The development of teachers’ knowledge in a lesson study

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Abstract

Purpose – This paper seeks to understand how teachers’ mathematical knowledge for teaching (MKT) is developed during a lesson study regarded as a professional development process that places particular emphasis on teachers’ knowledge of tasks, representations and students’ learning.

Design/methodology/approach – This study is based on the networking of two theories, namely the Interconnected Model of Teacher Professional Growth (IMTPG) and MKT. The methodology is qualitative and interpretative. The participants are three primary school teachers and team members involved in a lesson study. Data collection was conducted by means of participant observation, with audio recordings of the sessions and a final individual interview.

Findings – The participant teachers developed their knowledge of tasks and representations, which are essential elements of content and teaching knowledge, as well as knowledge of the curriculum, content and students. Additionally, they developed specialized knowledge of mathematics, the meaning of fraction as a measure and its representation on the number line. This development arose from activities conducted in the Domain of Practice, in many cases prompted by the External Domain, and was particularly important when these activities gave rise to unforeseen consequences. Connections among the various domains were established through enactment and reflection processes.

Originality/value – The study shows how the networking of two different theories, in this case the IMTPG and MKT, may enhance further understanding of educational phenomena. This networking involved the coordination of these two theories, which were superimposed in the Group Domain.

Keywords Lesson study, Professional development, Didactic knowledge, Interconnected model of teacher professional growth, Primary teachers

Paper type Research paper

Introduction

Lesson study is a professional development process used widely in Japan and China (Fujii, 2014; Huang et al., 2014) which has seen increased implementation in many other countries (Quaresma et al., 2018). This type of teachers’ professional development may comprehend cognitive, affective, social, cultural and organizational elements (Ponte et al., 2016), in close connection with the professional culture and institutional conditions of the country in which it takes place. This paper addresses teachers’ Mathematical Knowledge for Teaching (MKT) (Ball et al., 2008) in the dimensions of specialized content knowledge, knowledge of the curriculum, knowledge of content and students and knowledge of content and teaching.

Its particular aim is to ascertain how such knowledge may develop through a lesson study. This has been a subject of interest for many years (Lewis et al., 2019); however, to date, no convincing findings have been advanced. In order to understand this process, a modified version of the Interconnected Model of Teacher Professional Growth (IMTPG) (Clarke and Hollingsworth, 2002) was used. This model considers several interacting domains, assuming that teachers’ growth depends in great measure on their reflection on the Domain of

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Consequences, involving the perception of salient outcomes of activities conducted in the Domain of Practice, which includes personal experimentation. Building on the work advanced in the study by Ponte et al. (2022), this paper offers an adaptation of the IMPTG, considering the professional development of a group of teachers (and not of single teachers).

Hence, this study requires the coordination of the aforementioned theories (Kidron and Bikner-Ahsbahs, 2016; Prediger et al., 2008), the IMTPG (Clarke and Hollingsworth, 2002) and MKT (Ball et al., 2008) with further elaboration, as proposed by Ponte (2012). The research question is to ascertain how teachers' MKT develops during a lesson study regarded as a professional development process with particular emphasis on the knowledge of tasks, representations and students’ learning.

The interconnected model of teacher professional growth

The IMTPG (Clarke and Hollingsworth, 2002) is a particularly useful model to understand the process of teachers' knowledge development. This model considers four main domains, the Personal Domain (involving knowledge, beliefs and attitudes), the Domain of Practice (involving personal experimentation), the Domain of Consequences (involving salient outcomes) and the External Domain (the external source of information or stimulus, in this case the lesson study process).

