, we

will explain the Kerov-Vershik ergodic method, and establish the connection with $\mathbb{G}$-invariant measures on A°.

Finally, the standard supercharacters of an AF-algebra group G enjoy additional properties which are analogous to the case of finite groups. In fact, supercharacters may be seen as induced characters in the following sense: for every $\lambda \in A^\circ$, there are an algebra subgroup L_λ of G and a one-dimensional representation $(\widetilde{\lambda}, \mathbb{C})$ of L_λ (hence, $\widetilde{\lambda}$ is a character of L_λ) such that the induced representation (in the sense of Mackey) from L_λ to G is quasi-equivalent to the standard super-representation $(T^\lambda, \mathscr{H}^\lambda)$ (which is associated with the supercharacter $\chi^\lambda \in \mathscr{E}$).

The induced model for super-representations is essentially free of measure theory, which allows us to study super-representations without an explicit description of the corresponding $\mathbb{G}$-ergodic measures. Furthermore, by capitalizing on the algebraic nature of the induced model, it is possible to present a factorization of supercharacters as a product of *"elementary"* supercharacters. Such a decomposition is finite and essentially unique in the case where G is finite-dimensional; on the other hand, if this is not the case, then the factorization may be asymptotic and in general not unique.

At this point it is worth to mention that, in concrete examples, using the main results of [30] it may possible to use the induction property to determine whenever a supercharacter is associated with a representation of type *I* or *II*. This is accomplished in Chapter 6 for the two types of infinite unitriangular groups in positive characteristic.

4.1 $\mathbb{G}$-invariant measures and multiplicities

The standard supercharacter theory of the AF-algebra group $G = 1 + A$ is determined by the $\mathbb{G}$-invariant Borel measures on A°, and therefore the understanding of the topology on A° might provide some insight on the behavior of the supercharacters of G.

Since

$$1 + A = \varinjlim_{n\in\mathbb{N}} (1 + A_n) \simeq 1 + \varinjlim_{n\in\mathbb{N}} A_n,$$

we may realize the abelian group A as the direct limit

$$A = \varinjlim_{n\in\mathbb{N}} A_n,$$

where the direct limit (taken with respect to the natural inclusions $A_n \hookrightarrow A_{n+1}$) is considered in the category of topological abelian groups. On the other hand, in this category, we clearly have

$A^\circ = \mathrm{Hom}(A, \mathbb{S}^1)$ where $\mathbb{S}^1$ denotes the complex unit circle. Since the restriction functor is the adjoint of the inclusion functor, it follows that

$$A^\circ = \mathrm{Hom}\left(\varinjlim_{n\in\mathbb{N}} A_n, \mathbb{S}^1\right) \simeq \varprojlim_{n\in\mathbb{N}} \mathrm{Hom}(A_n, \mathbb{S}^1),$$

where the inverse limit is taken with respect to the restriction maps $\mathbf{Res}_n^{n+1} : A_{n+1}^\circ \to A_n^\circ$. Therefore, for every $n \in \mathbb{N}$, the n-projection $\mathbf{Res}_n : A^\circ \to A_n^\circ$ is simply the restriction of characters, and the pointwise convergence topology on A° coincides with the inverse limit topology, which implies that A° is a compact, totally disconnected Hausdorff space. For simplicity, we write

$$\alpha_{|n} = \mathbf{Res}_n^{n+1}(\alpha), \qquad \alpha \in A_{n+1}$$

and similarly

$$\lambda_{|n} = \mathbf{Res}_n(\lambda), \qquad \lambda \in A;$$

moreover, we consider the topological basis of clopen subsets (that is, subsets which are both open and closed) consisting of all *cylinders*: the cylinder associated with $n \in \mathbb{N}$ and $\gamma \in (A_n)^\circ$, is the set

$$[\gamma]_n = \{\lambda \in A^\circ \colon \lambda_{|n} = \gamma\}.$$

Lemma 4.1.1. *Every Borel measure on A° is fully determined by its values on cylinders. Furthermore, a sequence of measures $(\mu_m)_{m\in\mathbb{N}}$ weak*-converges to a measure μ if and only if, for every $n \in \mathbb{N}$,*

$$\lim_{m\to\infty} \mu_m([\gamma]_n) = \mu([\gamma]_n), \qquad \gamma \in (A_n)^\circ,\ n \in \mathbb{N}.$$

Proof. Since every cylinder $[\gamma]_n$, for $\gamma \in (A_n)^\circ$ and $n \in \mathbb{N}$, is a clopen set, the corresponding indicator function $\mathbb{I}_{[\gamma]_n}$ is continuous; moreover, the set $\left\{\mathbb{I}_{[\gamma]_n} \colon \gamma \in (A_n)^\circ,\ n \in \mathbb{N}\right\}$ separates points of A°: if $\lambda, \lambda' \in A^\circ$ are distinct, then there must exist $n \in \mathbb{N}$ such that $\lambda_{|n} \neq (\lambda')_{|n}$, and consequently

$$\mathbb{I}_{[\lambda_{|n}]_n}(\lambda) = 1 \qquad \text{and} \qquad \mathbb{I}_{[\lambda_{|n}]_n}(\lambda') = 0.$$

On the other hand, since cylinders are either disjoint or one is contained in the other, it is clear that for every $n, m \in \mathbb{N}$, every $\gamma \in (A_n)^\circ$, and every $\alpha \in (A_m)^\circ$

$$\mathbb{I}_{[\gamma]_n}\mathbb{I}_{[\alpha]_m} = \begin{cases} \mathbb{I}_{[\alpha]_m}, & \text{if } n < m \text{ and } \alpha_{|n} = \gamma, \\ 0, & \text{otherwise;} \end{cases}$$

consequently, the $\mathbb{C}$-linear span of the set $\{\mathbb{I}_{[\gamma]_n}\colon \gamma \in (A_n)^\circ,\ n \in \mathbb{N}\}$ is a unital C^*-subalgebra of $C(A^\circ)$ with unit $\mathbb{I}_{[0]_1}$, and in virtue of the Stone-Weierstrass theorem it is a dense subalgebra of $C(A^\circ)$.

To conclude the proof it is enough to identify the topological dual $C(A^\circ)^*$ with the space of measures on A°. □

Let $g \in G$ be arbitrary, and let $n \in \mathbb{N}$ be such that $g \in G_n$. Then, the function T_g can be written as the sum

$$T_g = \sum_{\gamma \in (A_n)^\circ} \gamma(g-1)\mathbb{I}_{[\gamma]_n},$$

and thus for every measure μ on A°

$$\int_{A^\circ} T_g \, d\mu = \sum_{\gamma \in (A_n)^\circ} \gamma(g-1)\mu([\gamma]_n).$$

In particular, if φ is a superclass function of G associated with a $\mathbb{G}$-invariant measure μ, then

$$\mu([\gamma]_n) = \mu([\alpha]_n), \qquad \alpha \in \mathscr{O}^\gamma,\ \gamma \in (A_n)^\circ$$

and so

$$\int_{A^\circ} T_g \, d\mu = \sum_{\gamma \in \Gamma_n} \Big(\sum_{\alpha \in \mathscr{O}^\gamma} \alpha(g-1) \Big) \mu([\gamma]_n) = \sum_{\gamma \in \Gamma} |\mathscr{O}^\gamma| \mu([\gamma]_n) \chi^\gamma(g),$$

where $\Gamma_n \subseteq (A_n)^\circ$ is a complete set of representatives of Ω_n (so that $\Omega_n = \{\mathscr{O}^\gamma\colon \gamma \in \Gamma\}$). Therefore, we get a relationship between multiplicities and $\mathbb{G}$-invariant measures, namely,

$$m(\chi^\gamma, \varphi_{|n}) = |\mathscr{O}^\gamma| \mu([\gamma]_n), \qquad \gamma \in (A_n)^\circ.$$

(In this sense, the $\mathbb{G}$-invariant measure μ encodes the multiplicities of the restrictions of φ to the finite subgroups G_n, for $n \in \mathbb{N}$, which "approximate" G.)

For every $\lambda \in A^\circ$ (and every $n \in \mathbb{N}$) we define the set

$$(S^\lambda)_n = \{\lambda'_{|n}\colon \lambda' \in \mathbb{G}\cdot\lambda\},$$

and we notice that

$$\mathscr{O}^\lambda = \bigcap_{n \in \mathbb{N}} \Big(\bigcup_{\gamma \in (S^\lambda)_n} [\gamma]_n \Big).$$

Consequently, if $\lambda, \lambda' \in A^\circ$ are such that $\mathbb{G}\cdot\lambda \neq \mathbb{G}\cdot\lambda'$, then

$$\mathscr{O}^\lambda = \mathscr{O}^{\lambda'} \quad \Longleftrightarrow \quad (S^\lambda)_n = (S^{\lambda'})_n \text{ for all } n \in \mathbb{N};$$

in this sense, if this condition holds, then we may say that the $\mathbb{G}$-orbits $\mathbb{G}\cdot\lambda$ and $\mathbb{G}\cdot\lambda'$ are *"locally equal"*. On the other hand, if $\lambda, \lambda' \in A^\circ$ are such that $\mathscr{O}^\lambda \neq \mathscr{O}^{\lambda'}$, then there are $n \in \mathbb{N}$ and $\gamma \in (A_n)^\circ$ such that, either $\gamma \in (S^\lambda)_n \setminus (S^{\lambda'})_n$ or $\gamma \in (S^{\lambda'}) \setminus (S^\lambda)_n$.

Proposition 4.1.2. *Let $\lambda \in A^\circ$, and let $\gamma \in (A_n)^\circ$ for $n \in \mathbb{N}$. Then,*

$$m(\chi^\gamma, (\chi^\lambda)_{|n}) \neq 0 \quad \Longleftrightarrow \quad \mathscr{O}^\lambda \subseteq (S^\lambda)_n.$$

Moreover, if $\lambda, \lambda' \in A^\circ$ are such that $\mathscr{O}^\lambda \neq \mathscr{O}^{\lambda'}$, then there are $n \in \mathbb{N}$ and $\gamma \in (A_n)^\circ$ such that one of the following exclusive conditions hold:

- $m(\chi^\gamma, (\chi^\lambda)_{|n}) \neq 0$ *and* $m(\chi^\gamma, (\chi^{\lambda'})_{|n}) = 0$*, or*
- $m(\chi^\gamma, (\chi^\lambda)_{|n}) = 0$ *and* $m(\chi^\gamma, (\chi^{\lambda'})_{|n}) \neq 0$*.*

Proof. It is clear that, if $\gamma \notin (S^\lambda)_n$, then $\omega_\lambda([\gamma]_n) = 0$. On the other hand, assume that $\gamma \in (S^\lambda)_n$. For every $m \in \mathbb{N}$, we define the set

$$(X_m)^\gamma = \begin{cases} \{\alpha \in (A_m)^\circ \colon \alpha_{|n} = \gamma\}, & \text{if } m \geq n, \\ \emptyset, & \text{otherwise,} \end{cases}$$

and let

$$F_m = \bigcap_{n \in \mathbb{N}} \Bigg(\bigcup_{\gamma' \in (S^\lambda)_n \setminus (X_m)^\gamma} [\gamma']_m \Bigg) = \mathscr{O}^\lambda \setminus [\gamma]_n;$$

notice that F_m is a closed subset of A°. If $\omega_\lambda([\gamma]_n) = 0$, then $\omega_\lambda(F_m) = 1$, and since F_m is closed this means that

$$\mathscr{O}^\lambda = \operatorname{supp}(\omega_\lambda) \subseteq F_m,$$

a contradiction. Therefore, $\omega_\lambda([\gamma]_n) \neq 0$, and thus the first assertion of the result is proved (because $m(\chi^\gamma, (\chi^\lambda)_{|n}) = |\mathscr{O}^\gamma| \omega_\lambda([\gamma]_n) \neq 0$).

For the second assertion, let $\lambda, \lambda' \in A^\circ$ be such that $\mathscr{O}^{\lambda'} \neq \mathscr{O}^\lambda$. Then, there is $n \in N$ such that $(S^\lambda)_n \neq (S^\lambda)_n$, and without loss of generality one may assume that there is $\gamma \in (S^\lambda)_n \setminus (S^{\lambda'})_n$. This implies that

$$m(\chi^\gamma, (\chi^\lambda)_{|n}) \neq 0 \qquad \text{and} \qquad m(\chi^\gamma, (\chi^{\lambda'})_{|n}) = 0,$$

which completes the proof. $\square$

For every $\lambda \in A^\circ$, the corresponding $\mathbb{G}$-ergodic measure ω_λ is, in a way, determined by the sets $(S^\lambda)_n$ for $n \in \mathbb{N}$; however, the relationship between the supercharacter χ^λ of G and the finite groups G_n, $n \in \mathbb{N}$, is deeper. Indeed, we can use supercharacters of these finite subgroups of G to approximate χ^λ. In order to explain this, we need a *pointwise ergodic theorem for amenable groups*.

Recall that a countable discrete group $\mathscr{G}$ is amenable if it admits *Følner sequence*, that is, a family $\{F_n\}_{n\in\mathbb{N}}$ of finite subsets of such that

$$\lim_{n\to\infty} \frac{|F_n\Delta(gF_n)|}{|F_n|} = 0, \qquad g \in G,$$

where Δ denotes the symmetric difference of sets; a Følner sequence $\{F_n\}_{n\in\mathbb{N}}$ is said to be *tempered* if there is a constant $C > 0$ such that

$$\left|\bigcup_{k\leq n} F_k^{-1} F_{n+1}\right| \leq C\,|F_{n+1}|, \qquad n \in \mathbb{N}.$$

According to Lindenstrauss [69, Theorem 1.3], the following result holds.

Theorem 4.1.3 (Lindenstrauss pointwise ergodic theorem)**.** *Let $\mathscr{G}$ be a countable discrete amenable group acting on a probability space (X,μ), and suppose that the measure μ is $\mathscr{G}$-ergodic. If there is a tempered Følner sequence $\{F_n\}_{n\in\mathbb{N}}$, then for μ-almost every point $x \in X$ and every $f \in L^1(X,\mu)$*

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} (g\cdot f)(x) = \lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} f(g^{-1}\cdot x) = \int_X f\, d\mu.$$

Since the group $\mathbb{G}$ admits the tempered Følner sequence $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$, Lindenstrauss' pointwise ergodic theorem, allows us to prove the following.

Proposition 4.1.4 (Finite approximation property)**.** *Let $\lambda \in A^\circ$, and for every $n \in \mathbb{N}$ let $\chi^{\lambda_{|n}} \in \mathscr{E}_n$ be the supercharacter of G_n which is associated with $\lambda_{|n} \in (A_n)^\circ$. Then,*

$$\lim_{n\to\infty} \chi^{\lambda_{|n}}(g) = \chi^\lambda(g), \qquad g \in G.$$

Proof. For every $\mathbf{k} \in \mathbb{G}$, let $X_\mathbf{k}$ be the set consisting of all characters $\mu \in A^\circ$ such that

$$\lim_{n\to\infty} \frac{1}{|\mathbb{G}_n|} \sum_{\mathbf{k}\in\mathbb{G}_n} T_g(\mathbf{k}^{-1}\cdot x) = \int_{A^\circ} T_g\, d\omega_\lambda.$$

According to Lindenstrauss' pointwise ergodic theorem, for every $\mathbf{k} \in \mathbb{G}$ the set $X_\mathbf{k}$ has full ω_λ-measure, and thus the intersection

$$X = \bigcap_{\mathbf{k}\in\mathbb{G}} X_\mathbf{k}$$

also as full ω_λ-measure; without loss of generality, we may assume that $\lambda \in X$.

Let

$$\mathrm{Stab}_n = \{\mathbf{k} \in \mathbb{G}_n \colon \mathbf{k}\cdot\lambda_{|n} = \lambda_{|n}\}$$

be the stabilizer in $\mathbb{G}_n$ of $\lambda_{|n} \in (A_n)^\circ$, so that $|\mathbb{G}_n| = |\mathrm{Stab}_n||\mathbb{G}_n \cdot \lambda_{|n}|$. Then,

$$\frac{1}{|\mathbb{G}_n|} \sum_{\mathbf{k} \in \mathbb{G}_n} T_g(\mathbf{k}^{-1} \cdot \lambda) = \frac{1}{|\mathrm{Stab}_n||\mathbb{G}_n \cdot \lambda_{|n}|} \sum_{\gamma \in \mathbb{G}_n \cdot \lambda_{|n}} \big|\{\mathbf{k} \in \mathbb{G}_n \colon \mathbf{k} \cdot \lambda_{|n} = \gamma\}\big| \, \gamma(g-1).$$

On the other hand, since

$$\big|\{\mathbf{k} \in \mathbb{G}_n \colon \mathbf{k} \cdot \lambda_{|n} = \gamma\}\big| = |\mathrm{Stab}_n|, \qquad \gamma \in \mathbb{G}_n \cdot \lambda_{|n},\ n \in \mathbb{N},$$

we conclude that

$$\lim_{n\to\infty} \chi^{\lambda_{|n}}(g) = \lim_{n\to\infty} \frac{1}{|\mathbb{G}_n \cdot \lambda_{|n}|} \sum_{\gamma \in \mathbb{G}_n \cdot \lambda_{|n}} T_g(\gamma) = \int_{A^\circ} T_g \, d\omega_\lambda,$$

which completes the proof. □

Notice that the approximation of the previous proposition does not depend on λ due to the "local equality" of $\mathbb{G}$-orbits with closure equal to $\mathscr{O}^\lambda$, that is, if $\lambda' \in A^\circ$ is such that the closure of $\mathbb{G} \cdot \lambda'$ equals $\mathscr{O}^\lambda$, then

$$\lim_{n\to\infty} \frac{1}{|\mathbb{G}_n \cdot \lambda'_{|n}|} \sum_{\gamma \in \mathbb{G}_n \cdot \lambda'_{|n}} \gamma(g) = \chi^\lambda(g), \qquad g \in G.$$

In this sense, the finite standard supercharacter theory of the finite groups G_n, for $n \in \mathbb{N}$, determine the nature of the supercharacters of G. Furthermore, the supercharacters of G can be understood as *asymptotic objects*, and this makes it possible to establish an asymptotic expression for the multiplicities of supercharacters.

Lemma 4.1.5. *For every $\alpha \in (A_{n+1})^\circ$ and every $\gamma \in (A_n)^\circ$, the multiplicity of χ^γ in the restriction $(\chi^\alpha)_{|n}$ of χ^α to G_n is given by the formula*

$$m\big(\chi^\gamma, (\chi^\alpha)_{|n}\big) = \frac{|\mathscr{O}^\gamma|}{|\mathscr{O}^\alpha|} \, \big|\{\beta \in \mathscr{O}^\alpha \colon \beta_{|n} = \gamma\}\big|.$$

Proof. Let us consider the set

$$S_\gamma = \{\beta \in \mathscr{O}^\alpha \colon \beta_{|n} = \gamma\};$$

we note that $\mathbf{k} \cdot S_\gamma = S_{\mathbf{k} \cdot \gamma}$ for all $\mathbf{k} \in \mathbb{G}_n$, and thus we may write $S_{\mathscr{O}}$ instead of S_γ where $\mathscr{O} \in \Omega$ is such that $\gamma \in \mathscr{O}$. For every $g \in G_n$ with $g = 1 + a$ for $a \in A_n$, we deduce that

$$\begin{aligned} (\chi^\alpha)_{|n}(g) &= \frac{1}{|\mathscr{O}^\alpha|} \sum_{\beta \in \mathscr{O}^\alpha} \beta_{|n}(a) \\ &= \frac{1}{|\mathscr{O}^\alpha|} \sum_{\mathscr{O} \in \Omega_n} \sum_{\gamma \in \mathscr{O}} |S_\gamma| \gamma(a) \\ &= \sum_{\mathscr{O} \in \Omega_n} \frac{|\mathscr{O}|}{|\mathscr{O}^\alpha|} |S_{\mathscr{O}}| \chi^{\mathscr{O}}(g); \end{aligned}$$

recall that $\chi^{\mathscr{O}} = \chi^\gamma$ for all $\mathscr{O} \in \Omega_n$ and all $\gamma \in \mathscr{O}$. The proof is complete. □

As a consequence of Proposition 4.1.4 we obtain the following.

Corollary 4.1.6. *Let $\lambda \in A^\circ$, and let $\gamma \in \Omega_n$ for some $n \in \mathbb{N}$. Then, the multiplicity of χ^γ in the restriction $(\chi^\lambda)_{|n}$ of χ^λ to G_n is asymptotically given by the formula*

$$m\big(\chi^\gamma, (\chi^\lambda)_{|n}\big) = \lim_{m\to\infty} \frac{|\mathscr{O}^\gamma|}{|\mathscr{O}^{\lambda_{|m}}|} \,\big|\{\alpha \in \mathscr{O}^{\lambda_{|m}} \colon \alpha_{|n} = \gamma\}\big|.$$

To conclude this section, let $\lambda \in A^\circ$ be associated with the $\mathbb{G}$-ergodic measure ω_λ. As we remarked before, for every $\gamma \in (A_n)^\circ$ we have

$$m(\chi^\gamma, (\chi^\lambda)_{|n}) = |\mathscr{O}^\gamma|\omega_\lambda([\gamma]_n).$$

Consequently, in virtue of the previous corollary, we conclude that

$$\omega_\lambda([\gamma]_n) = \lim_{m\to\infty} \frac{|\{\alpha \in \mathscr{O}^{\lambda_{|m}} \colon \alpha_{|n} = \gamma\}|}{|\mathscr{O}^{\lambda_{|n}}|},$$

and thus we may say that the value $\omega_\lambda([\gamma]_n)$ is the "average" over all elements $\lambda' \in \mathscr{O}^\lambda$ satisfying $\lambda'_{|n} = \gamma$.

4.2 The superbranching graph

One of the main features of AF-algebra groups is the existence of a canonical graded graph associated with its standard supercharacter theory; this will be called the *superbranching graph*.

The works of Elliott [37] and Bratteli [26] allowed a combinatorial description of the structure of locally semisimple C^*-algebras through what is now known as *Bratteli diagrams*; later Kerov and Vershik, in the context of the character theory of the infinite symmetric group S_∞, pioneered in [93] an ergodic-combinatorial approach by using Bratteli diagrams in order to introduce the language of dynamical systems into the representation theoretical framework of AF-groups.

In a more general context (see, for example, [25, 90]), if

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n \subseteq \cdots$$

is a sequence of finite groups, then the restrictions of characters of G_{n+1} to G_n determine a graded graph, which is called the *branching graph*, and the *Kerov-Vershik ergodic method* allows to formulate the character-theory of the direct limit

$$G = \varinjlim_{n\in\mathbb{N}} G_n = \bigcup_{n\in\mathbb{N}} G_n$$

in terms of *invariant measures* on the path space of the branching graph. In this setting, extreme characters are in correspondence with *Gibbs measures* (which are ergodic in a certain sense; we refer to [90] for more details and references therein).