Building on the study by Ponte et al. (2022), this paper offers an adaptation of the IMPTG, considering the work of the group of teachers as the unit of analysis. In this adapted model, instead of Personal Domain, Group Domain is considered as the knowledge shared during the group work [1]. In many studies that use the IMTPG (Clarke and Hollingsworth, 2002) the Domain of Practice is restricted to classroom experimentation. However, the original IMTPG does not observe that restriction, considering other potential forms of professional experimentation. In this study, the Domain of Practice follows the original model and comprehends the practice of the group in the lesson study sessions as well as the practice of the teachers in their classroom, including the research lesson in the lesson study. The External Domain is similar in both models, and comprises the elements brought to the work of the group by the facilitator. The Domain of Consequences is also similar and includes participants’ reflections on the effects of a planned activity practiced with their students or conducted in the lesson study group. The reflection and enactment processes establish connections among the four domains. In reflection, the teachers consider the effects of an occurrence in one domain on another, while in enactment, they transfer elements from one domain to another.

Several studies on teachers' professional development through a lesson study have used the IMTPG. For example, Verhoef et al. (2014) studied a group of secondary school teachers designing a lesson on the derivative using GeoGebra. The participants reported having learned to use visualizations and value interactions among the students. They claimed that recourse to GeoGebra to address the derivative had led them to reflect on how the students grapple with learning activities. In another study, Widjaja et al. (2017) addressed teachers' growth in their planning activity, as they participated in a lesson study. They analyzed teachers' growth in the participants' knowledge of lesson planning, concluding that the teachers had developed in several dimensions, namely in teachers' collaborative planning skills, attention to students' mathematical thinking, use of whole-class discussions and collaborative practices for teacher inquiry. Another study by Agricola et al. (2022), which also used the IMTPG, aimed to identify how the learning activities conducted in a lesson study contributed to how university supervisors perceived changes in their pedagogical content knowledge. The results showed that the lesson study contributed to the development of the supervisors' knowledge of instructional strategies and student understanding. The learning activity that contributed most to these changes was the participants' reflection on their own
practice and that of their students. Beyond the IMTPG, all these studies, whether explicitly or implicitly, call on some framework of Pedagogical Content Knowledge (PCK) or Didactic Knowledge but do not generate a discussion on the networking of theories.

**Teachers’ knowledge of mathematics teaching**

Teachers’ knowledge has featured as a highly active field of research in mathematics education since the seminal works of Shulman (1986, 1987), who coined the notion of “Pedagogical Content Knowledge”. He regarded this notion as a special blend of content and pedagogy, providing a key resource for teachers to design their teaching based on an understanding of students’ learning processes and difficulties. For mathematics teaching, this notion was further elaborated by Ball et al. (2008), who proposed the theory of “mathematical knowledge for teaching”, embracing content knowledge (CK) and PCK. These authors subdivided PCK into three domains, that is, knowledge of content and students, knowledge of content and teaching and knowledge of content and curriculum. They also divided the CK domain into three subdomains, namely common CK, specialized CK and horizon CK. The main feature of this theory is the division of CK and PCK into several domains, highlighting the critical importance of two domains, specialized CK and knowledge of content and students.

Ponte (2012) highlighted a connection between the notions of PCK and the notion of Didactic Knowledge (DK), with a long tradition in European countries (Margolinas et al., 2005). According to Ponte (2012), elements of teachers’ DK play a key role in the teaching activity, such as knowledge of the role and design of tasks, knowledge of representations, knowledge of students’ learning processes and difficulties and knowledge regarding the curriculum. It should be noted that his view on students’ learning processes and difficulties is in line with Ball et al’s (2008) knowledge of content and students, and knowledge of the curriculum is also similar in both cases. Knowledge of the role and design of tasks and of representations are two main elements highlighted by Ponte (2012), which are in keeping with Ball et al’s (2008) knowledge of content and teaching.