A similar approach can be (to a certain extent) adapted to deal with the supercharacter theory on a AF-algebra group. A particular supercharacter theory of the locally finite unitriangular group $U_n(\mathbb{F}_q)$ (known as the uncolored supercharacter theory), briefly described in Chapter 5, was completely described in this manner by André, Gomes and the author in [8].

In what follows, we give a brief introduction on the subject with the purpose of connecting the main concepts with our ergodic setting. Our main reference is the book [25] (Chapter 7) since it presents a very concise and structured survey of the existent literature.

4.2.1 Graded graphs and Gibbs measures

Let Γ be a graph and denote by V and E the sets consisting of all vertices and all edges of Γ, respectively. Following [25], we say that Γ is a *graded graph* if the following conditions are satisfied:

- V is countable and is partitioned into *levels* Γ_n, for $n \in \mathbb{N}$, in which $\Gamma_0 = \{\emptyset\}$ is a singleton;
- There are edges only between consecutive levels and multiple edges are allowed;
- For every vertex $v \in \Gamma_n$ with $n \geq 1$, the set of edges $\{(v,u) \in E\colon\ u \in V_{n+1}\}$ is non-empty and finite.

Most of the times we shall identify a graded graph Γ with its set of vertices $\bigcup_{n\in\mathbb{N}} \Gamma_n$. Notice that, if for all $n \in \mathbb{N}$ the set Γ_n is finite, then Γ is a Bratteli diagram; while we shall only deal with Bratteli diagrams, we have chosen to refer to them as graded graphs to be coherent with the language of [25].

Example 4.2.1. One of the most prominent graded graphs in representation theory is the Young graph $\mathbb{Y}$: for each $n \in \mathbb{N}$ the n^{th}-level of vertices is $\mathbb{Y}_n$, the set consisting of Young tableaux with n boxes, and there is a single vertex between $\lambda \in \mathbb{Y}_n$ and $\mu \in \mathbb{Y}_{n+1}$ if and only if μ is obtained from λ by adding one box. The Young graph encodes the (non-normalized) multiplicities of the restriction of irreducible characters of S_{n+1} to S_n; for more details we refer to [25].

We write $v \nearrow u$ to denote that there is an edge between $v \in \Gamma_n$ and $u \in \Gamma_{n+1}$, and by a finite path of length n on Γ we mean a sequence of vertices $v_0 \nearrow v_1 \nearrow \cdots \nearrow v_n$ such that $v_i \in \Gamma_i$ for $0 \leq i \leq n$; similarly, we define an infinite path on Γ to be an infinite sequence $v_0 \nearrow v_1 \nearrow \cdots \nearrow v_n \nearrow \cdots$ such that $v_i \in \Gamma_i$ for all $i \in \mathbb{N}_0$. The set consisting of all paths of length n will be denoted by $T_n(\Gamma)$ and the set consisting of all infinite paths by $T(\Gamma)$; if there is no risk of confusion, we omit the graph Γ and simply write T_n and T to denote the sets of all finite and infinite paths on Γ, respectively.

There are natural projections $p_n^{n+1} : T_{n+1} \to T_n$, for $n \in \mathbb{N}_0$, which are defined by simply forgetting the ending edge of the path, and it is clear that with respect to those projections T is the inverse limit

$$T = \varprojlim_{n \in \mathbb{N}} T_n,$$

and that for every $n \in \mathbb{N}_0$ the canonical projection $p_n : T \to T_n$ simply forgets all but the first n-edges of the path. The set T is then equipped with the inverse limit topology whose cylinders form a base of clopen sets; recall that for every $u \in T_n$, $n \in \mathbb{N}_0$, the corresponding cylinder is the set

$$[u]_n = \{t \in T : \, p_n(t) = u\}.$$

Similarly to Lemma 4.1.1, using the fact that cylinders are clopen sets and that the corresponding indicator functions span a dense subset of continuous functions, the following is true.

Lemma 4.2.2. *Any (Borel) measure on T is fully determined by its values on cylinders. Moreover, a sequence $(M_m)_{m \in \mathbb{N}}$ of measures on T weak*-converges to a measure M if and only if for any cylinder $[u]_n$, $n \in \mathbb{N}_0$, we have*

$$\lim_{n \to \infty} M_m([u]_n) = M([u]_n).$$

A measure M on T is said to be a *Gibbs measure* (Gibbs measures are called *ergodic* in the works of Kerov and Vershik due to their connections with dynamical systems; see [90] for more details) if for every two finite paths $t = u_1 \nearrow \cdots \nearrow u_n$ and $t' = u_1' \nearrow \cdots \nearrow u_n'$ ending at the same point (that is, such that $u_n = u_n'$)

$$M([t]_n) = M([t']_n).$$

For every two vertices $v \in \Gamma_n$ and $u \in \Gamma_m$, for some $n,m \in \mathbb{N}_0$ with $n < m$, we define the *relative dimension*, denoted by $\dim(v,u)$, as the number of paths starting in v and ending in u; for simplicity, we set $\dim(v) = \dim(\emptyset, v)$, and notice that

$$\dim(v) = \sum_{u \nearrow v} \dim(u) \dim(u,v).$$

For every $u \in \Gamma_n$ and $v \in \Gamma_{n+1}$, for $n \in \mathbb{N}_0$, we define

$$\Lambda_n^{n+1}(v,u) = \frac{\dim(u)\dim(u,v)}{\dim(v)},$$

and we say that a family of measures $\{M_n\}_{n \in \mathbb{N}}$, where M_n is a measure on Γ_n for all $n \in \mathbb{N}$, is a *coherent system* on Γ (in [25] the term coherent system is reserved for families of probability measures; however, this restriction is not needed) if for every $n \in \mathbb{N}$

$$M_{n+1}\Lambda_n^{n+1} = M_n \quad \iff \quad \sum_{v \in \Gamma_{n+1}} M_{n+1}(v)\Lambda_n^{n+1}(u,v) = M_n(u), \quad u \in \Gamma_n.$$

The set consisting of all families of coherent systems of probability measures will be denoted by $\mathscr{M}_\infty(\Gamma)$; it is a Choquet simplex and the corresponding set of extreme points is called *the boundary of* Γ, and is denoted by $\partial\Gamma$.

Proposition 4.2.3. *There is a correspondence between Gibbs measures on T and coherent systems on Γ. In particular, there is a bijection between probability Gibbs measures on T and probability measures on $\partial\Gamma$.*

Proof. Let $\{M_n\}_{n \in \mathbb{N}}$ be a coherent system, and for every path $t = v_0 \nearrow \cdots \nearrow v_n \in T_n$, $n \in \mathbb{N}$, define a measure M on T by setting

$$M([t]_n) = \frac{M_n(v_n)}{\dim(v_n)}.$$

It is straightforward to check that M is a Gibbs measure on T. Conversely, for every Gibbs measure M on T, we define a measure M_n on Γ_n, for $n \in \mathbb{N}$, as follows: for every $v \in \Gamma_n$

$$M_n(v) = \dim(v) M([t]_n),$$

where $t \in T_n$ is any path ending at v. One can check that $\{M_n\}_{n \in \mathbb{N}}$ is a coherent system on Γ.

The rest of the assertion follows from the fact that $\partial\Gamma$ is the set consisting of all indecomposable coherent systems of probability measures on Γ. $\square$

The cornerstone of the theory of Gibbs measures on graded graphs is the Kerov-Vershik ergodic theorem (see [25, Theorem 7.17]).

Theorem 4.2.4 (Kerov-Vershik ergodic theorem)**.** *Let $\{M_n\}_{n\in\mathbb{N}} \in \partial\Gamma$ with corresponding Gibbs measure M. Then, for M-almost every path $t = (v_n)_{n\in\mathbb{N}}$ the following limit exists for every $v \in \Gamma_n$*

$$\lim_{m\to\infty} \frac{\dim(v)}{\dim(v_m)} \dim(v, v_m) = M_n(v).$$

An important application arises in the classification of indecomposable characters of the infinite symmetric group S_∞: each probability Gibbs measure on the Young graph $\mathbb{Y}$ determines a unique indecomposable character of S_∞, and it follows from the Kerov-Vershik ergodic theorem that for every indecomposable character ξ of S_∞ there is at least one sequence $(\xi_n)_{n\in\mathbb{N}}$, where ξ_n is an irreducible character of S_n, such that

$$\lim_{n\to\infty} \frac{\xi_n(g)}{\xi_n(1)} = \xi(g), \qquad g \in S_\infty.$$

For all the details on the proof, we refer to [25, 93]. We also mention that an analogue result is true for groups obtained as direct limits of finite groups; the formalism and proofs can be found in [90].

4.2.2 The superbranching graph and multiplicities

For a fixed AF-algebra group

$$G = 1 + A = \varinjlim_{n\in\mathbb{N}} G_n, \qquad G_n = 1 + A_n,\ n \in \mathbb{N},$$

we define the *superbranching graph* $\Gamma = \Gamma(G)$ as follows: for every $n \in \mathbb{N}$ we set $\Gamma_n = \Omega_n$, and there is an edge $\mathscr{O} \nearrow \mathscr{O}'$ whenever there is $\alpha \in \mathscr{O}'$ such that $\alpha_{|n} \in \mathscr{O}$; in particular, we have

- $\dim(\mathscr{O}, \mathscr{O}') = \big|\{\alpha \in \mathscr{O}' \colon \alpha_{|n} \in \mathscr{O}\}\big|$ for all $\mathscr{O} \in \Gamma_n$, all $\mathscr{O}' \in \Gamma_{n+1}$ and all $n \in \mathbb{N}$;
- $\dim(\mathscr{O}) = |\mathscr{O}|$ for all $\mathscr{O} \in \Gamma_n$ and all $n \in \mathbb{N}$.

On the other hand, the operator Λ_n^{n+1}, for $n \in \mathbb{N}$, has a familiar form: for every $\mathscr{O} \in \Omega_n$ and every $\mathscr{O}' \in \Omega_{n+1}$, we have

$$\Lambda_n^{n+1}(\mathscr{O}, \mathscr{O}') = \frac{|\mathscr{O}|}{|\mathscr{O}'|} \big|\{\alpha \in \mathscr{O}' \colon \alpha_{|n} \in \mathscr{O}\}\big| = m\big(\chi^{\mathscr{O}}, (\chi^{\mathscr{O}'})_{|n}\big).$$

The relationship between the sequence $\{\Lambda_n^{n+1}\}_{n\in\mathbb{N}}$ and the the multiplicities of the restricted supercharacters allows us to relate superclass characters of G and Gibbs measures on the path space $T = T(\Gamma)$ of Γ.

Proposition 4.2.5. *There is an affine homeomorphism between Gibbs measures on T and superclass functions of G. Moreover, the set of probability Gibbs measures on T is affinely homeomorphic to the space of superclass characters of G.*

Proof. Let M be a Gibbs measure on T with corresponding coherent system $\{M_n\}_{n\in N}$. For every $n \in \mathbb{N}$ we define the superclass function $\varphi^{M_n} : G_n \to \mathbb{C}$ by

$$\varphi^{M_n} = \sum_{\mathscr{O}\in\Omega_n} M_n(\mathscr{O})\chi^{\mathscr{O}}.$$

Since $\{M_n\}_{n\in\mathbb{N}}$ is a coherent system, we deduce that

$$\begin{aligned}
(\varphi^{M_{n+1}})_{|n} &= \sum_{\mathscr{O}'\in\Omega_{n+1}} M_{n+1}(\mathscr{O}')(\chi^{\mathscr{O}'})_{|n} \\
&= \sum_{\mathscr{O}'\in\Omega_{n+1}} M_{n+1}(\mathscr{O}') \sum_{\mathscr{O}\in\Omega_n} \Lambda_n^{n+1}(\mathscr{O},\mathscr{O}')\chi^{\mathscr{O}} \\
&= \sum_{\mathscr{O}\in\Omega_n} \sum_{\mathscr{O}'\in\Omega_{n+1}} M_{n+1}(\mathscr{O}')\Lambda_n^{n+1}(\mathscr{O},\mathscr{O}')\chi^{\mathscr{O}} \\
&= \sum_{\mathscr{O}\in\Omega_n} M_n(\mathscr{O})\chi^{\mathscr{O}} = \varphi^{M_n},
\end{aligned}$$

and thus the sequence $\{\varphi^{M_n}\}_{n\in\mathbb{N}}$ uniquely determines a superclass character $\varphi^M : G \to \mathbb{C}$ such that

$$\left(\varphi^M\right)_{|n} = \varphi^{M_n}, \qquad n \in \mathbb{N}.$$

On the other hand, if φ is any superclass character of G, then for every $n \in \mathbb{N}$ we define the measure $(M^\varphi)_n$ on Ω_n by

$$(M^\varphi)_n(\mathscr{O}) = m(\chi^{\mathscr{O}}, \varphi_{|n}), \qquad \mathscr{O} \in \Omega_n.$$

For every $n \in \mathbb{N}$ and every $\mathscr{O} \in \Omega_n$, we see that

$$\begin{aligned}
\sum_{\mathscr{O}'\in\Omega_{n+1}} (M^\varphi)_{n+1}(\mathscr{O}')\,\Lambda_n^{n+1}(\mathscr{O},\mathscr{O}') &= \sum_{\mathscr{O}'\in\Omega_{n+1}} m(\chi^{\mathscr{O}'}, \varphi_{|n})\, m(\chi^{\mathscr{O}}, (\chi^{\mathscr{O}'})_{|n}) \\
&= m(\chi^{\mathscr{O}}, \varphi_{|n}) = (M^\varphi)_n(\mathscr{O}),
\end{aligned}$$

which means that $\{(M^\varphi)_n\}_{n\in\mathbb{N}}$ is a coherent system on Γ; we denote by M^φ the corresponding Gibbs measure on T.

If φ and φ' are distinct superclass functions of G, then there is $n \in \mathbb{N}$ such that $\varphi_{|n} \neq \varphi'_{|n}$, and so there is $\chi \in \mathscr{E}_n$ such that $m(\chi, \varphi_{|n}) \neq m(\chi_n, \varphi'_{|n})$, which implies that the mapping $\varphi \mapsto M^\varphi$

defines an injective map. Furthermore, the mapping $M \mapsto \varphi^M$ clearly defines the inverse map, and thus the superclass functions of G are in bijection with the Gibbs measures on T.

Let $(\varphi^{(m)})_{m\in\mathbb{N}}$ is a convergent sequence in $\mathrm{SCl}_{\mathscr{K}}(G)$ with limit point $\varphi \in \mathrm{SCl}_{\mathscr{K}}(G)$. For every $n \in \mathbb{N}$ and every $\chi \in \mathscr{E}_n$ we have

$$\lim_{m\to\infty} m\big(\chi, (\varphi^{(m)})_{|n}\big) = m(\chi, \varphi_{|n}),$$

and thus for every cylinder $[t]_n$, $t \in T_n$, $n \in \mathbb{N}$, we get

$$\lim_{m\to\infty} M_{\varphi^{(m)}}([t]_n) = M_\varphi([t]_n).$$

Conversely, if $(M^{(m)})_{m\in\mathbb{N}}$ is a convergent sequence of Gibbs measures on T with limit point M, then for every cylinder $[t]_n$, $t \in T_n$, $n \in \mathbb{N}$, the sequence $\big(M^{(m)}([t]_n)\big)_{m\in\mathbb{N}}$ is convergent with limit $M([t]_n)$, which means that if t is the path $t = \chi_1 \nearrow \cdots \nearrow \chi_n$, then

$$\lim_{m\to\infty} m\big(\chi_n, \varphi^{M^{(m)}}_{|n}\big) = m\big(\chi_n, (\varphi^M)_{|n}\big),$$

and thus the sequence $\big(\varphi^{M^{(m)}}\big)_{m\in\mathbb{N}}$ is convergent with limit φ^M. In this fashion, we conclude that the set of superclass functions (with the pointwise convergence) is homeomorphic to the set of Gibbs measures (with the weak* convergence).

Furthermore, it is clear that $\mathrm{SCl}^+_{\mathscr{K}}(G)$ is mapped onto the set of probability Gibbs measures on T and that the correspondence above defines an affine homeomorphism. $\square$

In conclusion, a Gibbs measure on T (or equivalently a coherent system on Γ) essentially encodes the multiplicities of the various restrictions of the corresponding superclass functions to the finite levels G_n.

Since Gibbs measures are "the same" as superclass functions, which in turn are "the same" as $\mathbb{G}$-invariant measures on A°, we shall refer to an indecomposable probability Gibbs measure as a $\mathbb{G}$-*ergodic Gibbs measure* on T.

Let M be a $\mathbb{G}$-ergodic Gibbs measure on T with corresponding coherent system $\{M_n\}_{n\in\mathbb{N}}$ and supercharacter χ^M. The Kerov-Vershik ergodic theorem states that for M-almost every path $t = (\mathscr{O}_m)_{m\in\mathbb{N}}$ in T and every $\mathscr{O} \in \Omega_n$, $n \in \mathbb{N}$,

$$\lim_{m\to\infty} \frac{|\mathscr{O}|}{|\mathscr{O}_m|} \big|\{\alpha \in \mathscr{O}_m \colon \alpha_{|n} \in \mathscr{O}\}\big| = M_n(\mathscr{O});$$

in particular, we see that

$$\lim_{m\to\infty} \chi^{\mathscr{O}_m} = \chi^M.$$

This is quite evocative of Proposition 4.1.4; indeed, it is the same phenomenon phrased in different settings.

Now, let φ be a superclass function with corresponding $\mathbb{G}$-invariant measure μ on A° and Gibbs measure M on T. For every $n \in \mathbb{N}$, every $\gamma \in (A_n)^\circ$ and every $t = (\mathscr{O}_j)_{1\leq j\leq n} \in T^n$ with $\mathscr{O}_n = \mathscr{O}^\gamma$, we have

$$\mu([\gamma]_n) = \frac{m(\chi^\gamma, \varphi_{|n})}{|\mathscr{O}^\gamma|} = \frac{M_n(\mathscr{O}^\gamma)}{|\mathscr{O}^\gamma|} = M([t]_n).$$

Finally, for every $\lambda \in A^\circ$, let M^λ denote the $\mathbb{G}$-ergodic Gibbs measure on T which is associated with the supercharacter χ^λ of G. Then, for every $\gamma \in (A_n)^\circ$

$$m\big(\chi^\gamma, (\chi^\lambda)_{|n}\big) = |\mathscr{O}^\gamma| M^\lambda([\gamma]_n) \neq 0 \qquad \Longleftrightarrow \qquad \gamma \in (S^\lambda)_n,$$

and therefore the set

$$T^\lambda = \{(\mathscr{O}_n)_{n\in\mathbb{N}} \in T \colon \mathscr{O}_n \subseteq (S^\lambda)_n\}$$

has full M^λ-measure. As consequence of Kerov-Vershik ergodic theorem, it follows that there is at least one path $t = (\mathscr{O}_n)_{n\in\mathbb{N}} \in T^\lambda$ such that

$$\lim_{n\to\infty} \chi^{\mathscr{O}_n} = \chi^\lambda.$$

It is obvious that we may choose a sequence $(\gamma_n)_{n\in\mathbb{N}}$ such that $\gamma_n \in \mathscr{O}_n$ and $(\gamma_{n+1})_{|n} = \gamma_n$ for all $n \in \mathbb{N}$; thus, we see that this sequence is convergent and that its limit $\lambda' = \lim_{n\to\infty} \gamma_n$ satisfies $\mathscr{O}^{\lambda'} = \mathscr{O}^\lambda$. In conclusion, the Kerov-Vershik ergodic theorem is nothing more than the finite approximation property established in Proposition 4.1.4.

4.3 Supercharacters as induced characters

In this section, for every standard supercharacter χ^λ, $\lambda \in A^\circ$, of an AF-algebra group

$$G = \varinjlim_{n\in\mathbb{N}} G_n,$$

we provide an induced representation which is quasi-equivalent the standard super-representation $(\mathscr{T}^\lambda, \mathscr{H}^\lambda)$, which should be understood as an analogue to [33, Theorem 5.4].

Let $H \subseteq G$ be a subgroup (which is necessarily an open subgroup), consider the quotient G/H equipped with the quotient topology it is a discrete countable topological space (in a more general setting, the quotient of a second countable group by an open subgroup is always a

countable discrete topological space). Let $(\tau, \mathscr{U})$ be a unitary representation of H, following [58], we define the *induced representation from an open subgroup* (in the sense of Mackey)

$$\mathrm{Ind}_H^G(\tau, \mathscr{U}) = (\tau^G, \mathscr{U}^G)$$

as follows:

- $\mathscr{U}^G$ is the vector space consisting of all functions $f : G \to \mathscr{U}$ satisfying

$$f(gh) = \tau(h^{-1})f(g), \qquad g \in G,\ h \in H,$$

  and such that

$$\sum_{gH \in G/H} \|f(g)\|^2 < \infty;$$

- the inner product on $\mathscr{U}^G$ is defined by

$$\langle f_1 | f_2 \rangle = \sum_{gH \in G/H} f_1(g)\overline{f_2(g)}, \qquad f_1, f_2 \in \mathscr{U}^G;$$

- the G-action on $\mathscr{U}^G$ is given by

$$(\tau^G(g)f)(x) = f(g^{-1}x), \qquad g, x \in G,\ f \in \mathscr{U}^G.$$

We observe that, if $f : G \to \mathscr{U}$ is such that $f(gh) = \tau(h^{-1})f(g)$ for all $g \in G$ and all $h \in H$, then the sum $\sum_{gH \in G/H} \|f(g)\|^2$ is well defined: if $g_1, g_2 \in G$ are such that $g_1H = g_2H$, then $g_1 = g_2h$ for some $h \in H$, and thus

$$\|f(g_1)\| = \left\|\tau(h^{-1})f(g_2)\right\| = \|f(g_2)\|$$

because the linear operator $\tau(h^{-1})\colon \mathscr{U} \to \mathscr{U}$ is unitary.

In what follows *Mackey's imprimitivity theorem* (see, for example, [58, Proposition 3.17] for a proof) is paramount.

Theorem 4.3.1 (Mackey's imprimitivity theorem)**.** *Let $(\mathscr{T}, \mathscr{H})$ be a unitary representation of G, and let H be a subgroup of G. Assume that there is a $*$-homomorphism $\Delta : C_0(G/H) \to$ $\mathrm{B}(\mathscr{H})$ (that is, an algebra homomorphism such that $\Delta(f^*) = (\Delta(f)^*)$), where $C_0(G/H)$ denotes the complex vector space consisting of all continuous functions $G/H \to \mathbb{C}$ with compact support, and $\mathrm{B}(\mathscr{H})$ denotes the group consisting of all bounded linear operators of $\mathscr{H}$, satisfying*

- *$\mathscr{T}(g)\Delta(f)\mathscr{T}(g^{-1}) = \Delta(gf)$ for all $g \in G$ and all $f \in C_0(G/H)$ (as usual, we define $gf \in C_0(G/H)$ by $(gf)(x) = f(g^{-1}x)$ for all $x \in G/H$);*
- *$\Delta(C_0(G/H))\mathscr{H}$ is dense in $\mathscr{H}$.*

Then, $(\mathscr{T}, \mathscr{H})$ is equivalent to the induced representation $\mathrm{Ind}_H^G(\sigma, \mathscr{U})$, where

$$\mathscr{U} = \{\Delta(1)v \colon v \in \mathscr{H}\} \qquad \textit{and} \qquad \sigma(h)\xi = \mathscr{T}(h)\xi, \quad h \in H,\ \xi \in \mathscr{U}.$$

We observe that, in our situation, since $G = 1 + A$ is a discrete group any subgroup H is an open subgroup, furthermore, $C_0(G/H)$ is simply the set consisting of all complex-valued functions on G/H having finite support.

Now, we fix $\lambda \in A^\circ$, and define

$$L_\lambda = \{g \in G : g\lambda = \lambda\} \qquad \text{and} \qquad \mathfrak{l}_\lambda = \{a \in A \colon \lambda(ax) = 1 \text{ for all } x \in A\}.$$

Notice that $\mathfrak{l}_\lambda$ is a left ideal of A, and that $L_\lambda = 1 + \mathfrak{l}_\lambda$; in particular, we see that the map $\widetilde{\lambda} : L_\lambda \to \mathbb{C}$ given by

$$\widetilde{\lambda}(h) = \lambda(h-1), \qquad h \in L_\lambda,$$

defines a one-dimensional unitary representation of L_λ (hence, a character of L_λ); indeed, for every $a, b \in L_\lambda$ we evaluate

$$\begin{aligned}\widetilde{\lambda}((1+a)(1+b)) &= \lambda(a+b+ab) = \lambda(a)\lambda(b)\lambda(ab)\\ &= \lambda(a)\lambda(b) = \widetilde{\lambda}(1+a)\tilde{\lambda}(1+b).\end{aligned}$$

Finally, we define (π^λ, V^λ) to be the induced representation

$$\mathrm{Ind}_{L_\lambda}^G(\widetilde{\lambda}, \mathbb{C});$$

our next goal is to show that the representations $(\mathscr{T}^\lambda, \mathscr{H}^\lambda)$ and (π^λ, V^λ) are quasi-equivalent. In order to achieve this, we need the following preliminary result.

Lemma 4.3.2. *For every $\lambda \in A^\circ$, the closure $\overline{\lambda G}$ of the right G-orbit $\lambda G \subseteq A^\circ$ has non zero ω_λ-measure.*

Proof. We fix $\lambda \in A^\circ$, and recall that

$$(S^\lambda)_n = \{\mu_{|n} \colon \mu \in \mathbb{G}\cdot\lambda\}, \qquad n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$, let

$$X_n = \{\mu_{|n} \colon \mu \in \lambda G\} \qquad \text{and} \qquad Y_n = (S^\lambda)_n \setminus X_n;$$

notice that

$$\overline{\lambda G} = \bigcap_{n \in \mathbb{N}} \Big(\bigcup_{\gamma \in X_n} [\gamma]_n \Big),$$

and that

$$\mathscr{O}^\lambda = \Big(\bigcap_{n \in \mathbb{N}} \Big(\bigcup_{\gamma \in X_n} [\gamma] \Big) \Big) \dot{\bigcup} \Big(\bigcap_{n \in \mathbb{N}} \Big(\bigcup_{\gamma' \in Y_n} [\gamma']_n \Big) \Big),$$

and so $\mathscr{O}^\lambda \setminus \overline{\lambda G}$ is a closed set. Consequently, if $\omega_\lambda(\overline{\lambda G}) = 0$, then

$$\operatorname{supp}(\omega_\lambda) = \mathscr{O}^\lambda \setminus \overline{\lambda G},$$

a contradiction. $\square$

Since $\omega_\lambda(\overline{\lambda G}) \neq 0$ and ω_λ is a $\mathbb{G}$-invariant measure, we conclude that for every $g \in G$ the set $\overline{g(\lambda G)} = g\big(\overline{\lambda G}\big)$ has non zero ω_λ-measure. Moreover, since $\bigcup_{g \in G} \overline{g(\lambda G)}$ is a $\mathbb{G}$-invariant set of positive ω_λ-measure, the ergodicity of ω_λ implies that

$$\omega_\lambda \Big(\bigcup_{g \in G} \overline{g(\lambda G)} \Big) = 1.$$

Proposition 4.3.3. *For every $\lambda \in A^\circ$, the representations (π^λ, V^λ) and $(\mathscr{T}^\lambda, \mathscr{H}^\lambda)$ of G are quasi-equivalent.*

Proof. Let $\lambda \in A^\circ$ be arbitrary; for simplicity, we set $L = L_\lambda$ and $(\mathscr{T}, \mathscr{H}) = (\mathscr{T}^\lambda, \mathscr{H}^\lambda)$. For every $g_0 L \in G/L$, we consider the Dirac function $\delta_{g_0 L} : G/L \to \mathbb{C}$; note that $C_0(G/L)$ is the $\mathbb{C}$-linear span of the set $\{\delta_{gL} \colon gL \in G/L\}$. By linear extension, we may define the map $\Delta : C_0(G/L) \to \mathrm{B}(\mathscr{H})$ by the rule

$$\Delta(\delta_{g_0 L})(f) = \mathbb{I}_{\overline{g_0 \lambda G}}\, f, \qquad g_0 L \in G/L,\ f \in \mathscr{H};$$

for simplicity, we write $\Delta_{g_0 L} = \Delta(\delta_{g_0 L})$ for $g_0 L \in G/L$. It is straightforward to check that Δ is a $*$-homomorphism.

We next show that the set $\{\Delta_{g_0 L} f \colon g_0 L \in G/L,\ f \in \mathscr{H}\}$ is dense in $\mathscr{H}$. Let $g_1, \ldots, g_n, \ldots$ be a family of pairwise distinct elements of G such that

$$\overline{g_i \lambda G} \neq \overline{g_j \lambda G}, \qquad i, j \in \mathbb{N},\ i \neq j.$$

For every $g \in G$, let T'_g denote the restriction of T_g to the union $\bigcup_{i\in\mathbb{N}} \overline{g_i\lambda G}$, and notice that since $\bigcup_{i\in\mathbb{N}} \overline{g_i\lambda G}$ has full ω_λ-measure, the images of T_g and T'_g in $\mathscr{H}$ are equal. For every $n \in \mathbb{N}$ and every $1 \leq i \leq n$, let

$$\alpha^{(n)}_{g_i} = \big|\big\{1 \leq j \leq n\colon\ j \neq i,\ \overline{g_j\lambda G} \cap \overline{g_i\lambda G} \neq \emptyset\big\}\big|.$$

For every $g \in G$, we define

$$T^{(n)}_g(\mu) = \sum_{i=1}^{n} \frac{1}{\alpha^{(n)}_{g_i}} \mathbb{I}_{\overline{g_i\lambda G}}(\mu)\, T_g(\mu), \qquad \mu \in A^\circ;$$

we note that the sequence $\big(T^{(n)}_g\big)_{n\in\mathbb{N}}$ is pointwise convergent with

$$\lim_{n\to\infty} T^{(n)}_g = T'_g.$$

On the other hand, we have

$$|T^{(n)}_g(\mu)| \leq 1, \qquad n \in \mathbb{N},\ \mu \in A^\circ;$$

since the function constantly equal to one is measurable and ω_λ-integrable, the dominated convergence theorem [17, Theorem 2.8.5] implies that the sequence $\big(T^{(n)}_g\big)_{n\in\mathbb{N}}$ converges in $\mathscr{H}^\lambda$ with limit

$$\lim_{n\to\infty} T^{(n)}_g = T_g \in \mathscr{H}.$$

It follows that the closure of $\{\Delta_{g_0L} f\colon\ g_0L \in G/L,\ f \in \mathscr{H}\}$ contains a dense subset, and this implies that $\{\Delta_{g_0L} f\colon\ g_0L \in G/L,\ f \in \mathscr{H}\}$ is dense in $\mathscr{H}$.

Let $g \in G$ and let $f \in \mathscr{H}$ be arbitrary. For every $\mu \in A^\circ$, we evaluate

$$\begin{aligned}
\big(\mathscr{T}(g)\Delta_{g_0L}\mathscr{T}(g^{-1})f\big)(\mu) &= T_g(\mu)\ \big(\big(\Delta_{g_0L}\mathscr{T}(g^{-1})\big)f\big)(g^{-1}\mu) \\
&= T_g(\mu)\ \mathbb{I}_{\overline{g_0\lambda G}}(g^{-1}\mu)\ T_{g^{-1}}(g^{-1}\mu)\ f(\mu) \\
&= \mathbb{I}_{\overline{g_0\lambda G}}(g^{-1}\mu)\ f(\mu) = \mathbb{I}_{\overline{gg_0\lambda G}}(\mu)\, f(\mu) \\
&= \big(\Delta_{gg_0L} f\big)(\mu),
\end{aligned}$$

and thus, according to Mackey's imprimitivity theorem,

$$(\mathscr{T}, \mathscr{H}) \simeq \operatorname{Ind}^G_L(\sigma, \mathscr{U}),$$

where

$$\mathscr{U} = \{\mathbb{I}_{\overline{\lambda G}} f\colon\ f \in \mathscr{H}\} \qquad \text{and} \qquad (\sigma(g)\xi)(\mu) = T_g(\mu)\,\xi(g^{-1}\mu), \quad g \in L,\ \xi \in \mathscr{U},\ \mu \in A^\circ.$$

Finally, let $\xi = \mathbb{I}_{\overline{\lambda G}} f \in \mathscr{U}$, $f \in \mathscr{H}$, and let $g \in L$. Since $g^{-1}\lambda = \lambda$, we see that

$$g\mu = \mu \qquad \text{and} \qquad \mu(g-1) = \lambda(g-1), \quad \mu \in \overline{\lambda G}.$$

Therefore, for every $\mu \in A^\circ$ we have

$$(\sigma(g)\xi)(\mu) = T_g(\mu)\,\mathbb{I}_{\overline{\lambda G}}(g^{-1}\mu)\, f(g^{-1}\mu) = \begin{cases} \lambda(g-1)\, f(\mu), & \text{if } x \in \overline{\lambda G}, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $\sigma(g)\xi = \tilde{\lambda}(g)\xi$, and hence the representations $(\tilde{\lambda}, \mathbb{C})$ and $(\sigma, \mathbb{C}\xi)$ of L_λ are equivalent. On the other hand, $(\sigma, \mathscr{U})$ is a direct sum of cyclic representations (see [35, 2.2.7]) each one of them equivalent to $(\tilde{\lambda}, \mathbb{C})$, and so $(\tilde{\lambda}, \mathbb{C})$ is quasi-equivalent to $(\sigma, \mathscr{U})$. Therefore, $(\mathscr{T}^\lambda, \mathscr{H}^\lambda)$ is equivalent to a direct sum of representations each one of them equivalent to (π^λ, V^λ), and this implies that $(\mathscr{T}^\lambda, \mathscr{H}^\lambda)$ is quasi-equivalent to (π^λ, V^λ), as stated. $\square$

In this fashion, we can think of the representation (π^λ, V^λ) as a *linearization* of the representation $(\mathscr{T}^\lambda, H^\lambda)$ of G, and since $\tilde{\lambda}$ is in fact a character of L_λ, the supercharacter χ^λ may be understood as the resulting *induced character*. Moreover, the representation (π^λ, V^λ) does not depend on the measure ω_λ, and hence properties of the supercharacter χ^λ may be extrapolated (at least in theory) without a description of the corresponding $\mathbb{G}$-ergodic measure on A°.

This induction property strengthens the claim that the standard supercharacter theory is a *cruder* version of Kirillov's orbit method, since in the case of nilpotent Lie groups (which are tame), irreducible representations are induced from one-dimensional representations of suitable subgroups (see [63] for all details); we also observe that, just as in the case of nilpotent groups arising from a rational Lie algebra, factor representations are also induced from linear representations (we refer to [31]).

4.3.1 A factorization of supercharacters

Let $G = 1 + A$ be an AF-algebra group over a field $\mathbb{K}$, and consider the unital $\mathbb{K}$-algebra $\mathscr{A} = \mathbb{K} \oplus A$, so that we can naturally identify G with the subgroup of all unipotent elements of $\mathscr{A}$. Furthermore, the *right* G-action on A° admits a natural extension to a right $\mathscr{A}$-action: for every $\mathbf{a} = \alpha + a \in \mathscr{A}$, with $\alpha \in \mathbb{K}$ and $a \in A$, and every $\lambda \in A^\circ$, we define $\lambda\mathbf{a} \in A^\circ$ by

$$(\lambda\mathbf{a})(x) = \lambda(\alpha x + xa), \qquad x \in A;$$

hence, we may consider the right (cyclic) $\mathscr{A}$-module $\lambda\mathscr{A}$. It is the algebraic nature of this right $\mathscr{A}$-module $\lambda\mathscr{A}$ that determines the factorization of the supercharacter χ^λ as a product of other supercharacters.

Recall that two idempotents $e_1, e_2 \in \mathrm{End}_{\mathscr{A}}(\lambda\mathscr{A})$ are said to be *orthogonal* if $e_1e_2 = e_2e_1 = 0$.

Proposition 4.3.4. *Let $\lambda \in A^\circ$ be arbitrary, and consider the right $\mathscr{A}$-module $\lambda\mathscr{A}$. Assume that there are two non-trivial orthogonal idempotents $e_1, e_2 \in \mathrm{End}_{\mathscr{A}}(\lambda\mathscr{A})$ such that $\mathrm{Id} = e_1 + e_2$. If $\lambda_1 = e_1(\lambda)$ and $\lambda_2 = e_2(\lambda)$, then*

$$\chi^\lambda = \chi^{\lambda_1}\chi^{\lambda_2}.$$

Proof. We first prove that $\lambda\mathscr{A} = \lambda_1\mathscr{A} \oplus \lambda_2\mathscr{A}$. Clearly, $\lambda_1\mathscr{A} + \lambda_2\mathscr{A} = \lambda\mathscr{A}$. On the other hand, let $\mathbf{a} \in \mathscr{A}$ be such that $\lambda\mathbf{a} \in \lambda_1\mathscr{A} \cap \lambda_2\mathscr{A}$, and let $\mathbf{a}_1, \mathbf{a}_2 \in \mathscr{A}$ be such that $\lambda\mathbf{a} = \lambda_1\mathbf{a}_1 = \lambda_2\mathbf{a}_2$. Then,

$$\lambda\mathbf{a} = e_1(\lambda\mathbf{a}) + e_2(\lambda\mathbf{a}) = e_1(\lambda_2\mathbf{a}_2) + e_2(\lambda_1\mathbf{a}_1).$$

Since e_1 and e_2 are orthogonal, we have $e_1(\lambda_2\mathbf{a}_2) = e_1e_2(\lambda)\mathbf{a}_2 = 0$, and similarly $e_2(\lambda_1\mathbf{a}_1) = 0$. It follows that $\lambda\mathbf{a} = 0$, and thus $\lambda_1\mathscr{A} \cap \lambda_2\mathscr{A} = 0$ proving the claim.

For every $M \subseteq A$ and every $N \subseteq A^\circ$, we define

$$M^\perp = \{\gamma \in A^\circ \colon \gamma(m) = 1 \text{ for all } m \in M\} \quad \text{and} \quad N^\perp = \{a \in A \colon \gamma(a) = 1 \text{ for all } \gamma \in N\}.$$

Let $i = 1, 2$, and let $x \in (\lambda_i\mathscr{A})^\perp$. Then, $\lambda_i(xa) = 1$ for all $a \in A$, and so $(\lambda_i\mathscr{A})^\perp \subseteq \mathfrak{l}_{\lambda_i}$. On the other hand,

$$A = \{0\}^\perp = (\lambda_1\mathscr{A} \cap \lambda_2\mathscr{A})^\perp = (\lambda_1\mathscr{A})^\perp + (\lambda_2\mathscr{A})^\perp \subseteq \mathfrak{l}_{\lambda_1} + \mathfrak{l}_{\lambda_2} \subseteq A,$$

and thus $\mathfrak{l}_{\lambda_1} + \mathfrak{l}_{\lambda_2} = A$. Since both $\mathfrak{l}_{\lambda_1}$ and $\mathfrak{l}_{\lambda_2}$ are left ideals of A, it follows that

$$L_{\lambda_1}L_{\lambda_2} = 1 + \mathfrak{l}_{\lambda_1} + \mathfrak{l}_{\lambda_2} = 1 + A.$$

On the other hand, we clearly have $L_\lambda = L_{\lambda_1} \cap L_{\lambda_2}$ and $\widetilde{\lambda} = (\widetilde{\lambda}_1)_{|L_\lambda} \otimes (\widetilde{\lambda}_2)_{|L_\lambda}$. According to [72, Theorem 7.2], we conclude that

$$(\pi^\lambda, V^\lambda) \simeq (\pi^{\lambda_1}, V^{\lambda_1}) \otimes (\pi^{\lambda_2}, V^{\lambda_2}),$$

and this implies that $(\mathscr{T}^\lambda, \mathscr{H}^\lambda)$ is quasi-equivalent to

$$(\mathscr{T}^{\lambda_1}, L^2(A^\circ, \omega_{\lambda_1})) \otimes (\mathscr{T}^{\lambda_2}, L^2(A^\circ, \omega_{\lambda_2})) \simeq (\mathscr{T}^{\lambda_1} \times \mathscr{T}^{\lambda_2}, L^2(\mathscr{O}^{\lambda_1} \times \mathscr{O}^{\lambda_2}, \omega_{\lambda_1} \otimes \omega_{\lambda_2})).$$

Therefore, $\chi^\lambda = \chi^{\lambda_1}\chi^{\lambda_2}$ (see [35, Proposition 13.4.9]), as required. $\square$

For every $\lambda \in A^\circ$, there are two non-trivial idempotents in $\mathrm{End}_{\mathscr{A}}(\lambda\mathscr{A})$ which decompose the identity if and only if the right $\mathscr{A}$-module $\lambda\mathscr{A}$ is decomposable; if this is not the case (that is, if $\lambda\mathscr{A}$ is indecomposable), then we shall say that the supercharacter χ^λ is *elementary*.

Corollary 4.3.5. *Let $\lambda \in A^\circ$ be arbitrary. If the right $\mathscr{A}$-module $\lambda\mathscr{A}$ is completely decomposable, the supercharacter χ^λ admits a unique factorization as a product of supercharacters*

$$\chi^\lambda = \chi^{\lambda_1} \cdots \chi^{\lambda_m}$$

where $\lambda_1, \ldots, \lambda_n \in A^\circ$ are such that $\lambda_i\mathscr{A}$, for $1 \leq i \leq m$, are indecomposable submodules of $\lambda\mathscr{A}$ and

$$\lambda\mathscr{A} = \lambda_1\mathscr{A} \oplus \cdots \oplus \lambda_m\mathscr{A}.$$

In particular, if G is finite dimensional (that is, $\dim_{\mathbb{K}}(A) < \infty$), then every supercharacter admits an essentially unique factorization as a product of elementary supercharacters.

Proof. If the $\mathscr{A}$-module $\lambda\mathscr{A}$ is fully decomposable, then it can be written as a finite direct sum

$$\lambda\mathscr{A} = \lambda_1\mathscr{A} \oplus \cdots \oplus \lambda_m\mathscr{A}$$

of indecomposable submodules where $\lambda_1, \ldots, \lambda_n \in \lambda\mathscr{A}$ are such that $\lambda = \lambda_1 + \cdots + \lambda_m$.

On the other hand, if G is finite dimensional then it is straightforward to check that the $\mathscr{A}$-module $\lambda\mathscr{A}$ is both Artinian and Noetherian, and so, according to the Krull-Schmidt theorem, the decomposition above is essentially unique. The result follows by the previous proposition using an inductive argument on the dimension of A. $\square$

In this sense for a finite dimensional discrete algebra group there is a canonical factorization of supercharacters as a finite product of elementary supercharacters. The infinite dimensional case is much more sensitive: it may happen that a right $\mathscr{A}$-module $\lambda\mathscr{A}$, for $\lambda \in A^\circ$, is not completely decomposable, that is, the identity $\mathrm{Id} \in \mathrm{End}_{\mathscr{A}}(\lambda\mathscr{A})$ might be written as an infinite sum of non-trivial pairwise orthogonal idempotents and, if any of the summands is not primitive, then the decomposition can be refined and this will lead to an *ill-behaved* factorization of supercharacters (it is asymptotic in nature and may not be unique).

Corollary 4.3.6. *Let $\lambda \in A^\circ$ be arbitrary. If the right $\mathscr{A}$-module $\lambda\mathscr{A}$ is not completely decomposable, then there is at least one family $(e_i)_{i\in\mathbb{N}}$ of non-trivial pairwise orthogonal idempotents such that $id = e_1 + \cdots + e_n$ for all $n \in \mathbb{N}$ and such that*

$$\chi^\lambda = \lim_{n\to\infty} \prod_{i=1}^{n} \chi^{\lambda_i}.$$

where $\lambda_i = e_i(\lambda)$ *for all* $i \in \mathbb{N}$.

Proof. If $\lambda\mathscr{A}$ is not completely decomposable, then there are at least two non-trivial orthogonal idempotents $e_1, e_2 \in \mathrm{End}_{\mathscr{A}}(\lambda\mathscr{A})$ such that $\mathrm{Id} = e_1 + e_2$, which implies that $\chi^{\lambda} = \chi^{\lambda_1}\chi^{\lambda_2}$ for $\lambda_1 = e_1(\lambda)$ and $\lambda_2 = e_2(\lambda)$. Since $\lambda\mathscr{A}$ is not completely decomposable, at least one of these idempotents is not primitive, and thus it can be decomposed as a sum of two non-trivial orthogonal idempotents which will refine the previous factorization. The result follows by an inductive argument. $\square$

This difference in behavior highlights how the algebraic anatomy of the algebra group influences the supercharacter structure. Heuristically, just as the regular representation, this shows that properties of the standard supercharacter theory do not depend on the class of discrete algebra groups, but rather on the algebraic structure of each individual group.

Chapter 5

Other supercharacter theories and Kirillov functions

All the ideas (and proofs) presented so far can be adapted to a more general framework, making it possible to consider different supercharacter theories and "special functions" on G.