Extensive research on the lesson study has sought to ascertain the effects of this professional development process on teachers and prospective teachers’ PCK or DK. For example, Leavy and Hourigan (2016) conducted a study with 25 prospective primary teachers using a lesson study to design, teach and reflect upon early number lessons. They claimed that the lesson study had promoted the development of PCK, especially in the domains of knowledge of content and students and knowledge of content and teaching. The authors also underlined that “[r]eflecting on classroom teaching facilitated growth across both knowledge subdomains and resulted in highly integrated and robust pedagogical understandings that transferred beyond the study context” (p. 161). In a large-scale project in the UK, Warwick et al. (2016) studied teacher discussions during a lesson study. They identified learning in knowledge of content and teaching (in topics such as purposes/importance of lesson objectives, lesson structures and suitability of teaching methods/activities) and learning regarding knowledge of content and students (specific success criteria, student learning, students’ mathematical skills and students’ needs). In an additional study, Clivaz and Ni Shuilleabhain (2019) investigated primary teachers’ use of CK and PCK elements at different levels of teacher activity in a lesson study. The authors concluded that all the domains of MKT, at each level of teacher activity, can occur across a cycle, albeit with varying degrees of emphasis, with knowledge of content and teaching being by far the most prevalent. It should be noted that in this study two different theories are explicitly networked, namely, MKT (Ball et al., 2008) and Levels of Teaching Activity (Margolinas et al., 2005). The most noteworthy aspects in the outcomes of lesson study regarding teachers’ PCK or DK are closely related to the emphasis of the activities and discussions conducted during the lesson study process.
In this paper, two theories are coordinated: the IMTPG and the MKT. The IMTPG comprehends the domains and processes involved in teachers’ professional development while the MKT covers the domains of teachers’ knowledge. The coordination of the two theories occurs mainly through the Group Domain which is related to teachers’ knowledge and is common to both theories. The MKT also provides elements for interpreting what occurs in the other domains; however it is in the Group Domain that this coordination is stronger. The two theories are coordinated (and thus combined) and not contrasted or integrated (Bikner-Ahsbahs et al., 2016; Prediger et al., 2008) since in our view they refer to different empirical elements that connect essentially at a point – the Group Domain.

Methodology
The methodology was qualitative and interpretative (Erickson, 1986). Data were taken from a 12-session lesson study with primary school teachers in which the main facilitator was the second author of this paper. The first and third author also participated in some sessions, such as the research lesson, the post-lesson reflection and the final reflection of the whole lesson study. The participant teachers were Irina, Manuela and Antónia (pseudonyms)[2], all tenured teachers with 10–15 years of experience. Manuela and Antónia had general primary teacher training, whereas Irina had a specialization in teaching mathematics and science.

The lesson study had nine work sessions, plus three follow-up sessions. Session 1 aimed to present the lesson study to all the participating teachers; Sessions 2 to 7 aimed to deepen their knowledge of the chosen topic and prepare a lesson on this topic, with Session 6 entirely dedicated to the design of the task to be used in the research lesson; Session 8 consisted of observing a lesson; and Session 9 was dedicated to reflecting on the lesson observed, and on the work achieved so far. In Sessions 10 to 12, the teachers were asked to plan, conduct and reflect on two lessons according to the work developed during the lesson study.

All the lesson study sessions (Sx) were audio recorded and individual interviews (I) were conducted with the participant teachers. For the purpose of this paper, specific episodes for analysis have been selected in which the abovementioned two theories are coordinated (Prediger et al., 2008), namely, the adapted IMTPG (Clarke and Hollingsworth, 2002), and MKT (Ball et al., 2008).

The data were analyzed through content analysis (Bardin, 1979) with a view to identifying illustrative aspects of mobilized or developed MKT in the teachers’ discourse concerning curriculum, tasks, representations (key elements of content and teaching knowledge) and knowledge of students. On the basis of the data, the importance of taking the specialized knowledge of mathematics category into account became apparent. Our consideration of the two theoretical domains (IMTPG and MKT) led to the construction of the categories of analysis indicated in Table 1.

<table>
<thead>
<tr>
<th>IMTPG</th>
<th>MKT</th>
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<tbody>
<tr>
<td>External domain</td>
<td>Knowledge of content and teaching</td>
</tr>
<tr>
<td>Domain of practice</td>
<td>(knowledge of tasks and representations)</td>
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<tr>
<td>Domain of consequences</td>
<td>Knowledge of content and students</td>
</tr>
<tr>
<td>Group domain</td>
<td>Knowledge of content and curriculum</td>
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<td></td>
<td>Specialized content knowledge</td>
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Table 1. Categories of analysis

<table>
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<tr>
<th>Enactment</th>
<th>Reflection</th>
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</table>
The lesson study

The lesson study focused on a Grade 3 topic as a result of the implementation of a new curriculum (Ministério da Educação [Ministry of Education], 2013) [3]. Thus, a striking aspect of this lesson study was the chosen topic, the addition and subtraction of rational numbers by rectilinear juxtaposition of collinear straight-line segments [4]. According to the teachers, this topic had undergone more significant changes in the new curriculum, which could pose more challenges to its teaching. A second important feature in the lesson study was the diagnosis of students’ knowledge. After some discussion, the teachers concluded that it was important to understand the extent of knowledge on rational numbers internalized by the students in the previous year.

Elaboration of the task for the research lesson was also a noteworthy feature. Irina was appointed to teach this lesson, [5] and she suggested dedicating a session to the preparation and discussion of the task. She was completely at ease with the mathematical content and felt confident to create the tasks. In addition, as the curriculum had been subject to significant changes, she felt that there was not yet sufficient diversity in the materials produced for primary school and was not satisfied with what she had found. Thus, Session 6 was entirely dedicated to the preparation of the task. This task was managed largely by Irina while the facilitator supported the discussion, challenging the teachers to reflect on the different aspects of the task and of the lesson. Antónia and Manuela participated actively in the discussion of the ideas presented by their peer but did not feel sufficiently confident to take the initiative and make very elaborate suggestions.

Preparing and reflecting on the research lesson on the addition of fractions

Preparing the research lesson

This lesson study was conducted within the scope of a change in the curriculum, the analysis of which occurred across several sessions. The teachers began by comparing the former and current curricula and identified important changes in the teaching and learning of rational numbers. More specifically, they realized that the learning of these numbers should depart from the measure meaning based on the representation on the number line. This was a totally novel approach for the teachers who had always worked on the concept of rational number in the part-whole and operator meanings [6].

Antónia: This is new ( . . ) The representation on the number line, isn’t it? That is the novelty.

Irina: You’re right, they begin fractions atypically. They don’t begin with the typical images [bars and circles]. They begin with the number line right away. No, but it would be more natural if the first page looked like this! This is what we would expect. A moment ago [the textbook] was open and I went back to see how it began, but it was really there. Pages and pages with the number line! This is going to be ugly . . . (S2)

The teachers considered the prevalence of the number line instead of the representations familiar to the students, such as bars or circular diagrams, to be very odd. In order to make the relationship between the measure meaning and its representation on the number line more understandable, the group also analyzed the representations more commonly used by the teachers and the meanings of the rational numbers they did not know.

Planning for the research lesson began in the fifth session. Throughout the previous sessions, the teachers highlighted the potential difficulties of the students in learning rational numbers based on the measure meaning with the representation on the number line and their willingness to gear the research lesson toward the topic “adding rational numbers by juxtaposition of collinear straight-line segments”.

However, in this session the teachers found that difficulties in maintaining their schedule were likely to compromise their work on this topic on the date set for the research lesson. Considering that the tasks had not yet been selected, the group considered the possibility of...
changing the topic of the research lesson. Irina disagreed with this suggestion since she deemed it highly challenging for both students and teachers, particularly as it was a topic introduced for the first time at this level of teaching in the new curriculum: “I think we should keep the topic . . . It is one of the most complicated . . .” (S5). Finally, all the teachers agreed with the decision to stick to the topic.

Irina also asked to dedicate one more session than initially foreseen to the planning of the lesson, so that the group could help her develop the task she had devised with a story about a sports competition with animals. Thus, the sixth session was given to preparing the task. Prior to the session, Irina emailed a list of ideas for the task that informed all the work in the session.