In what follows , we briefly explain how this can be achieved; we pay special attention to a particular family of functions to which Diaconis and Isaacs in [33] refer to as *Kirillov functions*. While in general Kirillov functions are not characters, they constitute an important family of class functions which is rich enough to (at least theoretically) describe every character. Furthermore, we will prove that a supercharacter is an indecomposable character if and only if it is a Kirillov function.

5.1 Special functions and other supercharacter theories

Let $G = 1 + A$ be a countable discrete algebra group, not necessarily amenable for the moment, and assume that there is a discrete amenable group $\mathscr{G}$ acting on A (on the left). Then, there is the natural corresponding contragradient action of $\mathscr{G}$ on A°: for every $\lambda \in A^\circ$ and every $\mathbf{g} \in \mathscr{G}$, we define

$$(\mathbf{g}\cdot\lambda)(a) = \lambda(\mathbf{g}^{-1}\cdot a), \qquad a \in A.$$

Then, the set

$$\mathscr{K}_{\mathscr{G}} = \{1 + \mathscr{G}\cdot a : \; a \in A\}$$

forms a partition of G, and the set consisting of all bounded functions defined on G which are constant on the members of $\mathscr{K}_{\mathscr{G}}$, equipped with the pointwise convergence topology, forms a family of *special functions* to which we will refer to as *$\mathscr{G}$-functions* on G.

By mimicking the previous proofs, it is possible to show that the $\mathscr{G}$-functions on G are in one-to-one correspondence with the $\mathscr{G}$-invariant measures on A°, and that every such measure has an integral decomposition with respect to a (complex) measure on the set of ergodic $\mathscr{G}$-measures.

For every $\lambda \in A^\circ$, let $\mathscr{O}_{\mathscr{G}}^{\lambda}$ denote the closure in A° of the $\mathscr{G}$-orbit $\mathscr{G}\cdot\lambda$, and let

$$\Omega_{\mathscr{G}} = \{\mathscr{O}_{\mathscr{G}}^{\lambda} \colon \lambda \in A^\circ\}.$$

Then, the amenability of $\mathscr{G}$ assures that every orbit closure $\mathscr{O}_{\mathscr{G}}^{\lambda}$, for $\lambda \in A^\circ$, supports a single $\mathscr{G}$-ergodic measure. We shall identify $\Omega_{\mathscr{G}}$ with the set consisting of all $\mathscr{G}$-functions: for every $\mathscr{O} \in \Omega$ let $\omega_{\mathscr{O}}$ be the unique $\mathscr{G}$-ergodic measure supported on $\mathscr{O}$, the corresponding $\mathscr{G}$-function, that we denote by $\chi^{\mathscr{O}}$, is given by

$$\chi^{\mathscr{O}}(g) = \int_{A^\circ} T_g \, d\omega_{\mathscr{O}}, \qquad g \in G.$$

According to this identification, if we equip $\Omega_{\mathscr{G}}$ with the pointwise convergence topology, then $\Omega_{\mathscr{G}}$ becomes a compact Hausdorff metrizable space whose $\mathbb{C}$-linear span is dense in the set of all $\mathscr{G}$-functions.

If we assume that $\mathscr{K}_{\mathscr{G}}$ is a family of superclasses, then the set of consisting of all characters which are constant on the elements of $\mathscr{K}_{\mathscr{G}}$ forms a Choquet simplex, and thus is fully determined by the set $\mathscr{E}_{\mathscr{G}}$ consisting of all indecomposable elements. However, elements in $\mathscr{E}_{\mathscr{G}}$ might not be in correspondence with the $\mathscr{G}$-ergodic measures on A°, nonetheless, every element of $\mathscr{E}_{\mathscr{G}}$ admits a unique integral decomposition over $\Omega_{\mathscr{G}}$ with respect to a probability measure.

Let $\mathscr{G}_1 \subseteq \mathscr{G}_2$ be amenable discrete countable groups (with $\mathscr{G}_1$ being a subgroup of $\mathscr{G}_2$). Then, if $\mathscr{G}_2$ acts on A, then $\mathscr{G}_1$ also acts on A. For every $a \in A$ the $\mathscr{G}_2$-orbit $\mathscr{G}_2\cdot a$ is clearly $\mathscr{G}_1$-invariant, and thus it must be a union of $\mathscr{G}_1$-orbits, which implies that every member of $\mathscr{K}_{\mathscr{G}_2}$ is a union of members of $\mathscr{K}_{\mathscr{G}_1}$. Furthermore, every $\mathscr{G}_2$-function on G can be decomposed in terms of indecomposable $\mathscr{G}_1$-functions on G, and for this reason we say that the pair $(\mathscr{K}_{\mathscr{G}_2}, \Omega_{\mathscr{G}_2})$ is *coarser* then $(\mathscr{K}_{\mathscr{G}_1}, \Omega_{\mathscr{G}_1})$ and write $(\mathscr{K}_{\mathscr{G}_2}, \Omega_{\mathscr{G}_2}) \preceq (\mathscr{K}_{\mathscr{G}_1}, \Omega_{\mathscr{G}_1})$ to indicate so. Notice that a similar relation holds if we replace $\Omega_{\mathscr{G}_1}$ and $\Omega_{\mathscr{G}_2}$ by $\mathscr{E}_{\mathscr{G}_1}$ and $\mathscr{E}_{\mathscr{G}_2}$, respectively.

Example 5.1.1. Let $G = 1 + A$ be an amenable countable discrete group over $\mathbb{K}$, and let $\mathbb{K}^\times$ denote the multiplicative group of $\mathbb{K}$ (notice that, being an abelian group, $\mathbb{K}^\times$ is amenable). Since $\mathbb{K}^\times$ acts naturally on both A and A°, the direct product $\mathscr{G} = \mathbb{K}^\times \times \mathbb{G}$ acts on both A and A°, and these actions extend in the natural way the actions of $\mathbb{G} = G \times G$. Since the product of amenable groups is amenable, the group $\mathscr{G}$ is amenable and it induces a supercharacter theory of G. In this case, for every $a \in A$ and every $\lambda \in A^\circ$ we have

$$\mathscr{G}\cdot a = \bigcup_{\alpha \in \mathbb{K}^\times} \mathbb{G}\cdot(\alpha a) \qquad \text{and} \qquad \mathscr{O}^\lambda_{\mathscr{G}} = \overline{\bigcup_{\alpha \in \mathbb{K}^\times} \mathscr{O}^{\alpha\lambda}}.$$

Example 5.1.2 (The uncolored supercharacter theory of unitriangular groups)**.** For $n \in \mathbb{N}$, let $G_n = U_n(\mathbb{K})$, and let $B_n = B_n(\mathbb{K})$ the group of all invertible uppertriangular matrices over $\mathbb{K}$, it acts naturally on $A_n = \mathfrak{u}_n(\mathbb{K})$ both on the left and the right (by multiplication), and thus the direct product $\mathbb{B}_n = B_n \times B_n$ (which is an amenable group) also acts on A_n. The resulting supercharacter theory is known as the *uncolored supercharacter theory* of $U_n(\mathbb{K})$ (the reason for this terminology will become clear in Section 6). Letting $\mathbb{B}_\infty = \varinjlim_{n\in\mathbb{N}} \mathbb{B}_n$, we may consider the *uncolored supercharacter theory* of $U_\infty(\mathbb{K})$ which arises from the action of the group $\mathbb{B}_\infty$.

We remark that in [8] this uncolored supercharacter theory of $U_\infty(\mathbb{F}_q)$ was considered and completely characterized.

5.2 Kirillov functions

Let $G = 1 + A$ be an amenable discrete countable algebra group. An important class of special functions arise when we consider $\mathscr{G} = G$ acting on A and A° *via* conjugation, that is, for every $a \in A$, every $\lambda \in A^\circ$ and every $g \in G$

$$g\cdot a = gag^{-1} \qquad \text{and} \qquad (g\cdot\lambda)(a) = \lambda(g^{-1}ag).$$

In this situation, $\mathscr{K}_G$ is simply the set consisting of all conjugacy classes of G, and the G-functions are the class functions of G.

For every $\lambda \in A^\circ$, we denote by O^λ the orbit closure of $G\cdot\lambda$, and by w_λ the corresponding G-ergodic measure on A° which is supported on O^λ. Furthermore, we refer to the class function determined by w_λ as the *Kirillov function* associated with O^λ; this terminology is borrowed from the finite group scenario (see for example [33, 78]).

Let $\mathbf{O} = \{O^\lambda : \lambda \in A^\circ\}$ denote the space of orbit closures, which we identify with the set consisting of all Kirillov functions of G; for every $O \in \mathbf{O}$ we denote by ψ^O the Kirillov function of G which corresponds to O, or (when appropriated) by ψ^λ for any $\lambda \in A^\circ$ such that $O = O^\lambda = \overline{G{\cdot}\lambda}$. For every (bounded) class function φ of G, there is a measure μ (in general complex) on $\mathbf{O}$ such that

$$\varphi(g) = \int_{\mathbf{O}} \psi^O(g)\, d\mu, \qquad g \in G;$$

however, it is not always true that Kirillov functions are characters of G (for the relationship between Kirillov and supercharacters of finite algebra groups we refer to [78]). By the way of example, in the case of the finite unitriangular group $U_n(\mathbb{F}_q)$, it is known that Kirillov functions are not always characters (see [56]), and this implies that there are Kirillov functions of the locally finite unitriangular group $U_\infty(\mathbb{F}_q)$ which are not characters. Therefore, in general the set $\Omega_G = \mathbf{O}$ (as a set of class functions) is not equal to $\mathscr{E}_G = \mathrm{Ex}(G)$. Nonetheless, since every character of G determines a unique G-invariant probability measure on A°, and since every Kirillov function is associated with a G-ergodic measure, if a Kirillov function is a character, then it must be indecomposable.

For every indecomposable character $\xi \in \mathrm{Ex}(G)$, let μ^ξ denote the corresponding G-invariant measure on A°; furthermore, for every (standard) supercharacter $\chi^{\mathscr{O}}$ of G, let $M_{\mathscr{O}} \in \mathscr{M}^+_{\mathbb{G}}(\mathrm{Ex}(G))$ be the Choquet measure associated with it. Then, for all $g \in G$ we have

$$\chi^{\mathscr{O}}(g) = \int_{\mathrm{Ex}(G)} \xi(g)\, dM_{\mathscr{O}} = \int_{\mathrm{Ex}(G)} \left(\int_{A^\circ} T_g\, d\mu^\xi \right) dM_{\mathscr{O}},$$

which implies that

$$\omega_{\mathscr{O}} = \int_{\mathrm{Ex}(G)} \mu^\xi\, dM_{\mathscr{O}},$$

and thus $\mathrm{supp}(\mu^\xi) \subseteq \mathscr{O}$ for $M_{\mathscr{O}}$-almost all $\xi \in \mathrm{supp}(M_{\mathscr{O}})$. Moreover, the following is true; here, we use the notation ψ^λ for the Kirillov function ψ^{O^λ}.

Proposition 5.2.1. *For every $\lambda \in A^\circ$, the supercharacter χ^λ of G is an indecomposable character if and only if $\mathscr{O}^\lambda = O^\lambda$; equivalently, if and only if $\chi^\lambda = \psi^\lambda$.*

Proof. Let $\lambda \in A^\circ$ be arbitrary, and notice that $\chi^\lambda = \psi^\lambda$ if and only if $\omega_\lambda = w_\lambda$, which is the case if and only if $\mathscr{O}^\lambda = O^\lambda$. Moreover, it is clear that, if $\chi^\lambda = \psi^\lambda$, then ψ^λ is a character of G, and therefore it must be indecomposable.

Now, assume that χ^λ is an indecomposable character of G. Then, the standard super-representation $(\mathscr{T}^\lambda, \mathscr{H}^\lambda)$ is a factor representation of G, and thus all of its subrepresentations

are quasi-equivalent to $(\mathscr{T}^\lambda, \mathscr{H}^\lambda)$ (see [35, Proposition 5.2.5]). Let X be a G-invariant (under conjugation) subset of $\mathscr{O}^\lambda$. If $\omega_\lambda(X) \neq 0$, then we define the function $f : A^\circ \to \mathbb{C}$ by

$$f(\lambda') = \frac{1}{\omega_\lambda(X)^{1/2}}\, \mathbb{I}_X(\lambda'), \qquad \lambda' \in A^\circ,$$

and denote by $\mathscr{H}_0$ the Hilbert space generated by the set $\{\mathscr{T}^\lambda(g)f \colon g \in G\}$. It is clear that the function f has norm one, and that it is a cyclic vector of the representation $(\mathscr{T}^\lambda, \mathscr{H}_0)$. On the other hand, $(\mathscr{T}^\lambda, \mathscr{H}_0)$ is a subrepresentation of the factor representation $(\mathscr{T}^\lambda, \mathscr{H}^\lambda)$, and consequently

$$\begin{aligned}\int_{A^\circ} \big(\mathscr{T}^\lambda(g)f\big)(\lambda')\overline{f(\lambda')}\, d\omega_\lambda &= \int_{A^\circ} T_g(\lambda')\, f(g^{-1}\lambda')\,\overline{f(\lambda')}\, d\omega_\lambda \\ &= \int_X T_g(\lambda') f(g^{-1}\lambda')\overline{f(\lambda')}\, d\omega_\lambda = \chi^\lambda(g).\end{aligned}$$

Using the Gelfand-Naimark-Segal construction (see [16, Section II.6.4] for details), it is possible to check that there is an invertible unitary intertwining operator $\Psi : \mathscr{H}^\lambda \to \mathscr{H}_0$ such that $\Psi(T_1) = f$. Since T_g is a non-zero function, its image $\Psi(T_g)$ is also non-zero for all $g \in G$; furthermore,

$$\Psi(T_g) = \Psi(\mathscr{T}^\lambda(g)T_1) = \mathscr{T}^\lambda(g)f$$

and

$$\begin{aligned} 1 = \langle T_g | T_g\rangle_{\mathscr{H}^\lambda} &= \langle \mathscr{T}^\lambda(g)f | \mathscr{T}^\lambda(g)f\rangle_{\mathscr{H}_0} \\ &= \int_X T_g(\lambda')\, f(g^{-1}\lambda')\,\overline{T_g(\lambda')\, f(g^{-1}\lambda')}\, d\omega_\lambda \\ &= \int_X f(g^{-1}\lambda')\, f(g^{-1}\lambda')\, d\omega_\lambda \\ &= \frac{1}{\omega_\lambda(X)}\, \omega_\lambda(g^{-1}X \cap X).\end{aligned}$$

Consequently, due to the fact that $\omega_\lambda(g^{-1}X \cap X) \neq 0$, the intersection $g^{-1}X \cap X$ is non-empty for all $g \in G$. Furthermore, since $\omega_\lambda(gX \cap X) = \omega_\lambda(X)$ for all $g \in G$, it follows that

$$\omega_\lambda\Big(\bigcap_{g\in G} gX\Big) = \omega_\lambda(X).$$

Let $h \in G$ be arbitrary and let $X_0 = \bigcap_{g\in G} gX$. Then,

$$hX_0 = \bigcap_{g\in G} hgX = X_0 \qquad \text{and} \qquad X_0 h = \bigcap_{g\in G} gh(h^{-1}Xh) = \bigcap_{g\in G} ghX = X_0,$$

which means that X_0 is a $\mathbb{G}$-invariant set. Since ω_λ is a $\mathbb{G}$-ergodic measure and X_0 as non-zero ω_λ-measure, we conclude that

$$1 = \omega_\lambda(X_0) \leq \omega_\lambda(X) \leq 1.$$

The above argument shows that, for an arbitrary G-invariant subset X of A°, either $\omega_\lambda(X) = 0$ or $\omega_\lambda(X) = 1$. Therefore, ω_λ is a G-ergodic measure, and thus we must have $\omega_\lambda = w_\lambda$, which completes the proof. $\square$

In a certain sense, the previous propositions shows that, in order for a supercharacter χ^λ of G to be an indecomposable character, the orbit closure $\mathscr{O}^\lambda$ must be small enough.

Chapter 6

The infinite unitriangular groups in positive characteristic

Having lay down a theoretical framework for supercharacter theories of amenable countable discrete algebra groups, in this section we describe and explore both the standard and uncolored supercharacter theories of the two types of infinite unitriangular groups in positive characteristic p, $U_n(\mathbb{F})$ and $U_\infty(\mathbb{F}_q)$; these may be considered the main prototypes of infinite discrete groups in non-zero characteristic. We should mention that Kirillov orbit method does not necessarily applies to these groups, and hence they present a relevant scenario for supercharacter theories; we also mention that, after understanding these supercharacter theories, it is fairly easy to describe the supercharacter theory of $U_\infty(\mathbb{F})$.

Both $U_n(\mathbb{F})$ and $U_\infty(\mathbb{F}_q)$ are AF-algebra groups and, since there is an explicit description of the supercharacter theory of the finite unitriangular group $U_n(\mathbb{F}_q)$ (see for example [1,6] for details), our main tool is the finite approximation property (Proposition 4.1.4), which allows us to take limits of (normalized) finite supercharacters to describe the required supercharacters. Using the explicit formula for the two supercharacter theories, we can asymptotically derive the corresponding formulas in the infinite scenario.

The different asymptotics are considered with respect to the field in the case of $U_n(\mathbb{F}) = \varinjlim_{m\in\mathbb{N}} U_n(\mathbb{F}_{p^m})$, and with respect to the dimension (of the algebra group) in the case of $U_\infty(\mathbb{F}_q) = \varinjlim_{n\in\mathbb{N}} U_n(\mathbb{F}_q)$, giving rise to structural differences in the corresponding supercharacter theories. The major difference is in the behavior of the regular character as a standard superclass character: in the case of $U_n(\mathbb{F})$ it is not a supercharacter, contrary to the case of $U_\infty(\mathbb{F}_q)$ where it is a supercharacter.

We begin by providing a very brief characterization of the standard supercharacter theory of the finite unitriangular group $U_n(\mathbb{F}_q)$ (all details can be found in [6, 7]): we explain how superclasses are encoded by $\mathbb{F}_q$-colored set partitions of $[n] = \{1, ..., n\}$, while supercharacters are parametrized by $\mathbb{F}_q^\circ$-colored set partitions, where $\mathbb{F}_q^\circ$ denotes the dual group of the additive group $\mathbb{F}_q^+$. From this characterization it follows that the uncolored supercharacter theory is parametrized simply by (uncolored) set partitions of $[n]$.

As it should be expected, given the asymptotic nature of the standard/uncolored supercharacter theory of $U_n(\mathbb{F})$, the standard superclasses of $U_n(\mathbb{F})$ are described by $\mathbb{F}$-colored set partitions and supercharacters are determined by $\mathbb{F}^\circ$-colored set partitions; on the other hand, the uncolored supercharacter theory is in one-to-one correspondence with set partitions of $[n]$, yielding a finite supercharacter theory.

As for the group $U_\infty(\mathbb{F}_q)$, supercharacters are dictated by the *limit shape* of either $\mathbb{F}_q^\circ$-colored or uncolored set partitions, which leads to *new* combinatorial objects: the ($\mathbb{F}_q^\circ$-colored) *augmented set partitions*. As a combinatorial object, augmented set partitions are somewhat uninteresting, however they provide a clear picture of the asymptotic nature of the supercharacters of $U_\infty(\mathbb{F}_q)$.

For both groups, using the induced model for standard super-representations, we are able to classify super-representations according to their type. This classification is achieved with Corwin's classification theorems [30, Theorems 5,6 and 7], which we now summarize. Let G be a countable discrete group, let H be a subgroup of G, and let $(\sigma, \mathscr{H})$ be a representation of H; furthermore, let

- $N_H = \{g \in G\colon gHg^{-1} = H,\ \sigma^g = \sigma\}$ where $\sigma^g(h) = \sigma(g^{-1}hg)$ for all $g \in G$ and all $h \in H$;
- $M_H = \{g \in N_H :$ the N_H-conjugates of g lie in finitely many H-cosets$\}$;
- $M'_H = [M_H, M_H]$, the commutator subgroup of M_H.

Proposition 6.0.1 (Corwin's classification theorem)**.** *With the notation as above, the induced representation* $\mathrm{Ind}_H^G(\sigma, \mathscr{H})$ *is type II if and only if one of the indexes* $|N_H : M_H|$ *or* $|M'_H : H \cap M'_H|$ *is infinite.*

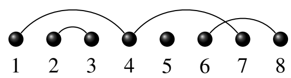
(In a way, it is the size of certain subgroups that determine the type of the supercharacters.)

6.1 The finite unitriangular group $U_n(\mathbb{F}_q)$

In this section, we consider the finite unitriangular group $U_n(\mathbb{F}_q) = 1 + \mathfrak{u}_n(\mathbb{F}_q)$ equipped with its standard supercharacter theory $(\mathscr{K}, \mathscr{E})$; for simplicity, we let $\mathbb{G} = U_n(\mathbb{F}_q) \times U_n(\mathbb{F}_q)$. In what follows, we aim to explain how $(\mathscr{K}, \mathscr{E})$ can be fully parametrized by *colorations* of set partitions of $[n] = \{1, ..., n\}$ and present a supercharacter formula depending only on them.

We denote by $\mathbf{SP}(n)$ the set consisting of all set partitions of $[n]$, and we write $\sigma \in \mathbf{SP}(n)$ as a sequence $\sigma = B_1/B_2/.../B_k$ where $B_1, \ldots, B_k$ are disjoint subsets of $[n]$ such that $[n] = B_1 \cup \cdots \cup B_k$; we refer to $B_1, \ldots, B_k$ as the *blocks* of σ. A pair (i, j) with $1 \leq i < j \leq n$ is said to be an *arc* of $\sigma \in \mathbf{SP}(n)$ if i and j both lie in the same block of σ; we denote by $D(\sigma)$ the set of arcs of σ.

For example, if $\sigma = 1,4,7/2,3/6,8/5$ is a set partition of $[8]$, then its set of arcs is $D(\sigma) = \{(1,4), (2,3), (4,7), (6,8)\}$; in pictorial terms we may represent σ as a graph:



.