Irina began with an exercise from the supporting material she deemed very difficult. She reported having found a task in the new curriculum support materials [7] involving the juxtaposition of collinear straight-line segments which she considered unsuitable since it involved no context. This idea was expressed as follows: “This was an exercise for the sake of an exercise” (S6). She proposed making the task more relevant and motivating for the students by means of a story about a competition between animals involving a relay race:

Irina: I think we can get their attention if there is an underlying logic. That will make them more willing to discover. Because this, as it stands, is quite complicated . . . So, the idea would be to mark the jumps on the number line and see how they get on in that regard. I do not know how many teams, but that is alright.

Marisa: OK, let’s do it now.

Irina: And then the teams . . . There were all teams . . . Let us think about having three teams with two members . . . And I thought that they could make different journeys since they’re not all the same size . . . [. . .]

Marisa: The first step would have to be the same for all of them, wouldn’t it?

Irina: No.

Marisa: In the relay race.

Antónia: First, there are the jumps, aren’t there?

Irina: Yes, but I’m already talking about the addition part.

Marisa: I don’t know! Do they know what a relay race is?

Irina: Yes!

Marisa: We have a problem with the relay race. In the relay race the distances are usually the same.

Irina: But this is different because it’s with animals. (S6)

Based on the context proposed by Irina, the task for the research lesson was designed (see Figure 1). In the first question, there was a long jump contest where the students were asked to mark a fraction on a number line already presented in the statement of the question and to conclude whether this fraction was greater or smaller than 1. The second question was related to the animal-adapted relay race and involved the addition of fractions.

The question was intended to lead the students to add fractions with different denominators by juxtaposition of collinear straight segments that corresponded to the distances traveled by the animals. In the seventh session, when the lesson plan was finalized, Irina stressed that “the number line is the basis” in all the task questions.

**Collective reflection**

After the research lesson (S8), a post-lesson reflection (S9) was conducted. In general, the students did not present difficulties in the first question and marked the fractions requested
on the number line correctly. However, one student who presented a solution calculated with a peer on the board displayed difficulty in marking the 5/8 fraction:

Irina: What they didn’t see, those who were a little closer, was that there were hardly any problems with 5/8 except for with Tiago and Madalena’s group. Because of what Tiago had said, that he got confused about when it is greater than one or smaller than one. I think it was on this one, wasn’t it?

Marisa: Exactly, it was there.

Irina: We began with something that was wrong which left us in a complete mess.

Antónia: Because he confused the denominator and numerator, right?

Irina: Yes, because he said when . . . He changed, instead of the denominator being larger, he said that when it was smaller it was larger than one. And then, he didn’t know where he should put . . . (S9)

Irina highlighted the students’ common confusion between the numerator and denominator of a fraction and referred to this being particularly problematic when using memorized rules to compare fractions.

This reflection prompted the group to review the meanings of the fractions involved in the tasks and to acknowledge the need for an understanding of the concept of fraction in these meanings. João Pedro indicated the involvement of two meanings of a rational number in the
task, part-whole and measure, and the understanding of the meaning of part-whole was crucial to students’ understanding of the concept of fraction beyond the memorization of rules. However, as far as Irina was concerned, this difficulty was caused by the use of the number line:

Irina: Yes, and I am almost sure that if it hadn’t been for the representation on the number line they would have all got this [question] of 5/8 wrong and would not have even thought of this rule.

Given that the new curriculum required in-depth work on the number line, the group considered deepening the discussion around the students’ work in the research lesson to be crucial:

João Pedro: I mean, the idea of dividing into eight parts and that . . . Basically, I think the problem is this: what is 1/8? Is 1/8 this point or is 1/8 this arc [connecting zero and the marked point]? What is 1/8?

Irina: 1/8 is each of the parts.

. . .

João Pedro: All right, I have measure and part-whole . . . My doubt is whether or not the students understood. Because I think when they were marking . . . I have a question . . . However, I would like to know what you think. I wonder if some of them were thinking about the line segment.

Manuela: I think that perhaps you are right, they focus more on the points and then they get lost with the length, ultimately with the measure. That’s what I think.

João Pedro: What do you think, Antónia?

Antónia: I think so. I mean, because 1/8 is not from zero to the end of the first bit, is it? from that part, because I can have it there . . . But I have 1/8 in all the other divisions, don’t I? It is that that they have to understand, that 1/8 is not just that first bit, but they have 1/8 there and they have 1/8 there and they may have 1/8 ahead, isn’t it?

Manuela and Antónia agreed with the interpretation of João Pedro regarding the students’ difficulty. Antónia stressed the need for the students to understand that 1/8 is the distance between any two consecutive points marked on that line and, therefore, indicates a measure that can be iterated on this line.

The second question of the task requested the addition by rectilinear juxtaposition of collinear straight-line segments. This question posed difficulties to many of the students. João Pedro suggested associating arcs with the measure of the straight-line segments related to the fractions that had to be added:

João Pedro: Halfway through the lesson, this idea of the segment represented by an arc began to appear.

Irina: In the relay races.

João Pedro: In the relay races, I think it appeared. Someone, probably Irina . . . At some point explained . . .

Antónia: There has to be a juxtaposition there. Therefore, we cannot put 1/8 [and then] 3/8 . . . And mark all that well, can we? Because we have a sum and a juxtaposition. Therefore, it is already different.

Antónia stressed the need for students to understand fractions as a measure of straight-line segments in order to understand the addition of fractions using the number line. Indeed, in the latter case they can no longer mark fractions from zero, and need to start from the end of the previous segment.
Individual reflections

The group concluded that in the planning of the lesson considerable focus had been placed on the number line representation and little attention had been paid to the idea of the segments' measure. This was clear in the individual interviews held with all the teachers following the reflection on the research lesson. In this interview, Irina reflected on the issue of marks on the number line:

Irina: I focused on the points and probably, for some [students], that didn't make as much sense as it should've . . . If we are really referring to distance, it is the line segment itself. (I)

In addition to highlighting the fractions as straight-line segments, Irina stressed that she had not been able to anticipate the students’ difficulty:

Irina: For us it was obvious! But for them it wasn't. They got it in the end, but if they had got it earlier . . . Without it, speaking of the point, of the point . . . For me, it was obvious that it would run until that point! But, for some of the kids no . . . It wasn't! It was the mark . . . That was what they were used to! The previous day, I had been asking them, on the same number line, for them to mark several points. Even with different denominators ( . . . ) They were stuck to what I had done on the previous day. It was probably because of that. And I did not mention distances . . . We didn't even speak about that. (I)

In addition to failing to anticipate this difficulty on the part of the students, Irina also identified that her own prior practice might have been a contributory factor, since she had been unaware of the need to help students understand fractions as a measure of straight-line segments. She mentioned that after the research lesson she had begun to consider marking the segments:

Irina: Today, for example, I took care to mark the distances they were talking about with different colors, so that they didn't think that was the point. However, on that day, I didn't. Not even in the planning had I thought [about that] . . . (I)

Sharing the same perspective, Antónia said that the discussion had allowed her to reflect on how she could work on the topic with her students to overcome the difficulties presented by Irina’s students:

Antonia: It helped me reflect. It helped . . . I think essentially it helped me to better reflect on the way to apply it in the classroom. And, perhaps, also to look at fractions a little bit differently, isn't it? Therefore . . . We are on the number line . . . That is a novelty, at the end of the day it’s a novelty. The new approach is in fact the number line, isn’t it? (I)

Antónia also stressed that the teaching and learning of rational numbers with a focus on representation on the number line was a major novelty of the new curriculum and that it was now also necessary to reflect on the demands and difficulties that this represented for the students.

Finally, Manuela framed her learning about teaching and learning rational numbers based on the number line in the entire lesson study:

Manuela: For me [the work done] helped a lot. That is, being able to see more clearly what a fraction is, the notion of proper and improper fractions, equivalent fractions, we visualized fractions on the number line, therefore, it covered everything. For me, it is a relief to feel, I mean, comfortable. And prepared . . . Because the curriculum is demanding and it increasingly demands more from us. (I)

In Manuela’s view, the work conducted in the lesson study allowed her to develop the knowledge needed to effectively implement the new curriculum that introduced various types of knowledge about rational numbers and to manage the teaching and learning involved.
The development of teachers’ knowledge

Based on the analysis of the new curriculum, the teachers decided to plan the research lesson on the topic of adding and subtracting rational numbers by juxtaposition of collinear straight-line segments, since it was a topic with which they were unfamiliar and for which they anticipated difficulties for the students. Given the teachers’ preference to deepen this important topic, they even decided to postpone the date of the research lesson. Based on this preference, in Session 5 the facilitator proposed to begin planning the research lesson (External Domain). Through an enactment process from the External Domain (Figure 2, Arrow 1), the planning of the research lesson began.