If $\sigma \in \mathbf{SP}(n)$, then a map $\phi : D(\sigma) \to \mathbb{F}_q \setminus \{0\}$ is called a $\mathbb{F}_q$-*coloration* of σ (to be rigorous, we may assume that the color $0 \in \mathbb{F}_q$ just deletes a possible arc). The set consisting of all $\mathbb{F}_q$-colorations of $\sigma \in \mathbf{SP}(n)$ will be denoted by $\mathrm{Col}_{\mathbb{F}_q}(\sigma)$, and we let

$$\Phi_n(\mathbb{F}_q) = \{(\sigma, \phi) \colon \sigma \in \mathbf{SP}(n),\ \phi \in \mathrm{Col}_{\mathbb{F}_q}(\sigma)\}$$

stand for the set consisting of all $\mathbb{F}_q$-colored set partitions of $[n]$. For every $(\sigma, \phi) \in \Phi_n(\mathbb{F}_q)$, we define $e_{\sigma,\phi} \in \mathfrak{u}_n(\mathbb{F}_q)$ to be the element

$$e_{\sigma,\phi} := \sum_{(i,j) \in D(\sigma)} \phi(i,j) e_{i,j},$$

where as usual $e_{i,j}$ stands for the matrix having 1 in the (i, j)-th entry and 0 in all other entries. It can be proven that for every superclass $K \in \mathscr{K}$ there is a unique $(\sigma, \phi) \in \Phi_n(\mathbb{F}_q)$ such that $1 + e_{\sigma,\phi} \in K$, thus establishing a bijection between the sets $\mathscr{K}$ and $\Phi_n(\mathbb{F}_q)$; we denote by $K_{\sigma,\phi}$ the superclass associated with $(\sigma, \phi) \in \Phi_n(\mathbb{F}_q)$.

Let $\mathbb{F}_q^\circ$ denote the dual group of the additive group $\mathbb{F}_q^+$ (notice that as a group $\mathbb{F}_q^\circ$ is isomorphic to $\mathbb{F}_q^+$). We identify the dual group $\mathfrak{u}_n(\mathbb{F}_q)^\circ$ of $\mathfrak{u}_n(\mathbb{F}_q)$ with the set $\mathfrak{u}_n(\mathbb{F}_q^\circ)$ consisting of all strictly uppertriangular $n \times n$ matrices with coefficients in $\mathbb{F}_q^\circ$: on the one hand, for every

$\psi = (\psi_{i,j}) \in \mathfrak{u}_n(\mathbb{F}_q^\circ)$ we define $\psi_0 \in \mathfrak{u}_n(\mathbb{F}_q)^\circ$ by

$$\psi_0(a) = \prod_{1\leq i<j\leq n} \psi_{i,j}(a_{i,j}), \qquad a = (a_{i,j}) \in \mathfrak{u}_n(\mathbb{F}_q);$$

on the other hand, for every $\lambda \in \mathfrak{u}_n(\mathbb{F}_q)^\circ$ we define $\lambda^0 = (\lambda^0_{i,j}) \in \mathfrak{u}_n(\mathbb{F}_q^\circ)$ by

$$\lambda^0_{i,j}(\alpha) = \lambda(\alpha e_{i,j}), \qquad 1 < i < j < n,\ \alpha \in \mathbb{F}_q.$$

It is obvious that the correspondences $\psi \mapsto \psi_0$ and $\lambda \mapsto \lambda^0$ are inverses of each other.

In order to understand the nature of the $\mathbb{G}$-action on $\mathfrak{u}_n(\mathbb{F}_q^\circ) = \mathfrak{u}_n(\mathbb{F}_q)^\circ$, it is useful to introduce the following notation: for every $\tau \in \mathbb{F}_q^\circ$ and every $a \in \mathfrak{u}_n(\mathbb{F}_q)$, we define $\tau \star a^* \in \mathfrak{u}_n(\mathbb{F}_q^\circ)$ by

$$(\tau \star a^*)(b) = \tau(\mathrm{Tr}(a^\intercal b)), \qquad b \in \mathfrak{u}_n(\mathbb{F}_q);$$

we recall that $(\tau+\tau')(a) = \tau(a)\tau'(a)$ for all $\tau, \tau' \in \mathbb{F}_q^\circ$ and all $a \in \mathbb{F}_q$. We observe that

$$(\tau \star e^*_{i,j})(b) = \tau(b_{i,j}), \qquad b = (b_{i,j}) \in \mathfrak{u}_n(\mathbb{F}_q),$$

and thus

$$\lambda = \sum_{1<i<j<n} \lambda_{i,j} \star e^*_{i,j}, \qquad \lambda \in \mathfrak{u}_n(\mathbb{F}_q^\circ).$$

On the other hand, straightforward calculations show that

$$(g,h)\cdot(\tau \star e^*_{i,j}) = \tau \star (g^{-\intercal} e_{i,j} h^\intercal)^*, \qquad (g,h) \in \mathbb{G},\ 1 < i < j < n$$

and consequently

$$(g,h)\cdot\lambda = \sum_{1<i<j<n} \lambda_{i,j} \star \sup\left((g^{-\intercal} e_{i,j} h^\intercal)^*\right), \qquad (g,h) \in \mathbb{G},\ \lambda \in \mathfrak{u}_n(\mathbb{F}_q^\circ),$$

where $\sup((g^{-\intercal} e_{i,j} h^\intercal)^*)$ denotes the uppertriangular matrix consisiting of all stricly uppertriangular entries of $(g^{-\intercal} e_{i,j} h^\intercal)^*$.

For every $\pi \in \mathbf{SP}(n)$ a map $\varphi : D(\pi) \to \mathbb{F}_q^\circ \setminus \{0\}$ is called a $\mathbb{F}_q^\circ$-*coloration* of π, and we denote by $\Phi_n(\mathbb{F}_q^\circ)$ the set of all $\mathbb{F}_q^\circ$-colored set partitions of $[n]$; if $(\pi,\varphi) \in \Phi_n(\mathbb{F}_q^\circ)$, then we set $\varphi(i,j) = \varphi_{i,j}$ for $1 \leq i < j \leq n$, and we define the element

$$e^*_{\pi,\varphi} = \sum_{(i,j)\in D(\pi)} \varphi_{i,j} \star e^*_{i,j} \in \mathfrak{u}_n(\mathbb{F}_q^\circ).$$

According to the way we realize the $\mathbb{G}$-action on $\mathfrak{u}_n(\mathbb{F}_q^\circ)$ (and in the same spirit as for superclasses), it can be shown that for every $\lambda \in \mathfrak{u}_n(\mathbb{F}_q^\circ)$ there is a unique $(\pi,\varphi) \in \Phi_n(\mathbb{F}_q^\circ)$ such

that $e^*_{\pi,\varphi} \in \mathscr{O}^\lambda = \mathbb{G} \cdot \lambda$, thus establishing a bijection between the sets $\mathscr{E}$ and $\Phi_n(\mathbb{F}_q^\circ)$; we denote by $\chi^{\pi,\varphi}$ the supercharacter associated with $(\pi,\varphi) \in \Phi_n(\mathbb{F}_q^\circ)$.

Let $\mathscr{A} = \mathbb{F}_q 1 + \mathfrak{u}_n(\mathbb{F}_q)$. It is straightforward to check that for every $\tau \in \mathbb{F}_q^\circ$ and every $1 \leq i < j \leq n$ the right $\mathscr{A}$-submodule $(\tau \star e^*_{i,j})\mathscr{A}$ of $\mathfrak{u}_n(\mathbb{F}_q)^\circ$ is indecomposable; indeed, it consists of all elements of the form

$$\sum_{i<t\leq j} (\tau_t \star e^*_{i,t}), \qquad \tau_t \in \mathbb{F}_q^\circ,\ i < t \leq j.$$

Furthermore, for every $\mathbb{F}_q^\circ$-colored set partition (π,φ), the $\mathscr{A}$-submodule $e^*_{\pi,\varphi}\mathscr{A}$ of $\mathfrak{u}_n(\mathbb{F}_q)^\circ)$ decomposes as the direct sum

$$e^*_{\pi,\varphi}\mathscr{A} = \bigoplus_{(i,j)\in D(\pi)} (\varphi_{i,j} \star e^*_{i,j})\mathscr{A}.$$

According to Corollary 4.3.5, it follows that the supercharacter $\chi^{\pi,\varphi}$ decomposes as a product

$$\chi^{\pi,\varphi} = \prod_{(i,j)\in D(\pi)} \chi^{(i,j),\varphi_{i,j}}$$

of elementary supercharacters.

In order to present a (normalized) supercharacter formula, we now define for every set partition $\pi \in \mathbf{SP}(n)$ the sets

$$\mathrm{Sing}(\pi) = \{(i,l),(k,j) \colon (i,j) \in D(\pi),\ j < l,\ k < i\}, \quad \text{and}$$

$$\mathrm{Reg}(\pi) = \{(i,j) \colon 1 \leq i < j \leq n\} \setminus \mathrm{Sing}(\pi);$$

moreover, for every $1 \leq i < j \leq n$ and every $\pi, \sigma \in \mathbf{SP}(n)$, we define the *nesting numbers*

$$\mathrm{nest}_{(i,j)}(\sigma) = |\{(k,l) \in D(\sigma) :\ i < k < l < j\}|, \quad \text{and}$$

$$\mathrm{nest}_\pi(\sigma) = \sum_{(i,j)\in D(\pi)} \mathrm{nest}_{(i,j)}(\sigma).$$

Then, the (normalized) supercharacter values are fully determined; as follows see [6].

Proposition 6.1.1. *Let $(\pi,\varphi) \in \Phi_n(\mathbb{F}_q^\circ)$ and $(\sigma,\phi) \in \Phi_n(\mathbb{F}_q)$ be arbitrary. Then, the value $\chi^{\pi,\varphi}(K_{\sigma,\phi})$ of the (normalized) supercharacter $\chi^{\pi,\varphi} \in \mathscr{E}$ on the superclass $K_{\sigma,\phi} \in \mathscr{K}$ is equal to* 0 *unless $D(\sigma) \nsubseteq \mathrm{Sing}(\pi)$, in which case it is given by the formula*

$$\chi^{\pi,\varphi}(K_{\sigma,\phi}) = \frac{1}{q^{\mathrm{nest}_\pi(\sigma)}} \prod_{(i,j)\in D(\pi)\cap D(\sigma)} \varphi_{i,j}(\phi(i,j)).$$

In this fashion, the standard supercharacter theory of $U_n(\mathbb{F}_q)$ is fully encoded by the set partitions of $[n]$ and by the corresponding $\mathbb{F}_q$-colorations and $\mathbb{F}_q^\circ$-colorations.

The uncolored supercharacter theory of $U_n(\mathbb{F}_q)$ is a combinatorial simplification of the standard supercharacter theory, and as hinted by its name it is fully parametrized by the *uncolored* set partitions.

Recall that $\mathbb{B} = B \times B$, where $B = B_n(\mathbb{F}_q)$ consists of all invertible uppertriangular $n \times n$ matrices over $\mathbb{F}_q$, and that the uncolored supercharacter theory is obtained by considering the $\mathbb{B}$-action on $\mathfrak{u}_n(\mathbb{F}_q)$ and the corresponding contragradient action on $\mathfrak{u}_n(\mathbb{F}_q)^\circ$.

Let $(\sigma, \phi) \in \Phi_n(\mathbb{F}_q)$ be arbitrary, and notice that

$$\mathbb{B}\cdot e_{\sigma,\phi} = \bigcup_{x \in T_n(\mathbb{F}_q)} x\cdot(\mathbb{G}\cdot e_{\sigma,\phi}) = \bigcup_{x \in T_n(\mathbb{F}_q)} \mathbb{G}\cdot(x\cdot e_{\sigma,\phi});$$

on the other and, if for every $x \in T_n(\mathbb{F}_q)$ we define the $\mathbb{F}_q$-coloration $x\cdot\phi : D(\sigma) \to \mathbb{F}_q \setminus \{0\}$ by

$$(x\cdot\phi)(i,j) = x_i\phi(i,j), \qquad (i,j) \in D(\sigma),$$

then $x\cdot e_{\sigma,\phi} = e_{\sigma,x\cdot\phi}$. Furthermore, the action of $T_n(\mathbb{F}_q)$ on $\mathrm{Col}_{\mathbb{F}_q}(\sigma)$ clearly permutes transitively the $\mathbb{F}_q$-colorations of σ, and thus we conclude that

$$1 + \mathbb{B}\cdot e_{\sigma,\phi} = \bigcup_{\phi' \in \mathrm{Col}_{\mathbb{F}_q}(\sigma)} K_{\sigma,\phi'}.$$

Consequently, the $\mathbb{B}$-superclasses of $U_n(\mathbb{F}_q)$ are parametrized by the set $\mathbf{SP}(n)$ of set partitions of $[n]$; for every $\sigma \in \mathbf{SP}(n)$ we denote by K_σ the $\mathbb{B}$-superclass of $U_n(\mathbb{F}_q)$ associated with σ.

Now, let $(\pi, \varphi) \in \Phi_n(\mathbb{F}_q^\circ)$ be arbitrary, and for every $x \in T_n(\mathbb{F}_q)$ define the $\mathbb{F}_q^\circ$-coloration $x\cdot\varphi : D(\pi) \to \mathbb{F}_q^\circ \setminus \{0\}$ by

$$(x\cdot\varphi)_{i,j}(\alpha) = \varphi_{i,j}(x_i^{-1}\alpha), \qquad (i,j) \in D(\pi),\ \alpha \in \mathbb{F}_q.$$

Then, we have

$$\mathbb{B}\cdot e^*_{\pi,\varphi} = \bigcup_{x \in T_n(\mathbb{F}_q)} \mathbb{G}\cdot e^*_{\pi,x\cdot\varphi},$$

and since the $T_n(\mathbb{F}_q)$-action on $\mathrm{Col}_n(\mathbb{F}_q^\circ)$ is transitive we conclude that

$$\mathbb{B}\cdot e^*_{\pi,\varphi} = \bigcup_{\varphi' \in \mathrm{Col}_n(\mathbb{F}_q^\circ)} \mathbb{G}\cdot e^*_{\pi,\varphi'}.$$

Therefore, the $\mathbb{B}$-orbits on $\mathfrak{u}_n(\mathbb{F}_q)^\circ$ are also in bijection with the set $\mathbf{SP}(n)$; for every $\pi \in \mathbf{SP}(n)$ we denote by χ^π the supercharacter of $U_n(\mathbb{F}_q)$ which is associated with π, and note that

$$\chi^\pi = \frac{1}{|\operatorname{Col}_{\mathbb{F}_q^\circ}(\pi)|} \sum_{\varphi \in \operatorname{Col}_n(\mathbb{F}_q^\circ)} \chi^{\pi,\varphi},$$

which allows us to obtain a formula for the supercharacter values similar to the one above (for details we refer to [7]).

Proposition 6.1.2. *Let $\pi, \sigma \in \boldsymbol{SP}(n)$ be arbitrary. Then, the value $\chi^\pi(K_\sigma)$ of the (normalized) uncolored supercharacter χ^π on the superclass K_σ is equal to* 0 *unless $D(\sigma) \not\subseteq \operatorname{Sing}(\pi)$, in which case we have*

$$\chi^\pi(K_\sigma) = \frac{1}{q^{\operatorname{nest}_\pi(\sigma)}} \left(\frac{-1}{q-1} \right)^{|D(\pi) \cap D(\sigma)|}.$$

In this way, the uncolored supercharacter theory only depends on q and on the combinatorics of set partitions. It is worth to mention that, contrary to the standard supercharacters, the uncolored supercharacters only have rational values.

6.2 The group $U_n(\mathbb{F})$

For every $m \in \mathbb{N}$ we set $\mathbb{F}_m = \mathbb{F}_{p^{m!}}$ where p is a prime number and $m!$ is the usual factorial of m; since $(m+1)! = (m+1)m!$, it follows that $\mathbb{F}_m$ is a subfield of $\mathbb{F}_{m+1}$, and this clearly allows us to consider $U_n(\mathbb{F}_m)$ as a subgroup of $U_n(\mathbb{F}_{m+1})$. The direct limit

$$\mathbb{F} = \varinjlim_{m \in \mathbb{N}} \mathbb{F}_m = \bigcup_{m \in \mathbb{N}} \mathbb{F}_m$$

is the algebraic closure of $\mathbb{F}_p$, and the unitriangular group $U_n(\mathbb{F})$ may be naturally identified with the direct limit

$$U_n(\mathbb{F}) = \varinjlim_{m \in \mathbb{N}} U_n(\mathbb{F}_m) = \bigcup_{m \in \mathbb{N}} U_n(\mathbb{F}_m).$$

Throughout the section, we consider each finite group $U_n(\mathbb{F}_m)$, for $m \in \mathbb{N}$, and the infinite group $U_n(\mathbb{F})$ equipped with the corresponding standard supercharacter theories $(\mathscr{K}_m, \mathscr{E}_m)$ and $(\mathscr{K}, \mathscr{E})$, respectively. For every $m \in \mathbb{N}$, we will also keep the notation $\Phi_n(\mathbb{F}_m)$ for the $\mathbb{F}_m$-colored set partitions of $[n]$, and will extend this notation to the field $\mathbb{F}$, that is, we will denote by $\Phi_n(\mathbb{F})$ the set consisting of all $\mathbb{F}$-colored set partitions of $[n]$. We note that $\Phi_n(\mathbb{F}_m) \subseteq \Phi_n(\mathbb{F}_{m+1})$ for all $m \in \mathbb{N}$, and thus the superclass $K^{(m)}_{\sigma,\phi} \in \mathscr{K}_m$ which is parametrized by $(\sigma, \phi) \in \Phi_n(\mathbb{F}_m)$

is obviously contained in the superclass $K^{(m+1)}_{\sigma,\phi} \in \mathscr{K}_{m+1}$ which is associated with $(\sigma,\phi) \in \Phi_n(\mathbb{F}_{m+1})$. Consequently, for every superclass $K \in \mathscr{K}$ there is a unique $\mathbb{F}$-colored set partition $(\sigma,\phi) \in \Phi_n(\mathbb{F})$ such that $1+e_{\sigma,\phi} \in K$, which means that, as in the case of the finite unitriangular groups, the superclasses of $U_n(\mathbb{F})$ are parametrized by the set $\Phi_n(\mathbb{F})$; as before, we denote by $K_{\sigma,\phi}$ the superclass in $\mathscr{K}$ which is associated with $(\sigma,\phi) \in \Phi_n(\mathbb{F})$.

For the characterization of $\mathscr{E}$ we use the finite approximation property (Proposition 4.1.4) and the supercharacter formula for the finite unitriangular groups (Proposition 6.1.1). Firstly, we mention that, similarly to the finite case, we may identify $\mathfrak{u}_n(\mathbb{F})^\circ$ with $\mathfrak{u}_n(\mathbb{F}^\circ)$, where $\mathbb{F}^\circ$ denotes the dual group of $\mathbb{F}^+$; furthermore, we let $\Phi_n(\mathbb{F}^\circ)$ stand for the set of all $\mathbb{F}^\circ$-colored set partitions of $[n]$.

Proposition 6.2.1. *The set $\mathscr{E}$ of supercharacters of $U_n(\mathbb{F})$ is in bijection with $\Phi_n(\mathbb{F}^\circ)$. For every $(\pi,\varphi) \in \Phi_n(\mathbb{F}^\circ)$ and every $(\sigma,\phi) \in \Phi_n(\mathbb{F})$, the value $\chi^{\pi,\varphi}(K_{\sigma,\phi})$ of the supercharacter $\chi^{\pi,\varphi} \in \mathscr{E}$ on the superclass $K_{\sigma,\phi} \in \mathscr{K}$ is given by*

$$\chi^{\pi,\varphi}(K_{\sigma,\phi}) = \begin{cases} 0, & \textit{if } D(\sigma) \subseteq \operatorname{Sing}(\pi) \textit{ or } \operatorname{nest}_\pi(\sigma) \neq 0, \\ \displaystyle\prod_{(i,j)\in D(\pi)\cap D(\sigma)} \varphi_{i,j}(\phi(i,j)), & \textit{otherwise;} \end{cases}$$

moreover, we have

$$\chi^{\pi,\varphi} = \prod_{(i,j)\in D(\pi)} \chi^{(i,j),\varphi_{i,j}}.$$

Proof. Let $\lambda \in \mathfrak{u}_n(\mathbb{F}^\circ)$ be arbitrary, and let $(\pi^m,\varphi^m) \in \Phi_n(\mathbb{F}^\circ_m)$ be the colored set partitions associated with the restriction $\lambda_{|m}$ of λ to $\mathfrak{u}_n(\mathbb{F}_m)$; without loss of generality we may assume that $\lambda_{|m} = e^*_{\pi^m,\varphi^m}$.

If $\tau \in \mathbb{F}^\circ$ is non-trivial, then there is $m_0 \in \mathbb{N}$ such that $\tau_{|m} \neq 0$ for all $m \geq m_0$, and consequently there is $M_0 \in \mathbb{N}$ such that for every $m \geq M_0$

$$\lambda_{i,j} \neq 0 \quad \Longrightarrow \quad (\lambda_{i,j})_{|m} \neq 0.$$

Therefore, there is a set partition $\pi \in \mathbf{SP}(n)$ such that

$$\pi^m = \pi^{M_0} = \pi, \qquad m \geq M_0.$$

According to Propositions 4.1.4 and 6.1.1, for every $m \geq M_0$, every $(i,j) \in D(\pi)$ and every $\alpha \in \mathbb{F}$, we have

$$\chi^{\pi,\varphi^m}(1+\alpha e_{i,j}) = \varphi^m_{i,j}(\alpha) \quad \text{and} \quad \lim_{m\to\infty} \chi^{\pi,\varphi^m}(1+\alpha e_{i,j}) = \chi^\lambda(1+\alpha e_{i,j}),$$

and so the sequence $(\varphi_{i,j}^m)_{m\in\mathbb{N}}$ converges to some element $\varphi_{i,j} \in \mathbb{F}^\circ$; moreover, the mapping $(i,j) \mapsto \varphi_{i,j}$ defines an $\mathbb{F}^\circ$-coloration $\varphi\colon D(\pi) \to \mathbb{F}^\circ \setminus \{0\}$ of π.

On the other hand, if $D(\sigma) \not\subseteq \mathrm{Sing}(\pi)$, then according to Proposition 6.1.1,

$$\chi^\lambda(K_{\sigma,\varphi}) = \lim_{m\to\infty} \chi^{\pi^m,\varphi^m}(K_{\sigma,\phi}) = \lim_{m\to\infty} \frac{1}{(p^m!)^{\mathrm{nest}_\pi(\sigma)}} \prod_{(i,j)\in D(\pi)\cap D(\sigma)} \varphi_{i,j}(\phi(i,j)),$$

which establishes the desired formula. □

Having a parametrization and a formula for supercharacters, we proceed to classify supercharacters using proposition 6.0.1. Let $(\pi,\varphi) \in \Phi_n(\mathbb{F}^\circ)$ and $\lambda = e^*_{\pi,\varphi} \in \mathfrak{u}_n(\mathbb{F})^\circ$ be arbitrary, and recall that the super-representation associated with (π,φ) is induced by the L_λ-representation $(\widetilde{\lambda},\mathbb{C})$, where

$$L_\lambda = 1 + \mathfrak{l}_\lambda = \{g \in U_n(\mathbb{F}):\ ge^*_{\pi,\varphi} = e^*_{\pi,\varphi}\}$$

and where $\widetilde{\lambda}: L_\lambda \to \mathbb{C}^\times$ is defined by

$$\widetilde{\lambda}(g) = \lambda(g-1), \qquad g \in L_\lambda.$$

For simplicity, we introduce the following notation:

- $N_\lambda = \{g \in U_n(\mathbb{F})\colon gL_\lambda g^{-1} = L_\lambda$ and $\widetilde{\lambda}^g = \widetilde{\lambda}\}$, and
- $M_\lambda = \{g \in N_\lambda:$ the N_λ-conjugates of g lie in finitely many L_λ-cosets$\}$.