In the planning of the research lesson (Domain of Practice), Irina, who was going to teach the topic, proposed that an entire session should be dedicated to the elaboration of a task to meet the objectives of the lesson. The nature of the task had been explored in the former lesson study sessions with activities proposed by the facilitator (External Domain). This work had led to Irina’s realization that more challenging tasks might foster the students’ discovery of new concepts (knowledge of teaching practice – tasks). As tasks of this nature were not available in the consulted materials, she requested help from the group to prepare an open task in an engaging and motivating context for the students. In the preparation of the task, she began with a simpler task with which most of the students were already familiar, involving marking fractions on the number line and comparing them. Only then would the students be challenged to use the number line to juxtapose different collinear straight-line segments to add the fractions with different denominators. The context chosen by Irina was a relay race with animals in an attempt to give meaning to the juxtaposition of the collinear segments. The preparation of the task and the planning of the lesson promoted in-depth reflection on the Domain of Practice that allowed for the development of new knowledge on the curriculum and on challenging tasks for teaching and learning the addition of fractions (knowledge of tasks) in the Group Domain (Figure 2, arrow 2).

Figure 2.
Teachers’ learning dynamics in working to assess students’ previous knowledge

<table>
<thead>
<tr>
<th>Group domain</th>
<th>External domain</th>
<th>Domain of practice</th>
<th>Domain of consequences</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Knowledge of the new curriculum and</td>
<td>- Proposal for planning the research lesson.</td>
<td>- Elaboration of the task for the research lesson with focus on the addition of fractions in the measure meaning represented in the number line;</td>
<td>- Students’ solutions and strategies;</td>
</tr>
<tr>
<td>challenging tasks for the teaching and learning of</td>
<td>- Information on different meanings of fraction.</td>
<td>- Research lesson;</td>
<td>- Students’ main difficulties in understanding fractions in the measure meaning using the number line;</td>
</tr>
<tr>
<td>the addition of fractions in the measure meaning represented on the number line;</td>
<td></td>
<td>- Post lesson discussion, with analysis of the work conducted by students in the research lesson.</td>
<td></td>
</tr>
<tr>
<td>- Knowledge of fractions in the measure meaning represented on the number line;</td>
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<tr>
<td>- Knowledge of the students’ difficulties, and the way they develop their understanding of fractions in this meaning and representation.</td>
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Change environment.

Enactment

Reflection

Reflection
Using the knowledge developed in the planning, by a process of enactment, Irina taught the research lesson which was observed by the remaining participants in the Domain of Practice (Figure 2, arrow 3). The post-lesson discussion on the Domain of Practice was then conducted, where the students’ efforts to solve the tasks were then analyzed. The teachers analyzed the difficulty of the student Tiago related to the meaning of the fraction, using their knowledge in the Group Domain. The discussion then focused more specifically on the difficulty displayed by the students in understanding the marking of fractions as a measure of segments, a process of reflection between the Domain of Practice and the Domain of Consequences (Figure 2, arrow 4).

The post-lesson discussion in the Domain of Practice prompted an in-depth reflection on the teaching and learning of fractions in measure meaning and its representation on the number line (Figure 2, arrow 5). This discussion considering the students’ difficulties led to further development of the teachers’ knowledge (Figure 2, arrow 6). This knowledge was related to the fraction in measure meaning and its representation on the number line (specialized knowledge of mathematics), students’ difficulties and the way students develop the understanding of fractions in this meaning and its representation (knowledge of students and their learning process).