Lemma 6.2.2. *If $\lambda = \tau \star e^*_{i,j}$ for $\tau \in \mathbb{F}^\circ \setminus \{0\}$ and $1 \le i < j \le n$, then*

$$L_\lambda = \{g \in U_n(\mathbb{F})\colon g_{i,t} = 0,\ i < t < j\}$$

and $N_\lambda = U_n(\mathbb{F})$.

Proof. Let $g \in L_\lambda$ be arbitrary. Then, for every $a \in \mathfrak{u}_n(\mathbb{F})$ we have

$$\lambda(g^{-1}a) = \lambda(a) \quad\Longleftrightarrow\quad \tau((g^{-1}a)_{i,j}) = \tau(a_{i,j}).$$

Moreover,

$$(g^{-1}a)_{i,j} = \sum_{i\le t<j} (g^{-1})_{i,t} a_{t,j};$$

therefore, if $i < t < j$ and $a = e_{t,j}$, then $(g^{-1}a)_{i,j} = (g^{-1})_{i,t}$, and thus for every $\alpha \in \mathbb{F}$

$$\lambda(g^{-1}\alpha e_{t,j}) = \lambda(\alpha e_{t,j}) = 1 \quad\Longleftrightarrow\quad \tau((g^{-1}\alpha e_{t,j})_{i,j}) = \tau(\alpha(g^{-1})_{t,j}) = 1,$$

which implies that $(g^{-1})_{t,j} = 0$ for all $i < t < j$ (because $\alpha \in \mathbb{F}$ is arbitrary and $\tau \in \mathbb{F}^\circ$ is non-trivial).

Conversely, if $g \in U_n(\mathbb{F})$ is such that $g^{-1}_{t,j} = 0$ for all $i < t < j$, then

$$(g^{-1}a)_{i,j} = \sum_{i \leq t < j} (g^{-1})_{i,t} a_{t,j} = a_{i,j},$$

and thus $g \in L_\lambda$.

Now, let $k \in U_n(\mathbb{F})$ be arbitrary, let $g \in L_\lambda$ and set $l = g - 1$. Then,

$$(k^{-1}lk)_{i,j} = \sum_{i \leq r < s \leq j} (k^{-1})_{i,r} l_{r,s} k_{s,j};$$

since $g = 1 + l \in L_\lambda$, for every $i \leq r < s \leq j$, the equality $l_{r,s} \neq 0$ holds if and only if $r = i$ and $s = j$, and thus

$$\widetilde{\lambda}((k^{-1}gk)) = \tau((k^{-1}lk)_{i,j}) = \tau((k^{-1})_{i,i} l_{i,j} k_{jj}) = \tau(l_{i,j}) = \widetilde{\lambda}(g(1+l)).$$

Consequently, $N_\lambda = U_n(\mathbb{F})$ and the proof is complete. □

This data, together with the fact that every supercharacter of $U_n(\mathbb{F})$ admits a unique finite factorization as a product of elementary supercharacters, is enough to classify all supercharacters.

Proposition 6.2.3. *For every $(\pi, \varphi) \in \Phi_n(\mathbb{F}^\circ)$, the supercharacter $\chi^{\pi,\varphi}$ is of type II if and only if there is $(i, j) \in D(\pi)$ with $j > i + 2$; otherwise, $\chi^{\pi,\varphi}$ is of type I.*

Proof. Let $\lambda = \tau \star e^*_{i,i+1}$ where $\tau \in \mathbb{F}^\circ \setminus \{0\}$ and $1 \leq i \leq n-1$. Since $L_\lambda = U_n(\mathbb{F}) = N_\lambda = M_\lambda$, it follows that $|N_\lambda : M_\lambda| = |M'_\lambda : M'_\lambda \cap L_\lambda| = 1$, and consequently χ^λ is type *I* (by Proposition 6.0.1). On the other hand, suppose that $\lambda = \tau \star e^*_{i,i+2}$ for $\tau \in \mathbb{F}^\circ \setminus \{0\}$ and $1 \leq i \leq n-2$. We claim that

$$kgk^{-1}L_\lambda = gL_\lambda, \qquad k, g \in U_n(\mathbb{F})$$

(recall that $N_\lambda = U_n(\mathbb{F})$ by the previous lemma). Let $k, g \in G$ be arbitrary. If $g \in L_\lambda$, then $kgk^{-1} \in L_\lambda$ (because L_λ is a normal subgroup of $U_n(\mathbb{F})$). On the other hand, if $g = 1 + \alpha e_{i,i+1} + l$ for some $l \in \mathfrak{l}_\lambda$, then $gL_\lambda = (1 + \alpha e_{i,i+1})L_\lambda$, and similarly

$$kgk^{-1}L_\lambda = k(1 + \alpha e_{i,i+1})k^{-1}L_\lambda = (1 + \alpha e_{i,i+1})L_\lambda$$

because $k(1 + \alpha e_{i,i+1})k^{-1} = 1 + \alpha e_{i,j} + l'$ for some $l' \in \mathfrak{l}_\lambda$. Therefore, $kgk^{-1}L_\lambda = gL\lambda$ and this implies that $U_n(\mathbb{F}) = N_\lambda = M_\lambda$. Moreover, since

$$U_n(\mathbb{F})' = \{g \in U_n(\mathbb{F})\colon g_{i,i+1} = 0,\ 1 \leq i \leq n-1\},$$

one can easily check that $M'_\lambda = M_\lambda \cap U_n(\mathbb{F})'$ and that $M'_\lambda \cap L_\lambda = M'_\lambda$. Therefore, $|N_\lambda : M_\lambda| = |M'_\lambda : M'_\lambda \cap L_\lambda| = 1$ which implies that χ^λ is type *I* (by Proposition 6.0.1).

Now, let $\lambda = \tau \star e^*_{i,j}$ for $\tau \in \mathbb{F}^\circ \setminus \{0\}$ and $1 \leq i < j \leq n$ with $j > i+2$. Then, $g = 1 + \alpha e_{i,j-2} \notin L_\lambda$ for all $\alpha \in \mathbb{F} \setminus \{0\}$. For every $\beta \in \mathbb{F} \setminus \{0\}$ we set $k_\beta = 1 + \beta e_{j-2,j-1}$, and note that $k_\beta^{-1} = 1 - \beta e_{j-2,j-1}$ and that

$$k_\beta^{-1} g k_\beta L_\lambda = (1 + \alpha e_{i,j-2} + \beta e_{i,j-1}) L_\lambda.$$

Since $\mathbb{F}$ is infinite, it follows that $1 + \alpha e_{i,j-2} \notin M_\lambda$, and so $|N_\lambda : M_\lambda| = |U_n(\mathbb{F}) : M_\lambda|$ is infinite. Therefore, χ^λ is type *II* (again by Proposition 6.0.1).

Finally, since every supercharacter decomposes uniquely as a finite product of elementary supercharacters, the result follows because the type of a finite product of characters is the highest type of the factors (see, for example, [16, Proposition III.2.5.27]). $\square$

In this setting, if we identify $\mathscr{E}$ with $\Phi_n(\mathbb{F}^\circ)$ (in the obvious way), then the topology of $\mathscr{E}$ is fairly easy to describe: a sequence $(\pi^m, \varphi^m)_{m\in\mathbb{N}}$ converges to (π, φ) if and only if there is $m_0 \in \mathbb{N}$ such that

$$\pi^m = \pi, \quad m \geq n_0 \qquad \text{and} \qquad \lim_{m\to\infty} \varphi^m_{i,j} = \varphi_{i,j}, \quad (i,j) \in D(\pi).$$

Moreover, the space $\mathscr{E}$ can be realized as a disjoint union of topological spaces: for every $\pi \in \mathbf{SP}(n)$ let

$$\Omega_\pi = \prod_{i=1}^{|D(\pi)|} \mathbb{F}^\circ \setminus \{0\},$$

which we naturally identify with the set $\{\chi^{\pi,\varphi} \colon \varphi \in \mathrm{Col}_\pi(\mathbb{F}^\circ)\}$; hence, $\mathscr{E}$ is identified with the disjoint union

$$\mathscr{E} \simeq \bigsqcup_{\pi\in\mathbf{SP}(n)} \Omega_\pi.$$

Having this identification in mind, the Haar measure ν on $\mathbb{F}^\circ$ induces a natural family of measures on $\mathscr{E}$ that we shall next describe; as it turns out, these measures will parametrize the uncolored supercharacter theory of $U_n(\mathbb{F})$.

For every $a \in \mathbb{F}$ we consider the function $\widehat{a} \colon \mathbb{F}^\circ \to \mathbb{C}$ given by

$$\widehat{a}(\tau) := \tau(a), \qquad \tau \in \mathbb{F}^\circ.$$

This function is continuous and the $\mathbb{C}$-span of $\{\widehat{a} \colon a \in \mathbb{F}\}$ is dense in $C(\mathbb{F}^\circ)$; therefore, every measure ν on $\mathbb{F}^\circ$ is uniquely determined by the the integration of all such functions. Further-

more, we have

$$\mathbb{F}^\circ = \varprojlim_{m\in\mathbb{N}} \mathbb{F}_m^\circ,$$

where the inverse limit is taken with respect to the restriction maps, and so we consider the topological basis consisting of all cylinders: for every $\gamma \in \mathbb{F}_m^\circ$, the corresponding cylinder

$$[\gamma]_m = \{\tau \in \mathbb{F}^\circ \colon \tau_{|m} = \gamma\}$$

has measure

$$\nu([\gamma]_m) = 1/|\mathbb{F}_m^\circ| = 1/|\mathbb{F}_m| = p^{-m!}.$$

Let $a \in \mathbb{F}$ be arbitrary. Then, for every $m \in \mathbb{N}$ such that $a \in \mathbb{F}_m$, we can write

$$\widehat{a} = \sum_{\gamma\in\mathbb{F}_m^\circ} \widehat{a}(\gamma)\mathbb{I}_{[\gamma]_m},$$

and thus

$$\int_{\mathbb{F}^\circ} \widehat{a}\, d\nu = \frac{1}{|\mathbb{F}_m^\circ|} \sum_{\gamma\in\mathbb{F}_m^\circ} \gamma(a) = \begin{cases} 0, & \text{if } a \neq 0, \\ 1, & \text{otherwise.} \end{cases}$$

For every $m \in \mathbb{N}$ we define a probability measure ν_m on $\mathbb{F}_m^\circ$ by

$$\nu_m(\gamma) = \begin{cases} \frac{1}{|\mathbb{F}_m^\circ \setminus \{0\}|} = \frac{1}{p^{m!}-1}, & \text{if } \gamma \in \mathbb{F}_m \setminus \{0\}, \\ 0, & \text{if } \gamma = 0, \end{cases}$$

and extend it to a measure $\widehat{\nu}_m$ on $\mathbb{F}^\circ$ by setting

$$\widehat{\nu}_m([\alpha]_{m_0}) = \begin{cases} \nu_m(\gamma), & \text{if } m_0 \leq m \text{ and } \gamma_{|m} = \alpha, \\ \frac{1}{|\{\alpha'\in\mathbb{F}_m^\circ\colon \alpha'_{|m}=\gamma\}|} \cdot \nu_m(\gamma), & \text{if } m < m_0 \text{ and } \alpha_{|m} = \gamma. \end{cases}$$

Lemma 6.2.4. *In the notation as above, the sequence $(\widehat{\nu}_m)_{m\in\mathbb{N}}$ weak*-converges to ν.*

Proof. For every $a \in \mathbb{F} \setminus \{0\}$ and every $m \in \mathbb{N}$ such that $a \in \mathbb{F}_m$, we have

$$\int_{\mathbb{F}^\circ} \widehat{a}\, d\widehat{\nu}_m = \frac{1}{|\mathbb{F}_m^\circ \setminus \{0\}|} \sum_{\gamma\in\mathbb{F}_m^\circ\setminus\{0\}} \gamma(a) = \frac{-1}{|\mathbb{F}_m^\circ \setminus \{0\}|},$$

which converges to zero as m goes to infinity. On the other hand,

$$\int_{\mathbb{F}_m^\circ} \widehat{0}\, d\widehat{\nu}_m = 1, \qquad m \in \mathbb{N},$$

and thus

$$\lim_{m\to\infty} \int_{\mathbb{F}^\circ} \widehat{a}\, d\widehat{\nu}_m = \int_{\mathbb{F}^\circ} \widehat{a}\, d\nu, \qquad a \in \mathbb{F}.$$

The lemma follows. □

Now, for every $\pi \in \mathbf{SP}(n)$, we define the probability measures on Ω_π

$$\eta_\pi^m = \bigotimes_{(i,j)\in D(\pi)} \widehat{\nu}_m, \quad m \in \mathbb{N}, \qquad \text{and} \qquad \eta_\pi = \bigotimes_{(i,j)\in D(\pi)} \nu;$$

notice that, according to the previous lemma, the sequence $(\eta_\pi^m)_{m\in\mathbb{N}}$ weak*-converges to η_π. On the other hand, we define the superclass character

$$\zeta_\pi = \int_{\Omega_\pi} \chi^{\pi,\varphi}\, d\eta_\pi,$$

and note that

$$\zeta_\pi = \lim_{m\to\infty} \int_{\Omega_\pi} \chi^{\pi,\varphi}\, d\eta_\pi^m = \lim_{m\to\infty} \frac{1}{|\mathbb{F}_m^\circ \setminus \{0\}|^{|D(\pi)|}} \sum_{\varphi\in \mathrm{Col}_{\mathbb{F}_m^\circ}(\pi)} \chi_m^{\pi,\varphi}.$$

Since $|\mathbb{F}_m^\circ \setminus \{0\}|^{|D(\pi)|} = |\mathrm{Col}_{\mathbb{F}_m^\circ}(\pi)|$, we conclude that

$$\zeta_\pi = \lim_{m\to\infty} \frac{1}{|\mathrm{Col}_{\mathbb{F}_m^\circ}(\pi)|} \sum_{\varphi\in \mathrm{Col}_{\mathbb{F}_m^\circ}(\pi)} \chi_m^{\pi,\varphi}.$$

As we already mentioned, the uncolored supercharacter theory $(\mathscr{K}_\mathbb{B}, \mathscr{E}_\mathbb{B})$ of $U_n(\mathbb{F})$ is associated with the action of $\mathbb{B} = T_n(\mathbb{F}) \times \mathbb{G}$ both on $\mathfrak{u}_n(\mathbb{F})$ and on $\mathfrak{u}_n(\mathbb{F})^\circ$. The superclasses in $\mathscr{K}_\mathbb{B}$ are in one-to-one correspondence with the set partitions of $[n]$: for every $\sigma \in \mathbf{SP}(n)$, the superclass $K_\sigma \in \mathscr{K}_\mathbb{B}$ is the union

$$K_\sigma = \bigcup_{\phi\in \mathrm{Col}_\mathbb{F}(\sigma)} K_{\sigma,\phi};$$

in particular, it contains the element $1 + e_{\sigma,\phi}$ for all $\phi \in \mathrm{Col}_\mathbb{F}(n)$. We are now able to describe the set of supercharacters $\mathscr{E}_\mathbb{B}$.

Proposition 6.2.5. *The uncolored supercharacters of $U_n(\mathbb{K})$ are precisely the characters ζ_π for $\pi \in \boldsymbol{SP}(n)$.*

Proof. The proof is simply a consequence of the finite approximation property. Let $\Omega_\mathbb{B}$ denote the set of uncolored orbit closures, and note that for every $\mathscr{O} \in \Omega_\mathbb{B}$ there is $\pi \in \mathbf{SP}(n)$ such that

$$\mathscr{O} \subseteq \overline{\bigcup_{\varphi\in \mathrm{Col}_{\mathbb{F}^\circ}(\pi)} \mathscr{O}^{\pi,\varphi}}.$$

Therefore, $\Omega_\mathbb{B}$ is in bijection with $\mathbf{SP}(n)$, and furthermore $\mathscr{O}$ is contained in the closure of the uncolored orbit of the element

$$\lambda = \sum_{(i,j)\in D(\pi)} e_{i,j}^* \in \mathfrak{u}_n(\mathbb{F})^\circ.$$

For every $\pi \in \mathbf{SP}(n)$ and every $m \in \mathbb{N}$, the uncolored supercharacter of $U_n(\mathbb{F}_m)$ associated with $\lambda_{|m}$ is given by

$$\chi_m^\pi = \frac{1}{|\operatorname{Col}_{\mathbb{F}_m^\circ}(\pi)|} \sum_{\varphi \in \operatorname{Col}_{\mathbb{F}_m^\circ}(\pi)} \chi_m^{\pi,\varphi},$$

and thus it follows from Proposition 4.1.4 that

$$\chi^\lambda = \lim_{m\to\infty} \frac{1}{|\operatorname{Col}_{\mathbb{F}_m^\circ}(\pi)|} \sum_{\varphi \in \operatorname{Col}_{\mathbb{F}_m^\circ}(\pi)} \chi_m^{\pi,\varphi} = \zeta_\pi$$

which completes the proof. □

For the uncolored supercharacter theory of $U_n(\mathbb{F})$, the number of superclasses is equal to the number of supercharacters, which is finite. If we are to understand a supercharacter theory as an approximation of the indecomposable character theory, the finiteness of the uncolored supercharacter theory yields a "rough approximation"; moreover, the set of uncolored supercharacters is homeomorphic to $\mathbf{SP}(n)$ equipped with the discrete topology. Nevertheless, we next present an uncolored supercharacter formula, which is simply consequence of the finite approximation property and of the uncolored supercharacter formula given in Proposition 6.1.2.

Proposition 6.2.6. *Let $\pi, \sigma \in \mathbf{SP}(n)$ be arbitrary. Then, the value $\chi^\pi(K_\sigma)$ of the supercharacter χ^π on the superclass K_σ is* 0 *whenever $D(\sigma) \subseteq \operatorname{Sing}(\pi)$, $\operatorname{nest}_\pi(\sigma) \neq 0$ or $|D(\pi) \cap D(\sigma)| \neq 0$; otherwise, we have $\chi^\pi(K_\sigma) = 1$.*

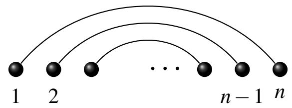
In a way, the uncolored supercharacters are measuring a particular kind of interaction between set partitions of $[n]$. For $\pi, \sigma \in \mathbf{SP}(n)$, we shall say that π *encompasses an arc* of σ if there are arcs $(i, j) \in D(\pi)$ and $(k,t) \in D(\sigma)$ such that $i \leq k < t \leq j$, equivalently, if $D(\sigma) \subseteq \operatorname{Sing}(\pi)$, $\operatorname{nest}_\pi(\sigma) \neq 0$, or $|D(\pi) \cap D(\sigma)| \neq 0$. In this fashion, $\chi^\pi(K_\sigma) = 1$ if and only if π does not encompass any arc of σ.

Next we turn our attention to the regular character of $U_n(\mathbb{F})$. Its nature (as a superclass character) varies depending on which supercharacter theory is being considered: for the standard supercharacter theory it is a decomposable superclass character, while for the uncolored one it is indeed a supercharacter.

We set

$$N(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd,} \end{cases}$$

and denote by $\pi_0 \in \mathbf{SP}(n)$ the (unique) set partition such that $D(\pi_0) = \{(i, n-i+1) :\ 1 \leq i \leq N(n)\} \in \mathbf{SP}(n)$, that is, the partition with diagram



Proposition 6.2.7. *The regular character ρ of $U_n(\mathbb{F})$ decomposes as an integral over Ω_{π_0} with respect to the measure η_{π_0}; indeed,*

$$\rho = \int_{\Omega_{\pi_0}} \chi^{\pi_0,\varphi}\, d\eta_{\pi_0} = \zeta_{\pi_0}.$$

In particular, ρ is an uncolored supercharacter of $U_n(\mathbb{F})$.

Proof. This is simply a consequence of the formula of the previous proposition. We remark that for every non-trivial set partition $\sigma \in \mathbf{SP}(n)$ and every $(k,t) \in D(\sigma)$ there is an arc $(i, j) \in D(\pi_0)$ such that $i \leq k < t \leq j$, and thus $\chi^{\pi}(K_\sigma) = 0$; since $\chi^{\pi_0}(1) = 1$, we conclude that χ^{π_0} is the regular character. $\square$

In terms of orbit closures, this result means that, while there are no dense colored orbits, the uncolored orbit of the element

$$\lambda = \sum_{i=1}^{N(n)} e^*_{i,n-i+1} \in \mathfrak{u}_n(\mathbb{F})$$

is dense.

The behavior of the regular character provides a good example of how different phenomena arises in the representation of *big groups*: in the finite group scenario, the regular character is always a decomposable superclass character and it is a convex sum of *all* supercharacters; on the other hand, for the group $U_n(\mathbb{F})$, the regular character is still a decomposable standard superclass character, although it is only the "convex sum" of the supercharacters having the shape π_0, and as an uncolored superclass character it is indecomposable.

Nonetheless, we can still find a superclass character that is the "convex sum" of all supercharacters. Let $\mathbf{u}$ be the measure on $\mathscr{E}$ defined as the sum

$$\mathbf{u} = \frac{1}{|\mathbf{SP}(n)|} \sum_{\pi \in \mathbf{SP}(n)} \eta_\pi;$$

we refer to $\mathbf{u}$ as the *uniform* measure. It is a measure whose support is $\mathscr{E}$, and the corresponding super-class character $\chi^{\mathbf{u}}$ can be decomposed as the sum

$$\chi^{\mathbf{u}} = \frac{1}{|\mathbf{SP}(n)|} \sum_{\pi \in \mathbf{SP}(n)} \zeta_\pi.$$

The uncolored supercharacter formula allow us to compute $\chi^{\mathbf{u}}$: for every $\sigma \in \mathbf{SP}(n)$, we have

$$\chi^{\mathbf{u}}(K_\sigma) = \frac{1}{|\mathbf{SP}(n)|} \cdot |\{\pi \in \mathbf{SP}(n) \colon \chi^{\pi}(K_\sigma) = 1\}.$$

For example, for $n = 3$, $|\mathbf{SP}(3)| = 5$ and some of the values of $\chi^{\mathbf{u}}$ are

$$\chi^{\mathbf{u}}(K_{1,2}) = \frac{1}{5}\chi^{2,3}(K_{1,2}) = \frac{1}{5}, \quad \text{and}$$

$$\chi^{\mathbf{u}}(K_{1,3}) = \frac{1}{5}\Big(\chi^{1,2}(K_{1,3}) + \chi^{2,3}(K_{1,3}) + \chi^{1,2,3}(K_{1,3})\Big) = \frac{3}{5}.$$

Essentially, the value of $\chi^{\mathbf{u}}(K_\sigma)$ provides the average number of the set partitions that do not encompass any arc of σ. One should think of $\chi^{\mathbf{u}}$ as the "average" of *all* supercharacters, being in this sense an analogue of the regular character.

6.3 The group $U_\infty(\mathbb{F}_q)$

In this section, we fix a power $q = p^e$ of a prime number p, and consider the group

$$U_\infty(\mathbb{F}_q) = \varinjlim_{n\in\mathbb{N}} U_n(\mathbb{F}_q) = \bigcup_{n\in\mathbb{N}} U_n(q),$$

which consists of all locally finite unitriangular square matrices over $\mathbb{F}_q$; hence, every $g \in U_\infty(\mathbb{F}_q)$ has only a finite number of non-zero entries above the diagonal. For every $n \in \mathbb{N}$, we consider $U_n(\mathbb{F}_q)$ equipped with the corresponding standard supercharacter theory $(\mathscr{K}_n, \mathscr{E}_n)$; similarly, $U_\infty(\mathbb{F}_q)$ is also equipped with its standard supercharacter theory $(\mathscr{K}, \mathscr{E})$. We observe that for every $n \in \mathbb{N}$ a superclass $K_n \in \mathscr{K}_n$ is contained in a unique superclass $K_{n+1} \in \mathscr{K}_{n+1}$; consequently, if we define

$$\Phi_\infty(\mathbb{F}_q) = \bigcup_{n\in\mathbb{N}} \Phi_n(\mathbb{F}_q),$$

then for every $K \in \mathscr{K}$ there is a unique $(\sigma, \phi) \in \Phi_\infty(\mathbb{F}_q)$ such that $1 + e_{\sigma,\phi} \in K$, and this provides a one-to-one correspondence between $\mathscr{K}$ and the set $\Phi_\infty(\mathbb{F}_q)$.

On the other hand, the description of $\mathscr{E}_n$, for $n \in \mathbb{N}$, ensures that the inclusion $\Phi_n(\mathbb{F}_q^\circ) \subseteq \Phi_{n+1}(\mathbb{F}_q^\circ)$ induces a (non-natural) inclusion $\mathscr{E}_n \subseteq \mathscr{E}_{n+1}$ where a supercharacter $\chi^{\pi,\varphi}$ of $U_n(\mathbb{F}_q)$, for $(\pi, \varphi) \in \Phi_n(\mathbb{F}_q^\circ)$, corresponds to the supercharacter $\chi^{\pi,\varphi}$ of $U_{n+1}(q)$, for $(\pi, \varphi) \in \Phi_{n+1}(\mathbb{F}_q^\circ)$; we shall use the same notation, but we must be aware that the supercharacters are in fact distinct. We define

$$\mathscr{E}^0 = \bigcup_{n\in\mathbb{N}} \mathscr{E}_n,$$

and note that, not only $\mathscr{E}^0 \subseteq \mathscr{E}$, but also $\mathscr{E}^0$ is a dense subset of $\mathscr{E}$ (according to Proposition 4.1.4).

The description of $\mathscr{E}$ is achieved by means of *augmented set partitions of* $\mathbb{N}$; in analogy with finite set partitions, for every π a set partition of $\mathbb{N}$ we denote by $D(\pi)$ the corresponding set of arcs. By an *augmented set partition* of $\mathbb{N}$ we mean a set partition π of $\mathbb{N}$ together with an subset $I \subseteq \mathbb{N}$ satisfying

$$(i,j) \in D(\pi) \quad \Longrightarrow \quad i \notin I;$$

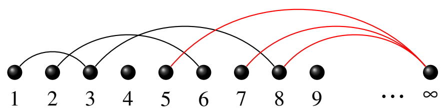
we denote by $\pi \sqcup I$ this augmented set partition. For every augmented set partition $\pi \sqcup I$ of $\mathbb{N}$ we set

$$D(I) = \{(i,\infty) : \ i \in I\},$$

and define the set of arcs of $\pi \sqcup I$ as being the set

$$D(\pi \sqcup I) = D(\pi) \cup D(I).$$

By the way of example, let $\pi = 1,3,8/2,6$ be a finite set partition of $\mathbb{N}$ and let $I = \{5,7,8\}$, then the augmented set partition $\pi \sqcup I$ is represented diagrammatically as



We denote **ASP** the set of all augmented set partitions of $\mathbb{N}$, and extend the definition of the sets $\mathrm{Sing}(\pi)$ and $\mathrm{Reg}(\pi)$ to $\mathrm{Sing}(\pi \sqcup I)$ and $\mathrm{Reg}(\pi \sqcup I)$ in the most natural fashion; in a similar way, we also define the nesting numbers $\mathrm{nest}_{(i,j)}(\pi \sqcup I)$ and $\mathrm{nest}(\pi \sqcup I)$ of an augmented set partition $\pi \sqcup I$. For every augmented set partition $\pi \sqcup I$ of $\mathbb{N}$, an $\mathbb{F}_q^\circ$*-coloration of* $\pi \sqcup I$ is simply a $\mathbb{F}_q^\circ$-coloration of π, that is, we assume that all *"infinite arcs"* in $D(I)$ are uncolored; we denote $\Phi_\infty(\mathbb{F}_q^\circ)$ the set consisting of all $\mathbb{F}_q^\circ$-colored augmented set partitions (this should not be confused with the union of all $\Phi_n(\mathbb{F}_q^\circ)$).

Proposition 6.3.1. *The supercharacters of $U_\infty(\mathbb{F}_q)$ are in bijection with $\Phi_\infty(\mathbb{F}_q^\circ)$; for every $\pi \sqcup I \in \Phi_\infty(\mathbb{F}_q^\circ)$ we denote by $\chi^{\pi\sqcup I,\varphi}$ the supercharacter of $U_\infty(F_q)$ associated with $(\pi \sqcup I, \varphi)$. Moreover, for every $(\pi \sqcup I, \varphi) \in \Phi_\infty(\mathbb{F}_q^\circ)$ and every $(\sigma,\phi) \in \Phi_\infty(\mathbb{F}_q)$, the value $\chi^{\pi\sqcup I,\varphi}(K_{\sigma,\phi})$ of supercharacter $\chi^{\pi\sqcup I,\varphi}$ on the superclass $K_{\sigma,\phi} \in \mathscr{K}$ is zero unless $D(\sigma) \nsubseteq \mathrm{Sing}(\pi \sqcup I)$, in which case it is given by*

$$\chi^{\pi\sqcup I,\varphi}(K_{\sigma,\phi}) = \frac{1}{q^{\mathrm{nest}_{\pi\sqcup I}(\sigma)}} \prod_{(i,j)\in D(\pi)\cap D(\sigma)} \varphi_{i,j}(\phi(i,j)).$$

Proof. We note that

$$\mathfrak{u}_\infty(\mathbb{F}_q)^\circ = \varprojlim_{n\in\mathbb{N}} \mathfrak{u}_n(\mathbb{F}_q^\circ);$$

moreover, if $\lambda \in \mathfrak{u}_n(\mathbb{F}_q^\circ)$ and $(\pi^n, \varphi^n) \in \Phi_n(\mathbb{F}_q^\circ)$, for every $n \in \mathbb{N}$, is the $\mathbb{F}_q^\circ$-colored set partition associated with $\lambda_{|n}$, then

$$(i,j) \in D(\pi^n) \quad \Longrightarrow \quad (i,j) \in D(\pi^{n+1}) \text{ or } (i,n+1) \in D(\pi^{n+1}).$$

Therefore, Propositions 4.1.4 and 6.1.1 imply the desired bijection and formula. □

The nature of the supercharacter formula explain why infinite arcs are uncolored. An infinite arc only contributes to the value of the supercharacter by means of the nesting number; in a way, it is "too faraway" for its color to have any impact.

In this fashion, the set of supercharacters $\mathscr{E}$ of $U_\infty(\mathbb{F}_q)$ can be identified with the set $\Phi_\infty(\mathbb{F}_q^\circ)$; under the identification, the dense subset $\mathscr{E}^0 \subseteq \mathscr{E}$ is identified with the union

$$\Phi_\infty^0(\mathbb{F}_q^\circ) = \bigcup_{n\in\mathbb{N}} \Phi_n(\mathbb{F}_q^\circ),$$

which may be viewed as a dense subset of $\Phi_\infty(\mathbb{F}_q^\circ)$.

Next, we classify supercharacters according to their type, again using Proposition 6.0.1.

Proposition 6.3.2. *For every $(\pi \sqcup I, \varphi) \in \Phi_\infty(\mathbb{F}_q^\circ)$ the supercharacter $\chi^{\pi\sqcup I,\varphi}$ is of type I if and only if $\chi^{\pi\sqcup I,\varphi} \in \mathscr{E}^0$ (that is, $I = \emptyset$ and $\pi \in \boldsymbol{SP}(n)$ for some $n \in \mathbb{N}$); otherwise, $\chi^{\pi\sqcup I,\varphi}$ is of type II.*

Proof. Let $\lambda \in \mathfrak{u}_\infty(\mathbb{F}_q)^\circ$ and let $(\pi \sqcup I, \varphi) \in \Phi_\infty(\mathbb{F}_q^\circ)$ be such that $\chi^\lambda = \chi^{\pi\sqcup I,\varphi}$. Similarly to the case of $U_n(\mathbb{F})$, one can check that

$$L_\lambda = \{g \in U_\infty(\mathbb{F}_q) \colon g_{i,s} = 0,\ (i,j) \in D(\pi \sqcup I),\ i < s < j\} \quad \text{and} \quad N_\lambda = U_\infty(\mathbb{F}_q).$$

Furthermore, since $\mathbb{F}_q$ is finite, if $I = \emptyset$, then $M_\lambda = U_\infty(\mathbb{F}_q)$, and thus $|N_\lambda : M_\lambda| = 1$; on the other hand, $M'_\lambda = N'_\lambda = U_\infty(\mathbb{F}_q)' = \{g \in U_\infty(\mathbb{F}_q) \colon g_{i,i+1} = 0,\ i \in \mathbb{N}\}$ and $U_\infty(\mathbb{F}_q)' \cap L_\lambda = L'_\lambda$, which implies that

$$|M'_\lambda : L'_\lambda| < \infty \qquad \Longleftrightarrow \qquad \pi \in \mathbf{SP}(n) \ \text{ for some } n \in \mathbb{N}.$$

If $I \neq \emptyset$, let i_0 be the smallest element of I. Then,

$$1 + e_{i_0,s} \in N_\lambda \setminus M_\lambda, \qquad s > i_0.$$

Furthermore, if $i_0 < s < t$, then

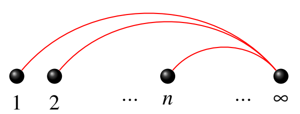
$$(1+ge_{i,s}g^{-1})L_\lambda \neq (1+he_{i,t}h^{-1})L_\lambda, \qquad g,h \in U_\infty(\mathbb{F}_q),$$

and thus $|N_\lambda : M_\lambda| = \infty$. The result follows (by Proposition 6.0.1). $\square$

Now, consider the regular character ρ of $U_\infty(\mathbb{F}_q)$. Let

$$\lambda = \sum_{1\leq i<j<\infty} e^*_{i,j} \in \mathfrak{u}_\infty(\mathbb{F}_q)^\circ;$$

this element as a dense $U_\infty(\mathbb{F}_q)\times U_\infty(\mathbb{F}_q)$-orbit which means that $\rho = \chi^\lambda$ is a supercharacter, and therefore it is associated with a unique element in $\Phi_\infty(\mathbb{F}_q^{;\circ})$. In virtue of Proposition 6.3.1, one can check that ρ must be associated with $\emptyset \sqcup \mathbb{N}$ where $\emptyset$ denotes the unique partition of $\mathbb{N}$ such that $D(\emptyset) = \emptyset$; diagrammatically, $\emptyset \sqcup \mathbb{N}$ is represented as



Indeed, $D(\sigma) \subseteq \mathrm{Sing}(\emptyset \sqcup \mathbb{N})$ whenever $\sigma \in \mathbf{SP}(n)$, for any $n \in \mathbb{N}$, is such that $D(\sigma) \neq \emptyset$, and thus

$$\chi^{\emptyset\sqcup\mathbb{N}}(g) = 0, \qquad g \in G,\ g \neq 1.$$

Next, we let $\mathbb{G} = U_\infty(\mathbb{F}_q)\times U_\infty(\mathbb{F}_q)$ and

$$T_\infty = \prod_{n\in\mathbb{N}}^{\infty} \mathbb{F}_q^\times$$

(hence, T_∞ is the group of all invertible infinite diagonal matrices over $\mathbb{F}_q$), and we consider the group $\mathbb{B} = T_\infty \times \mathbb{G}$. Recall that uncolored supercharacter theory $(\mathscr{K}_\mathbb{B}, \mathscr{E}_\mathbb{B})$ of $U_\infty(\mathbb{F}_q)$ is the one induced by the action of $\mathbb{B}$ on both $\mathfrak{u}_\infty(\mathbb{F}_q$ and $\mathfrak{u}_\infty(\mathbb{F}_q)^\circ$; for every superclass $K \in \mathscr{K}_\mathbb{B}$ there are unique $n \in \mathbb{N}$ and $\sigma \in \mathbf{SP}(n)$ such that

$$K = K_\sigma = \bigcup_{\phi\in\mathrm{Col}_{\mathbb{F}_q}(\sigma)} K_{\sigma,\phi}.$$

Therefore, $\mathscr{K}_\mathbb{B}$ is in bijection with the (disjoint) union

$$\mathbf{SP} = \bigcup_{n\in\mathbb{N}} \mathbf{SP}(n).$$

For the description of the uncolored supercharacters of $U_\infty(\mathbb{F}_q)$ (using Propositions 4.1.4 and 6.1.2), one can easily prove the following result.

Proposition 6.3.3. *The uncolored supercharacters of $U_\infty(\mathbb{F}_q)$ are in bijection with the set* **ASP** *of all augmented set partitions of $\mathbb{N}$; for every $\pi \sqcup I \in$* **ASP** *we denote by $\chi^{\pi\sqcup I}$ the uncolored supercharacter of $U_\infty(\mathbb{F}_q)$ which is associated with $\pi \sqcup I$. Moreover, for every $\pi \sqcup I \in$* **ASP** *and every $\sigma \in$* ***SP****, the value $\chi^{\pi\sqcup I}(K_\sigma)$ of the supercharacter $\chi^{\pi\sqcup I}$ on the superclass $K_\sigma \in \mathscr{K}_{\mathbb{B}}$ is zero unless $D(\sigma) \not\subseteq \mathrm{Sing}(\pi \sqcup I)$, in which case it is given by*

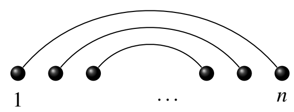
$$\chi^{\pi\sqcup I}(K_\sigma) = \prod_{(i,j)\in D(\pi\sqcup I)} \chi^{i,j}(K_\sigma) = \frac{1}{q^{\mathrm{nest}_{\pi\sqcup I}(\sigma)}} \left(\frac{-1}{q-1}\right)^{|D(\pi)\cap D(\sigma)|}.$$

Consequently, the set $\mathscr{E}_{\mathbb{B}}$ of uncolored supercharacters of $U_\infty(\mathbb{F}_q)$ can be identified with **ASP**, and the supercharacter theory $(\mathscr{K}_{\mathbb{B}}, \mathscr{E}_{\mathbb{B}})$ is fully described by the combinatorics of all "finite" set partitions and all the augmented set partitions of $\mathbb{N}$.

Notice that the regular character $\rho = \chi^{\emptyset\sqcup\mathbb{N}}$ of $U_\infty(\mathbb{F}_q)$ has the same description in both supercharacter theories, which becomes an interesting parametrization in the light of De Stavola's paper [32]. For every $n \in \mathbb{N}$, let ρ_n denote the regular character of $U_n(\mathbb{F}_q)$, and let $\mathbf{SPl}_n$ denote the super-Plancherel measure on $\mathbf{SP}(n)$ associated with the uncolored supercharacter theory for $U_n(\mathbb{F}_q)$, that is,

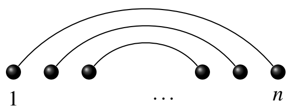
$$\rho_n = \sum_{\pi\in\mathbf{SP}(n)} \mathbf{SPl}_n(\pi)\chi^\pi.$$

Let $\mathscr{M}^+(\mathbf{SP}(n))$ denote the set consisting of all probability measures on $\mathbf{SP}(n)$. De Stavola defines an embedding of $\mathscr{M}^+(\mathbf{SP}(n))$ into the set of sub-probability measures[1] on the unit square $[0,1]\times[0,1]$; with respect to this inclusion, he has proved that the sequence $(\mathbf{SPl}_n)_{n\in\mathbb{N}}$ weak*-converges to the uniform measure on the set $\{(x,1-x) : x \in [0,1/2]\}$; the representation theoretical interpretation of this is not clear. However, De Stavola argues that such measure should be thought as the *limit shape*, as n goes to infinity, of the sequence of partitions $(\pi_n)_{n\in\mathbb{N}}$ where π_n has shape

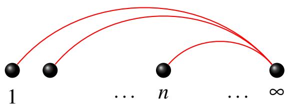


that is, "$\lim\limits_{n\to\infty} \mathbf{SPl}_n = \lim\limits_{n\to\infty} \pi_n$". This statement admits a concrete meaning in terms of uncolored supercharacters and augmented set partitions.The supercharacter of $U_n(\mathbb{F}_q)$ associated with the super-Plancherel measure $\mathbf{SPl}_n$ is precisely the regular character ρ_n, and the sequence $(\rho_n)_{n\in\mathbb{N}}$ converges pointwise to ρ. On the other hand, the sequence $(\chi^{\pi_n})_{n\in\mathbb{N}}$ also converges pointwise to ρ. Therefore, in **ASP**, as n goes to ∞, the shapes

[1] A *sub-probability measure* is a measure with total weight less than or equal to 1.



converge to the shape



At this point, we remark the similarity between the regular characters of $U_n(\mathbb{F})$ and $U_\infty(\mathbb{F}_q)$: both are somehow associated with *some* limit of the shape , a phenomenon that appears to be typical of the infinite versions of the unitriangular group.

6.3.1 The Kingman graph and supercharacters

The *Kingman graph*, whose boundary is called the *Kingman simplex*, appeared firstly in [62] in the context of population genetics. Its n-th level of vertices is the set $\mathbb{Y}_n$ of Young tableaux with n boxes and, as mentioned in [59], it describes the branching rule of certain characters of the finite symmetric groups S_n: with every integer partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0)$ of n (equivalently, with every Young tableaux) we associated the Young subgroup

$$S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k}$$

of S_n, and denote by τ^λ the character of S_n obtain from the induction of the trivial character of S_λ. For every $\lambda \in \mathbb{Y}_n$ and every $\mu \in \mathbb{Y}_{n+1}$, the dimension $\dim(\lambda, \mu)$ is defined to be the multiplicity of τ^λ in the restriction $(\tau^\mu)_{|S_n}$ of τ^μ to S_n.

However, no representation theoretical interpretation of the Kingman simplex has been presented so far, albeit extensively studied by some authors in different contexts (essentially due to its similarities with the Young graph) such as in [20, 59, 74]). In what follows, we interpret the Kingman graph in the uncolored supercharacter theory context, namely, we identify the Kingman simplex with a family of superclass characters of $U_\infty(\mathbb{F}_q)$ which is invariant under a particular action of the infinite symmetric group S_∞.

The action of the symmetric group S_n on $[n]$ induces a natural action on $\mathbf{SP}(n)$, and we say that two set partitions in $\mathbf{SP}(n)$ are S_n*-equivalent* if they lie in the same S_n-orbit. Notice that the S_n-orbits on $\mathbf{SP}(n)$ are indexed by the Young tableaux $\mathbb{Y}_n$: for every $\pi = B_1/\ldots/B_k \in \mathbf{SP}(n)$, we

may assume without any loss of generality that $|B_1| \geq |B_2| \geq \ldots \geq |B_k|$; since $|B_1| + \cdots + |B_k| = n$, we see that

$$\lambda_\pi = (|B_1|, |B_2| \ldots, |B_k|)$$

is an integer partition of n. It is straightforward to check that $\pi, \pi' \in \mathbf{SP}(n)$ are S_n-equivalent if and only if $\lambda_\pi = \lambda_{\pi'}$, and for this reason we identify the S_n-orbits on $\mathbf{SP}(n)$ with $\mathbb{Y}_n$; moreover, given $\lambda \in \mathbb{Y}_n$ we shall denote by $\mathfrak{o}_\lambda$ the corresponding S_n-orbit.

Every set partition of $[n+1]$ induces by restriction a set partition on $[n]$, and thus for every $n \in \mathbb{N}$ there is a well-defined map $P_n^{n+1} : \mathbf{SP}(n+1) \to \mathbf{SP}(n)$. In terms of arcs, the “projection” $P_n^{n+1}(\pi) \in \mathbf{SP}(n)$ of $\pi \in \mathbf{SP}(n+1)$ is uniquely determined by

$$D(P_n^{n+1}(\pi)) = \{(i, j) \in D(\pi) : \ j \neq n+1\}.$$

For every $\sigma \in \mathbf{SP}(n+1)$ and every $\tau \in S_n$, we have

$$P_n^{n+1}(\tau \cdot \sigma) = \tau \cdot P_n^{n+1}(\sigma),$$

and consequently

$$\tau \cdot ((P_n^{n+1})^{-1}(\pi) \cap \mathfrak{o}_\mu) = (P_n^{n+1})^{-1}(\tau \cdot \pi) \cap \mathfrak{o}_\mu, \qquad \pi \in \mathbf{SP}(n),\ \mu \in \mathbb{Y}_{n+1}.$$

Therefore,

$$\left|(P_n^{n+1})^{-1}(\pi) \cap \mathfrak{o}_\mu\right| = \left|(P_n^{n+1})^{-1}(\tau \cdot \pi) \cap \mathfrak{o}_\mu\right|, \qquad \pi \in \mathbf{SP}(n),\ \mu \in \mathbb{Y}_{n+1},\ \tau \in S_n.$$

The Kingman graph, which we will denote by $\mathbf{K}$, is a graded graph such that the n-th level of vertices is $\mathbb{Y}_n$ and for every $\lambda \in \mathbb{Y}_n$ and every $\mu \in \mathbb{Y}_{n+1}$ the number of edges between λ and μ is equal to $\left|(P_n^{n+1})^{-1}(\pi) \cap \mathfrak{o}_\mu\right|$ for any $\pi \in \mathfrak{o}_\lambda$; in other words,

$$\dim(\lambda, \mu) = \left|(P_n^{n+1})^{-1}(\pi) \cap \mathfrak{o}_\mu\right|.$$

It is clear that

$$\dim(\lambda) = |\mathfrak{o}_\lambda|, \qquad \lambda \in \mathbb{Y}_n;$$

moreover, it can be can be proved that for every $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{Y}_n$

$$\dim(\lambda) = \frac{n!}{\lambda_1! \cdots \lambda_l!}$$

(see for example [59]).