Conclusion
The IMTPG (Clarke and Hollingsworth, 2002) enabled the identification of important aspects that contributed decisively to the growth of knowledge of the group of teachers who participated in this lesson study. The most observed salient aspects of knowledge development were mainly knowledge of the new curriculum (knowledge of the curriculum), knowledge of tasks and representations (aspect of knowledge of content and teaching) and knowledge of students’ knowledge, difficulties, and learning (knowledge of content and students). Additionally, knowledge of the meanings of fractions, namely, the measure meaning, also emerged as an important feature, which the teachers had not known (specialized knowledge of the content). It should be noted that the learning of this meaning by the teachers was only fully clarified after the post-lesson discussion, given that initially their focus was geared toward the representation of points on the number line and not on the measurements of the segments represented thereon.

The preparation and teaching of the lesson and the post-lesson reflection highlight the role of the various domains of the modified IMTPG. The role of the External Domain is evident in the successive proposals and questions of the facilitator and, also, in the resources accessed by the teachers to organize their work. The Domain of Practice is where all the work occurs, in this case the lesson study sessions and the research lesson. The Group Domain comprehends the teachers’ prior knowledge and the new knowledge they acquire with their participation in the lesson study. Finally, in the Domain of Consequences the salient aspects of the actions taken in the Domain of Practice that make teachers reflect and produce new knowledge are highlighted. These aspects are mainly related to the students’ difficulties regarding the concept of fraction in measure meaning, which led the teachers to acknowledge the need to use representation on the number line differently, in order to clarify this concept. The reflection and enactment processes play a fundamental role in establishing connections between these domains. Reflection leads to a consideration of the actions taken and an evaluation of their consequences. Enactment, in turn, triggers actions, frequently giving rise to both predicted and unforeseen outcomes, which then become a matter for later reflection.

Previous research by Verhoef et al. (2014), Widjaja et al. (2017) and Agricola et al. (2022) has already shown how the IMTPG (Clarke and Hollingsworth, 2002) can be useful to analyze lesson study processes. This study adds to that body of knowledge by highlighting the importance of the knowledge shared and constructed by the group of teachers and by
underlining the key role of tasks and representations used in mathematics teaching – two notions that emerge in this study but which were very much in the background of previous studies. The main limitation in this study is that in view of space restrictions, only the analysis of a single situation is provided. The analysis of other situations may provide further elements for an enhanced understanding of the MKT elements that may develop in a lesson study and the specific mechanisms by which they may take place.

The coordination of two theories, the IMTPG and MKT, promoted an in-depth understanding of the learning process experienced by this group of teachers. The coordination occurred through the Group Domain, namely with regard to the teachers’ knowledge of mathematics and the teaching of mathematics relevant to this study. This may be regarded as a good example since the networking of two theories, through their coordination, may offer helpful support to understanding educational phenomena related to the development of teachers’ knowledge.

Notes
1. For simplification purposes, beliefs and attitudes are not addressed.
2. Tiago and Madalena, the names of the students in the excerpts, are also pseudonyms.
3. This new curriculum was viewed with considerable concern by most teachers since it represented a significant shift from a perspective of mathematics learning, with students’ work and exploration at the core, to a rather formal approach to mathematics activity, with a focus on algorithms and symbolic representations.
4. This new curriculum emphasized the measure meaning of rational numbers (Charalambous and Pitta-Pantazi, 2007). This feature contrasts with the former curriculum that placed an emphasis first on the part-whole meaning and, later on the operator meaning.
5. After some discussion, Irina volunteered to lead the research lesson as her two peers had firmly refused to do so. This process is explained in detail in the study by Quaresma and Ponte (2021).
6. Primary teachers in Portugal teach Grades 1–4. Under the former curriculum, only the part-whole and operator meanings were addressed for rational numbers.
7. Along with the new curriculum, the Ministry of Education provided support materials on its website to help teachers with its implementation. These materials were designed from a rather formal perspective, providing tasks that used mostly symbolic representations.

References
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