We provide a (succinct) description of the boundary of $\mathbf{K}$, which can be identified with the set

$$\Delta = \left\{ \alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq 0) \colon \sum_{n\in\mathbb{N}} \alpha_n \leq 1 \right\}$$

(more details can be found in [59, 62, 74]). For every $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l) \in \mathbb{Y}_n$, we denote the usual *monomial symmetric function* m_λ on an infinite number of commutative variables by

$$m_\lambda = \sum_{i_1,\dots,i_l} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \dots x_{i_l}^{\lambda_l}$$

where the sum is taken over all subsets $\{i_1,\dots,i_l\}$ of $\mathbb{N}$ with size l. These monomial symmetric functions are to be understood as polynomial functions $m_\lambda : \Delta \to \mathbb{R}$; for the general definitions and formalism of the ring of symmetric functions we refer to [70]. We consider $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0) \in \mathbb{Y}_n$ in its *exponential notation*

$$\lambda = (1^{r_1(\lambda)}, 2^{r_2(\lambda)}, \dots, n^{r_n(\lambda)})$$

where $r_i(\lambda)$, for $1 \leq i \leq n$, denotes the number of occurrences of i in λ; by the way of example, if $n = 11$ and $\lambda = (3,2,2,2,1,1)$, then $r_1(\lambda) = 2$, $r_2(\lambda) = 3$ and $r_3(\lambda) = 1$, and $\lambda = (1^2, 2^3, 3^1)$. Following [59], for every $\alpha \in \Delta$, with corresponding coherent system $\{M_n^\alpha\}_{n\in\mathbb{N}}$, for every $n \in \mathbb{N}$ and every $\lambda = (1^{r_1}, 2^{r_2}, \dots) \in \mathbb{Y}_n$ we have

$$M_n^\alpha(\lambda) = \sum_{k=0}^{r_1} \frac{(1 - \sum_{i=1}^\infty \alpha_i)^k}{k!} m_\lambda(\alpha).$$

Now, we consider the groups $U_n(\mathbb{F}_q)$, for $n \in \mathbb{N}$, and $U_\infty(\mathbb{F}_q)$ equipped with the corresponding uncolored supercharacter theories, and define

$$\mathscr{E}_{\mathbf{SP}} = \{\chi^\pi \colon \pi \in \mathbf{SP}(\mathbb{N})\},$$

which we naturally equip with the pointwise convergence topology. The projections $\{P_n^{n+1}\}_{n\in\mathbb{N}}$ allow us to identify $\mathbf{SP}(\mathbb{N})$ with the inverse limit

$$\mathbf{SP}(\mathbb{N}) = \varprojlim_{n\in\mathbb{N}} \mathbf{SP}(n),$$

endowed with canonical projections $\{P_n\}_{n\in\mathbb{N}}$ and equipped with the inverse limit topology, for which the family of all cylinders is a topological basis of clopen sets; we observe that for every $\pi \in \mathbf{SP}(\mathbb{N})$ and every $n \in \mathbb{N}$ the projection $P_n(\pi)$ is defined to be the unique set partition of $[n]$ having

$$D(P_n(\pi)) = \{(i,j) \in D(\pi) \colon 1 \leq i < j \leq n\}$$

as its set of arcs.

Proposition 6.3.4. *The inverse limit* $\mathbf{SP}(\mathbb{N}) = \varprojlim_{n\in\mathbb{N}} \mathbf{SP}(n)$ *is homeomorphic to* $\mathscr{E}_{\mathbf{SP}}$.

Proof. Let $(\pi^m)_{m\in\mathbb{N}}$ be a convergent sequence in $\mathbf{SP}(\mathbb{N})$ with limit point π. According to the uncolored supercharacter formula presented in Proposition 6.3.3, for every $n \in \mathbb{N}$ the restriction $(\chi^\pi)_{|n}$ is fully determined by the set of arcs

$$D_n(\pi) = \{(i,j) \in D(\pi) \colon i \leq n\};$$

furthermore, we can choose $j_0 \in \mathbb{N}$ such that $(i,j) \notin D_n(\pi)$ for all $i \leq n$ and all $j > j_0$, and such that

$$(\chi^\pi)_{|n} = (\chi^{P_{j_0}(\pi)})_{|n}.$$

On the other hand, there is an order $m_0 \in \mathbb{N}$ such that $P_{j_0}(\pi^m) = P_{j_0}(\pi)$ for all $m \geq m_0$, and thus

$$(\chi^{\pi^m})_{|n} = (\chi^{P_{j_0}(\pi^m)})_{|n} = (\chi^\pi)_{|n}.$$

Consequently, the sequence $(\chi^{\pi^m})_{m\in\mathbb{N}}$ converges to χ^π in $\mathscr{E}_{\mathbf{SP}}$.

Conversely, let $(\chi^{\pi^m})_{m\in\mathbb{N}}$ be a convergent sequence in $\mathscr{E}_{\mathbf{SP}}$ with limit point χ^π, and let $(i,j) \in D(P_n(\pi))$ for $n \in \mathbb{N}$. Then,

$$\lim_{m\to\infty} \chi^{\pi_m}(K_{(i,j)}) = \chi^\pi(K_{(i,j)}) = \frac{-1}{q-1},$$

and as a consequence of the uncolored supercharacter formula, we conclude that there is an order $m_0 \in \mathbb{N}$ such that $(i,j) \in D(\pi^m)$ for all $m \geq m_0$; hence, $D(P_n(\pi)) \subseteq D(P_n(\pi^m))$ for all $m \gg 0$.

Finally, assume that for every $m_0 \in \mathbb{N}$ there is $m > m_0$ such that $D(P_n(\pi)) \neq D(P_n(\pi^m))$. Then, the finiteness of $\{(i,j) \colon 1 \leq i < j \leq n\}$ allow us to choose a subsequence $(\pi^{t_m})_{m\in\mathbb{N}}$ of $(\pi^m)_{m\in\mathbb{N}}$ such that

$$D(P_n(\pi)) \neq D(P_n(\pi^{t_m})) = D(P_n(\pi^{t_{m+1}})).$$

For every $(i,j) \in D(P_n(\pi^{t_m})) \setminus D(P_n(\pi))$ we have

$$\chi^\pi(K_{(i,j)}) = \lim_{m\to\infty} \chi^{\pi_{t_m}}(K_{(i,j)}) = \frac{1}{q-1},$$

which implies that $(i,j) \in D(P_n(\pi))$, a contradiction. Therefore, we conclude that for every $n \in \mathbb{N}$ there is $m_0 \in \mathbb{N}$ such that $P_n(\pi^m) = P_n(\pi)$ for all $m \geq m_0$, which means that the sequence $(\pi^m)_{m\in\mathbb{N}}$ converges to π in $\mathbf{SP}(\mathbb{N})$, and this establishes the desired homeomorphism. $\square$

The action of the infinite symmetric group S_∞ on $\mathbb{N}$ induces a natural action on $\mathbf{SP}(\mathbb{N}) \simeq \mathscr{E}_{\mathbf{SP}}$, and thus we may consider the set $\mathscr{M}_{S_\infty}(\mathscr{E}_{\mathbf{SP}})$ of S_∞-invariant measures on $\mathscr{E}_{\mathbf{SP}}$, equipped with the weak*-convergence topology, and this defines a family of superclass functions of $U_\infty(\mathbb{F}_q)$.

Proposition 6.3.5. *There is an homeomorphism between $\mathscr{M}_{S_\infty}(\mathscr{E}_{\mathbf{SP}})$ and the Gibbs measures on the Kingman graph $\mathbf{K}$. Furthermore, this homeomorphism induces an affine homeomorphism between S_∞-invariant probability measures on $\mathscr{E}_{\mathbf{SP}}$ and probability Gibbs measures on $\mathbf{K}$.*

Proof. Let $\mathbf{M}$ be a S_∞-invariant measure on $\mathscr{E}_{\mathbf{SP}}$. Then, $\mathbf{M}$ is fully determined by its values on cylinders; for every $\pi \in \mathbf{SP}(n)$ and every $\tau \in S_n \subseteq S_\infty$ we have

$$\mathbf{M}([\pi]_n) = \mathbf{M}(\tau\cdot[\pi]_n) = \mathbf{M}([\tau\cdot\pi]_n)$$

and thus for every $\lambda \in \mathbb{Y}_n$, we may define

$$M_n(\lambda) = \left|S_n\cdot\pi\right| \mathbf{M}([\pi]_n) = \dim(\lambda)\mathbf{M}([\pi]_n), \qquad \pi \in \mathfrak{o}_\lambda.$$

We claim that $\{M_n\}_{n\in\mathbb{N}}$ is a coherent system for $\mathbf{K}$: for every $\lambda \in \mathbb{Y}_n$ and every $\pi \in \mathfrak{o}_\lambda$ we evaluate

$$\begin{aligned}
M_n(\lambda) &= \dim(\lambda)\mathbf{M}([\pi]_n) = \dim(\lambda) \sum_{\sigma\in(P_n^{n+1})^{-1}(\pi)} \mathbf{M}([\sigma]_{n+1}) \\
&= \dim(\lambda) \sum_{\mu\in\mathbb{Y}_{n+1}} \left(\sum_{\sigma\in(P_n^{n+1})^{-1}(\pi)\cap\mathfrak{o}_\mu} \frac{1}{\dim(\mu)} M_n(\mu) \right) \\
&= \sum_{\mu\in\mathbb{Y}_{n+1}} \frac{\dim(\lambda)}{\dim(\mu)} \dim(\lambda,\mu) M_n(\mu).
\end{aligned}$$

Conversely, suppose that $\{M_n\}_{n\in\mathbb{N}}$ is a coherent system for $\mathbf{K}$. For every $\pi \in \mathbf{SP}(n)$ we define

$$\mathbf{M}([\pi]_n) = \frac{1}{\dim(\lambda)} M_n(\lambda), \qquad \text{if } \pi \in \mathfrak{o}_\lambda.$$

Since $\{M_n\}_{n\in\mathbb{N}}$ is a coherent system, it follows that for every $\pi \in \mathbf{SP}(n)$

$$\begin{aligned}
\mathbf{M}([\pi]_n) &= \frac{1}{\dim(\lambda)} \sum_{\mu\in\mathbb{Y}_{n+1}} \frac{\dim(\lambda)}{\dim(\mu)} \dim(\lambda,\mu) M_n(\mu) \\
&= \sum_{\mu\in\mathbb{Y}_{n+1}} \left(\sum_{\sigma\in\mathfrak{o}_\mu} \dim(\lambda,\mu)\mathbf{M}([\sigma]_n) \right) \\
&= \sum_{\mu\in\mathbb{Y}_{n+1}} \left(\sum_{\sigma\in(P_n^{n+1})^{-1}(\pi)\cap\mathfrak{o}_\mu} \mathbf{M}([\sigma]_{n+1}) \right) \\
&= \sum_{\sigma\in(P_n^{n+1})^{-1}(\pi)} \mathbf{M}([\sigma]_{n+1}),
\end{aligned}$$

and hence $\mathbf{M}$ is in fact a measure, and S_∞-invariant by construction.

It is clear that the above maps $\{S_\infty\text{-invariant measures}\} \mapsto \{\text{coherent systems}\}$ and $\{\text{coherent systems}\} \mapsto \{S_\infty\text{-invariant measures}\}$ are inverses of each other, and thus it only remains to show that they are continuous.

Since $\mathscr{E}_{\mathbf{SP}} \simeq \mathbf{SP}(\mathbb{N})$ is realized as an inverse limit, a sequence $(\mathbf{M}^m)_{m\in\mathbb{N}}$ of S_∞-invariant measures converges to $\mathbf{M}$ if and only if $\mathbf{M}^m([\pi]_n) \longrightarrow \mathbf{M}([\pi]_n)$ for all $n \in \mathbb{N}$ and all $\pi \in \mathbf{SP}(n)$. For every $m \in \mathbb{N}$, let $\{M_n^m\}_{n\in\mathbb{N}}$ be the coherent system with Gibbs measure M^m associated with $\mathbf{M}^m$, and let $\{M_n\}_{n\in\mathbb{N}}$ be the coherent system with Gibbs measure M associated with $\mathbf{M}$. For every $n \in \mathbb{N}$ and every $\pi \in \mathbf{SP}(n)$ with $\pi \in \mathfrak{o}_\lambda$ we have

$$\lim_{m\to\infty} \mathbf{M}^m([\pi]_n) = \mathbf{M}([\pi]_n) \quad \iff \quad \lim_{m\to\infty} \big(\dim(\lambda) M_n^m(\lambda)\big) = \dim(\lambda) M_n(\lambda),$$

and thus $(\mathbf{M}^m)_{m\in\mathbb{N}}$ converges to $\mathbf{M}$ if and only if $(M^m)_{m\in\mathbb{N}}$ converges to M, which means that $\mathscr{M}_{S_\infty}(\mathscr{E}_{\mathbf{SP}})$ is homeomorphic to the space of Gibbs measures on $\mathbf{K}$.

The last assertion of the proposition is now clear. $\square$

In this fashion, the Kingman simplex Δ can be identified with the indecomposable S_∞-invariant superclass characters in $\mathscr{E}_{\mathbf{SP}}$. For every $\alpha \in \Delta$, with corresponding S_∞-invariant measure $\mathbf{M}^\alpha$, the representation theoretical meaning of the values $\mathbf{M}^\alpha([\pi]_n)$ are not clear. Nevertheless, it is possible to get a glimpse of the behavior of the superclass function χ^α associated with α. The superclass character χ^α is fully determined by its restrictions $\{(\chi^\alpha)_{|n}\}_{n\in\mathbb{N}}$; on the other hand, for every $n \in \mathbb{N}$, the restriction $(\chi^\alpha)_{|n}$ is given by

$$(\chi^\alpha)_{|n} = \int_{\mathscr{E}_{\mathbf{SP}}} (\chi^\sigma)_{|n}\, dM^\alpha = \sum_{\pi\in\mathbf{SP}(n)} \int_{[\pi]_n} (\chi^\sigma)_{|n}\, dM^\alpha.$$

Therefore, we focus our attention to the values of the last integral.

Firstly, we define the *set of rows* of a set partition $\sigma \in \mathbf{SP}(\mathbb{N})$ to be the set

$$I^\sigma = \{i \in \mathbb{N} \colon (i,j) \in D(\sigma) \text{ for some } \in \mathbb{N}\},$$

and we notice that the values of $(\chi^\sigma)_{|n}$ are completely determined by the set

$$\{(i,j) \in D(\sigma) \colon 1 \le i \le n < j\}.$$

Let $\pi \in \mathbf{SP}(n)$ be arbitrary, and define

$$\mathbf{SP}_n^\pi = \{\sigma \in \mathbf{SP}(\mathbb{N}) \colon (i,j) \in D(\sigma) \;\Rightarrow\; 1 \le i \le n < j,\, i \notin I^\pi\};$$

accordingly, for every $\sigma \in [\pi]_n$ there is a unique $\sigma_0 \in \mathbf{SP}_n^\pi$ such that

$$D(\sigma_0) \subseteq D(\sigma) \qquad \text{and} \qquad (\chi^\sigma)_{|n} = (\chi^{\pi/\sigma_0})_{|n},$$

where $\pi/\sigma_0 \in \mathbf{SP}(\mathbb{N})$ is such that $D(\pi/\sigma_0) = D(\pi) \cup D(\sigma_0)$. In virtue of the factorization of supercharacters, it follows that

$$(\chi^\sigma)_{|n} = \chi^\pi (\chi^{\sigma_0})_{|n},$$

and thus

$$\int_{[\pi]_n} (\chi^\sigma)_{|n} \, dM^\alpha = \chi^\pi \int_{[\pi]_n} (\chi^{\sigma_0})_{|n} \, dM^\alpha.$$

For every $\sigma, \sigma' \in \mathbf{SP}_n^\pi$ the equality $I^\sigma = I^{\sigma'}$ holds if and only if $(\chi^\sigma)_{|n} = (\chi^{\sigma'})_{|n}$. Let us consider the equivalence relation on $\mathbf{SP}_n^\pi$ given by

$$\sigma \sim \sigma' \quad \Longleftrightarrow \quad I^\sigma = I^{\sigma'},$$

and let $V^\pi = \mathbf{SP}_n^\pi / \sim$ denote the set of equivalence classes. For every $v \in V^\pi$ and every $\sigma \in v$ we set $\chi^v = (\chi^\sigma)_{|n}$, and let n_σ be the smallest positive integer satisfying $(i,j) \in D(\sigma) \implies j \le n_\sigma$. Then,

$$\int_{[\pi]_n} (\chi^{\sigma_0})_{|n} \, dM^\alpha = \sum_{v \in V^\pi} \chi^v \sum_{\sigma \in v} M^\alpha([\sigma]_{n_\sigma}),$$

and thus we obtain the following (rather unpleasant) relationship between χ^α and the values of M^α on cylinders:

$$\chi^\alpha(g) = \sum_{\pi \in \mathbf{SP}(n)} \sum_{v \in V^\pi} \sum_{\sigma \in v} \chi^\pi(g) \chi^v(g) M^\alpha([\sigma]_{n_\sigma}), \qquad g \in U_n(\mathbb{F}_q).$$

A brief note on some combinatorial aspects

Let **NCSym** denote the $\mathbb{R}$-algebra of symmetric functions on an infinite number of non-commuting variables $X = \{x_1, \dots, x_n, \dots\}$, that is, the algebra of formal power series over X which are invariant under permutation of the variables (for more details see [81] and references therein). For every $\pi \in \mathbf{SP}(n)$ the *non-commuting symmetric monomial function* $\mathbf{m}_\pi$ is defined as the sum of all monomials of length n over X where the variables in each monomial are equal if and only if the corresponding positions lie in the same block of π. The set of all such monomials is linearly independent and **NCSym** is linearly spanned by $\{\mathbf{m}_\pi \colon \pi \in \mathbf{SP}(n),\ n \in \mathbb{N}\}$.

On the other hand, denote by $\mathscr{E}_n$ the uncolored supercharacters of $U_n(\mathbb{F}_q)$ and by $\mathbf{SC}_n$ the $\mathbb{R}$-linear span of $\mathscr{E}_n$, and let

$$\mathbf{SC} = \bigoplus_{n\in\mathbb{N}} \mathbf{SC}_n;$$

for every $\pi \in \mathbf{SP}(n)$ let $k_\pi : U_n(\mathbb{F}_q) \to \mathbb{R}$ be the function defined for every $g \in U_n(\mathbb{F}_q$ by

$$k_\pi(g) = \begin{cases} 1, & \text{if } g \in K_\pi, \\ 0, & \text{otherwise.} \end{cases}.$$

It is clear that $\{k_\pi \colon \pi \in \mathbf{SP}(\mathbb{N})\}$ forms a basis for $\mathbf{SC}$. In [1] both $\mathbf{SC}$ and $\mathbf{NCSym}$ where equipped with a Hopf algebra structure, and it was proved that the mapping $k_\pi \mapsto \mathbf{m}_\pi$ defines an isomorphism of Hopf algebras (the exact Hopf algebra structure on both $\mathbf{SC}$ and $\mathbf{NCSym}$ is not relevant in the discussion that follows).

The combinatorial relationship between $\mathbf{SC}$ and $\mathbf{NCSym}$ yields a parallel with the representation theory of the symmetric groups: let $\mathrm{Irr}(S_n)$ denote the set of irreducible characters of S_n, and let R_n denote the $\mathbb{R}$-span of $\mathrm{Irr}(S_n)$. Then, the set

$$R = \bigoplus_{n\in\mathbb{N}} R_n$$

can be equipped with a Hopf algebra structure which is isomorphic to the $\mathbb{R}$-algebra $\mathbf{Sym}$ consisting of all symmetric functions on infinitely countable commutative variables. For this reason, the uncolored supercharacter theory fo $U_n(\mathbb{F}_q)$ may be seen as a "non-commutative version" of the irreducible character theory of S_n.

Furthermore, one can consider a different product on $\mathbf{SC}$ (related to the product of the Hopf dual $\mathbf{SC}^*$) and its relationship with the superbranching graph of $U_\infty(\mathbb{F}_q)$ (details can be found in [8]) is an analogue of the relationship between $\mathbf{Sym}$ and the Young graph, which is the branching graph of S_∞ (for a detailed discussion on the subject we refer to [25]), stretching the aforementioned analogy.

The most natural way to relate $\mathbf{NCSym}$ and $\mathbf{Sym}$ is *via* the projection map $\rho : \mathbf{NCSym} \to \mathbf{Sym}$ which simply allow the variables to commute. While ρ is well understood in terms of combinatorics (see [81]), its representation theoretical meaning is not clear.

However, if m_λ is the monomial symmetric function on commutative variables associated to $\lambda \in \mathbb{Y}_n$, [81, Theorem 2.1] implies that for every $\pi \in \mathfrak{o}_\lambda$

$$\rho(\mathbf{m}_\pi) = \dim(\lambda) m_\lambda,$$

which should, not only explain how S_∞-invariant properties on the superbranching graph of $U_\infty(\mathbb{F}_q)$ are translated into the Kingman graph, but also provide some more knowledge on the representation theoretical nature of the map ρ.

This facts, together with the Hopf algebra isomorphism given by $k_\pi \mapsto \mathbf{m}_\pi$ suggest that, for every $\alpha \in \Delta$, the values of the corresponding coherent system on cylinders describe the values of the restriction of χ^α to $U_n(\mathbb{F}_q)$ in terms of the basis $\{k_\pi \colon \pi \in \mathbf{SP}(n)\}$ rather than its values on the uncolored supercharacter basis; further analysis is required.

We strongly believe that the understanding of such combinatorial aspects would bring a rich knowledge both combinatorial and representation theoretical, and thus more investigation is required.

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